第 14 章　數學附錄

一　$\hat{\alpha}$ 與 $\hat{\beta}$ 的分配

由(15.11)式可知：

$$\hat{\beta}=\frac{\sum xy}{\sum x^2}=\frac{\sum x(Y-\overline{Y})}{\sum x^2}=\frac{\sum xY}{\sum x^2}-\frac{\overline{Y}\sum x}{\sum x^2}$$

$$=\frac{\sum xY}{\sum x^2}=\frac{\sum x(\alpha+\beta X+\varepsilon)}{\sum x^2}=\frac{\alpha\sum x}{\sum x^2}+\frac{\beta\sum x\cdot X}{\sum x^2}+\frac{\sum x\varepsilon}{\sum x^2}$$

$$=\beta+\frac{\sum x\varepsilon}{\sum x^2}\ (\text{因}\sum x^2=\sum x\cdot X\ )(15A.1)$$

當 ε 為常態分配時，$\hat{\beta}$ 為 ε 的線性函數，根據常態分配的加法定理，$\hat{\beta}$ 是常態分配。

□ $\hat{\beta}$ 的不偏性

由(15A.1)可得知：

$$E(\hat{\beta})=E(\beta+\frac{\sum x\varepsilon}{\sum x^2})=\beta+E(\frac{\sum x\varepsilon}{\sum x^2})$$

$$=\beta+\frac{\sum xE(\varepsilon)}{\sum x^2}\qquad(\text{因 } X \text{ 為固定變數})$$

$$=\beta+\frac{x_1E(\varepsilon_1)+x_2E(\varepsilon_2)+...+x_nE(\varepsilon_n)}{\sum x^2}\qquad(\text{因 } E(\varepsilon_i)=0\ )$$

由此證明過程可知，在 X 為固定變數，$E(\varepsilon_i)=0$ 的條件下，$\hat{\beta}$ 才具有不偏性。

□ $\hat{\beta}$ 的變異數

$$V(\hat{\beta})=\sigma_{\hat{\beta}}^2=E[\hat{\beta}-E(\hat{\beta})]^2=E(\hat{\beta}-\beta)^2$$

$$=E(\frac{\sum x\varepsilon}{\sum x^2})^2\qquad(\text{由(15A.1)式可知})$$

$$=E[\frac{(\sum x\varepsilon)^2}{(\sum x^2)^2}]$$

$$=\frac{1}{(\sum x^2)^2}E[x_1^2\varepsilon_1^2+x_2^2\varepsilon_2^2+...+x_n^2\varepsilon_n^2+2x_1x_2\varepsilon_1\varepsilon_2+...+2x_{n-1}x_n\varepsilon_{n-1}\varepsilon_n]$$

$$=\frac{1}{(\sum x^2)^2}[x_1^2\sigma^2+x_2^2\sigma^2+...+x_n^2\sigma^2+0...0]$$

(因 $V(\varepsilon_i)=\sigma^2, E(\varepsilon_i\varepsilon_j)=0, i\neq j$)

$$=\frac{1}{(\sum x^2)^2}\sigma^2\sum x^2\ =\frac{\sigma^2}{\sum x^2}$$

由以上知，$V(\varepsilon_i)=\sigma^2$ (變異數齊一性)及 $E(\varepsilon_i\varepsilon_j)=0$ (無自我相關)的假設下，可得：

$V(\hat{\beta})=\dfrac{\sigma^2}{\sum x^2}$

□ $\hat{\beta}$ 的分配

由上述的證明可知在假設 $E(\varepsilon_i)=0, V(\varepsilon_i)=\sigma^2, E(\varepsilon_i\varepsilon_j)=0$，$\varepsilon_i$ 為常態分配以及 X 為固定變數條件下，$\hat{\beta}$ 的分配為：

$$\hat{\beta}\sim N(\beta,\frac{\sigma^2}{\sum x^2})$$

因此由上可知迴歸分析的這些假設會影響$\hat{\beta}$的分配。

□ $\hat{\beta}$ 為一 BLUE

根據 Gauss-Markov 定理，$\hat{\beta}$ 為一 BLUE，即在假設 $E(\varepsilon_i)=0$ ，$V(\varepsilon_i)=\sigma^2$，$E(\varepsilon_i\varepsilon_j)=0$，$E(x\varepsilon_i)=0$ 下，$\hat{\beta}$為所有不偏估計式中變異數最小者。

證明：設$\tilde{\beta}$為任一線性估計式，表為：

$$\tilde{\beta}=\sum_{i=1}^{n}C_iY_i \qquad (\tilde{\beta} \text{ 為 } Y_i \text{ 的一次函數})$$

若 $\tilde{\beta}$ 為 β 之不偏估計式，則

$$E(\tilde{\beta})=E(\sum_{i=1}^{n}C_iY_i)=E[\sum_{i=1}^{n}C_i(\alpha+\beta X_i+\varepsilon_i)]$$
$$=\alpha E(\sum_{i=1}^{n}C_i)+\beta E(\sum_{i=1}^{n}C_iX_i)+E(\sum_{i=1}^{n}C_i\varepsilon_i)=0+\beta\cdot 1+0=\beta$$

由上式可知，若 $E(\tilde{\beta})=\beta$，則 $\sum C_i=0, \sum C_iX_i=1$

$$V(\tilde{\beta})=V(\sum_{i=1}^{n}C_iY)=V(C_1Y_1+...+C_nY_n)$$
$$=C_1^2V(Y_1)+...+C_n^2V(Y_n)+\sum_{i=1}^{n}\sum_{j=1}^{n}C_iC_jCov(Y_iY_j)$$
$$=C_1^2\sigma^2+...+C_n^2\sigma^2+0=\sigma^2\sum_{i=1}^{n}C_i^2$$

$Cov(Y_iY_j)=E[(Y_i-E(Y_i))(Y_j-E(Y_j))]=E(\varepsilon_i\varepsilon_j)=0$

若證明$V(\tilde{\beta})\geq V(\hat{\beta})$，則可得證 $\hat{\beta}$ 為 BLUE，求算$V(\tilde{\beta})$。

$$\hat{\beta}=\frac{\sum x_iy_i}{\sum x_i^2}=\sum w_iy_i$$

令$W_i=\dfrac{x_i}{\sum x_i}$　　$c_i=W_i+d_i$則

$$V(\tilde{\beta}) = \sigma^2 \sum (W_i + d_i)^2 = \sigma^2 \sum (W_i^2 + 2W_i d_i + d_i^2)$$

$$= \sigma^2 (\sum W_i^2 + \sum d_i^2) = \sigma^2 [\frac{1}{\sum x^2} + \sum d_i^2] \geq \frac{\sigma^2}{\sum x^2}$$

即 $V(\tilde{\beta}) \geq V(\hat{\beta})$

上式中：　$\sum W_i d_i = \sum W_i (c_i - W_i) = \sum W_i c_i - \sum W_i^2 = 0$

其中：

$$\sum W_i C_i = \frac{\sum x_i C_i}{\sum x^2} = \frac{1}{\sum x_i^2}，\quad \sum W_i^2 = \frac{\sum (x_i)^2}{(\sum x^2)^2} = \frac{1}{\sum x_i^2}。$$

由上式可知，任何一不偏線性估計式 $\hat{\beta}$ 其變異數均大於等於 OLSE $\hat{\beta}$ 的變異數。因此 Gauss-Markov 定理可得證。

同理可證明

$$\hat{\alpha} \sim N(\alpha, \frac{\sum X^2}{n \sum X^2} \sigma^2)$$

$\hat{\alpha}$ 根據 Gauss-Markov 定理亦為一 BLUE。

<二>　$S_{Y|X}^2$ 為 σ^2 的不偏估計式

$$S_{Y|X}^2 = \frac{1}{n-2} \sum e^2$$

$$= \frac{1}{n-2} \sum (Y - \hat{Y})^2 = \frac{1}{n-2} \sum (\alpha + \beta X + \varepsilon - \hat{\alpha} - \hat{\beta} X)^2$$

$$= \frac{1}{n-2} \sum [-(\hat{\alpha} - \alpha) - (\hat{\beta} - \beta) X + \varepsilon]^2$$

$$= \frac{1}{n-2} \sum [(\hat{\beta} - \beta) \overline{X} - \overline{\varepsilon} - (\hat{\beta} - \beta) X + \varepsilon]^2 \qquad = \frac{1}{n-2} \sum [-(\hat{\beta} - \beta) X + \varepsilon']^2$$

$$= \frac{1}{n-2} [(\hat{\beta} - \beta)^2 \sum x^2 + \sum \varepsilon'^2 - 2(\hat{\beta} - \beta) \sum x \varepsilon'] = \frac{1}{n-2} [-(\hat{\beta} - \beta)^2 \sum x^2 + \sum \varepsilon'^2]$$

其中 $\varepsilon' = \varepsilon - \overline{\varepsilon}$

$$E(S_{Y|X}^2) = \frac{1}{n-2} E[-(\hat{\beta} - \beta)^2 \sum x^2 + \sum \varepsilon'^2] = \frac{1}{n-2} [-\sigma^2 + (n-1)\sigma^2] = \sigma^2$$

因 $E(\hat{\beta} - \beta)^2 = \frac{\sigma^2}{\sum x^2}$，故

$$E[\sum \varepsilon'^2] = E[\sum (\varepsilon - \overline{\varepsilon})^2] = E[\sum \varepsilon^2 - n\overline{\varepsilon}^2] = E(\sum \varepsilon^2) - nE(\overline{\varepsilon}^2)$$

$$= n\sigma^2 - n \cdot \frac{\sigma^2}{n} = (n-1)\sigma^2$$

<三>　證明 $\frac{(n-2) S_{Y|X}^2}{\sigma^2} \sim \chi_{n-2}^2$

$$\frac{(n-2)S_{Y|X}^2}{\sigma^2}=\frac{\sum e^2}{\sigma^2}=\frac{-(\hat{\beta}-\beta)^2\sum x^2+\sum \varepsilon'^2}{\sigma^2}$$

$$=\frac{-(\hat{\beta}-\beta)^2}{\frac{\sigma^2}{\sum x^2}}+\frac{\sum \varepsilon'^2}{\sigma^2}=-\chi_1^2+\frac{\sum \varepsilon^2-n\bar{\varepsilon}^2}{\sigma^2}$$

$$=-\chi_1^2+\sum(\frac{\varepsilon-0}{\sigma})^2-\frac{(\bar{\varepsilon}-0)^2}{(\frac{\sigma}{\sqrt{n}})^2}=-\chi_1^2+\chi_n^2-\chi_1^2=\chi_{n-2}^2$$

<四> 證明 $\frac{\hat{\beta}-\beta}{\frac{S_{Y|X}}{\sqrt{\sum x^2}}}\sim t_{n-2}$

$$\frac{\hat{\beta}-\beta}{\frac{S_{Y|X}}{\sqrt{\sum x^2}}}=\frac{\hat{\beta}-\beta\Big/\sqrt{\frac{\sigma^2}{\sum x^2}}}{\sqrt{\frac{S_{Y|X}^2}{\sigma^2}}}=\frac{N(0,1)}{\sqrt{\frac{\sum e^2}{n-2}\Big/\sigma^2}}=\frac{N(0,1)}{\sqrt{\frac{\sum e^2}{\sigma^2}\Big/n-2}}$$

$$=\frac{N(0,1)}{\sqrt{x_{n-2}^2/n-2}}\sim t_{n-2}$$

(根據前述(三)已證明 $\frac{\sum e^2}{\sigma^2}\sim\chi_{n-2}^2$)

<五> 證明 $\sum(Y-\bar{Y})^2=\sum(Y-\hat{Y})^2+\sum(\hat{Y}-\bar{Y})^2$

因

$$\sum(Y-\bar{Y})^2=\sum[(Y-\hat{Y})^2+(\hat{Y}-\bar{Y})^2]$$

$$=\sum(Y-\hat{Y})^2+\sum(\hat{Y}-\bar{Y})^2+2\sum(Y-\hat{Y})(\hat{Y}-\bar{Y})$$

上式中：

$$\sum(Y-\hat{Y})(\hat{Y}-\bar{Y})=\sum e[\hat{\alpha}+\hat{\beta}X-(\hat{\alpha}+\hat{\beta}\bar{X})]=\sum e\cdot\hat{\beta}(X-\bar{X})$$

$$=\hat{\beta}\sum e(X-\bar{X})=\hat{\beta}\sum ex=0$$

因此可證明得：$\sum(Y-\bar{Y})^2=\sum(Y-\hat{Y})^2+\sum(\hat{Y}-\bar{Y})^2$

<六> 證明 $\frac{\sum\hat{y}^2}{\sum e^2/n-2}\sim F_{1,n-2}$

$$\frac{\sum\hat{y}^2}{\sum e^2/n-2}=\frac{\hat{\beta}^2/\sum x^2}{\sum e^2/n-2}=\frac{(\hat{\beta}/\frac{\sigma}{\sqrt{\sum x^2}})^2}{\frac{\sum e^2}{\sigma^2}\Big/n-2}=\frac{\chi_1^2/1}{\chi_{n-2}^2/n-2}\sim F_{1,n-2}$$

因當 H_0 為真時

$$\frac{\hat{\beta}-0}{\frac{\sigma}{\sqrt{\sum x^2}}} \sim N(0,1) \text{ 又 } \frac{\sum e^2}{\sigma^2} \sim \chi^2_{n-2}$$

<七> 證明 $V(\hat{Y}_0)=\sigma^2[\frac{1}{n}+\frac{(X_0-\overline{X})^2}{\sum x^2}]$

$$V(\hat{Y}_0)=V(\hat{\alpha}+\hat{\beta}X_0)=V(\hat{\alpha})+X_0^2V(\hat{\beta})+2X_0Cov(\hat{\alpha},\hat{\beta})$$

$$=\frac{\sigma^2\sum X^2}{n\sum x^2}+\frac{X_0^2\sigma^2}{\sum x^2}+2\cdot\frac{X_0(-\overline{X}\sigma^2)}{\sum x^2}=\frac{\sigma^2}{n\sum x^2}[\sum X^2+nX_0^2-2nX_0\overline{X}]$$

$$=\frac{\sigma^2}{n\sum x^2}[\sum x^2+n\overline{X}^2+nX_0^2-2nX_0\overline{X}]=\frac{\sigma^2}{n\sum x^2}[\sum x^2+n(X_0-\overline{X})^2]$$

上式中

$$=\sigma^2[\frac{1}{n}+\frac{(X_0-\overline{X})^2}{\sum x^2}]$$

$$Cov(\hat{\alpha},\hat{\beta})=E[(\hat{\alpha}-\alpha)(\hat{\beta}-\beta)]=E[-(\hat{\beta}-\beta)\overline{X}-\bar{\varepsilon}](\hat{\beta}-\beta)$$

$$=E[-(\hat{\beta}-\beta)^2\overline{X}+(\hat{\beta}-\beta)\bar{\varepsilon}]=-\overline{X}\frac{\sigma^2}{\sum x^2}+E[(\hat{\beta}-\beta)\bar{\varepsilon}]$$

$$=-\overline{X}\frac{\sigma^2}{\sum x^2} \qquad (\text{因}E[(\hat{\beta}-\beta)\bar{\varepsilon}]=0)$$

此外，亦可得 $\hat{Y}_0$ 之分配為：

$$\hat{Y}_0 \sim N(E(Y\mid X_0),\sigma^2_{\hat{Y}_0})$$

$\hat{Y}_0$ 為 $E(Y\mid X_0)$ 最小變異不偏估計式。

<八>　二元常態分配的機率密度函數

相關分析時，我們假設二元隨機變數 X 、Y 間的聯合機率函數為一常態分配，稱為二元常態分配，其機率函數為：

$$f(X,Y)=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp(-\frac{q}{2})$$

其中　$q=\frac{1}{1-\rho^2}[(\frac{X-\mu_X}{\sigma_X})^2-2\rho(\frac{X-\mu_X}{\sigma_X})(\frac{Y-\mu_Y}{\sigma_Y})+(\frac{Y-\mu_Y}{\sigma_Y})^2]$

且 $-\infty<X<\infty$，$-\infty<Y<\infty$，$\sigma_X>0$，$\sigma_Y>0$，$-1<\rho<1$，

ρ 代表 X 、Y 的母體相關係數。