# Mean Value Theorem(MVT) 與 L'Hôpital Rule、不定式

**Rolle Theorem** If $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$ with $f(a)=f(b)$, then $\exists\, c\in(a,b)$ such that $f'(c)=0$.

(*pf.*) Let $m := \min\limits_{x\in[a,b]} f(x), \leq M := \max\limits_{x\in[a,b]} f(x)$,

1. If $m=M$, then $f(x)=m=M \forall x\in[a,b]$(a horizontal line segment), then $f'(x)=0 \forall x \in (a,b)$.
2. If $m<M$, then either $m\neq f(a)=f(b)$ or $M\neq f(a)=f(b)$. Suppose $M\neq f(a)=f(b)$ and $M$ is assumed at $c\in(a,b)$. Then $f(c+h)-f(c)\leq 0$ for $h\approx 0$, $\Longrightarrow 0\leq \lim\limits_{h\to 0^+}\frac{f(c+h)-f(c)}{h} = \lim\limits_{h\to 0^+}\frac{f(c+h)-f(c)}{h} \leq 0$ (the equality $=$ holds since $f$ is differentiable), i.e. $f'(c)=0$.

Proof of the case $m\neq h(a)=h(b)$ is similar.

**Lagrange MVT** $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$, then $\exists\, c\in(a,b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

(*pf.*) Let $g(x) := f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$ (secant line). Then $f-g$ is differentiable on $(a,b)$, $(f-g)(a)=(f-g)(b)=0$, and $(f-g)'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}$. By *Rolle's Theorem*, $\exists c\in(a,b)$ such that $0=(f-g)'(c)=f'(c)-\frac{f(b)-f(a)}{b-a}$.

**Cauchy MVT** If $f,g$ both continuous on $[a,b]$ and both differentiable on $(a,b)$, with $g'(x)\neq 0\ \forall\, x\in(a,b)$ in addition, then $\exists\, c\in(a,b)$ such that $\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$.
( 註 : “ $g'(x)\neq 0\ \forall\, x\in(a,b)$ ” 的條件暗示著 $g$ 在 $(a,b)$ 上 不是嚴格遞增 就是 嚴格遞減,
自然 $g'(c)\neq 0$, $g(a)\neq g(b)$, 不必擔心分母爲 0。

(*pf.*) Let $h(x) := f(x)-\left[f(a)+\frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))\right]$, then $h(x)$ is differentiable on $(a,b)$, $h(a)=h(b)=0$ and $h'(x)=f'(x)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(x)$. By *Rolle's Theorem*, $\exists c\in(a,b)$ such that $h'(c)=0$, i.e. $\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$.

我們將用 Cauchy MVT 證明 L'Hôpital Rule.

## L'Hôpital Rule — about $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ indeterminate form

**(assume $f,g$ differentiable and no common factor)**

> **“$\frac{0}{0}$ form”** — If $\lim\limits_{x\to a} f(x)=0$ and $\lim\limits_{x\to a} g(x)=0$. Then $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{\spadesuit\to a}\frac{f'(\spadesuit)}{g'(\spadesuit)}$.

(*pf.*) Without lose of generality, say $f(a)=0$ and $g(a)=0$, then $\frac{f(x)}{g(x)}=\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(c)}{g'(c)}$ for some $c$ (depends on $x$) between $x$ and $a$ by Cauchy MVT. So, if $x\to a$, $c\to a$ as well. Hence, $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{c\to a}\frac{f'(c)}{g'(c)}=\lim\limits_{\spadesuit\to a}\frac{f'(\spadesuit)}{g'(\spadesuit)}$.

"$\frac{\infty}{\infty}$ **form**" — If $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$. Then $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{\spadesuit\to a}\frac{f'(\spadesuit)}{g'(\spadesuit)}$.

(*pf.*) Assume $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = c$ exists. Then $c = \lim\limits_{x\to a}\frac{f(x)}{g(x)} = \underbrace{\lim\limits_{x\to a}\frac{(\frac{1}{g(x)})}{(\frac{1}{f(x)})}}_{\frac{0}{0}\text{ form}} = \lim\limits_{\spadesuit\to a}\frac{-\frac{g'(\spadesuit)}{g^2(\spadesuit)}}{-\frac{f'(\spadesuit)}{f^2(\spadesuit)}} = \lim\limits_{\spadesuit\to a}\frac{g'(\spadesuit)}{f'(\spadesuit)}\underbrace{\frac{f^2(\spadesuit)}{g^2(\spadesuit)}}_{c^2}$.

Therefore, if $\lim\limits_{\spadesuit\to a}\frac{f'(\spadesuit)}{g'(\spadesuit)}$ exists, then $\lim\limits_{\spadesuit\to a}\frac{f'(\spadesuit)}{g'(\spadesuit)} = c = \lim\limits_{x\to a}\frac{f(x)}{g(x)}$.

## 如何將其他 limit 問題轉換成 $\frac{0}{0}$ 或 $\frac{\infty}{\infty}$ 標準型：

1. " $\lim \underbrace{f\cdot g}_{0\cdot\infty} = \lim \underbrace{\frac{f}{(\frac{1}{g})}}_{\frac{0}{0}} = \lim \underbrace{\frac{g}{(\frac{1}{f})}}_{\frac{\infty}{\infty}}$ "

2. " $\lim \underbrace{f-g}_{\pm\infty-\pm\infty} = \lim \underbrace{\frac{\frac{1}{g}-\frac{1}{f}}{\frac{1}{f}\frac{1}{g}}}_{\frac{0}{0}}$ "

3. " $\lim f^g = \lim e^{(g\cdot\ln f)} = e^{\lim(g\cdot\ln f)}$ ", 再視 $\lim(g\cdot\ln f)$ 的型 選擇適合的方法。

Remark: 1, 就像以前講的, 解任何問題前, 包括 limit 問題, 一點要先觀察、化簡/約分, 必要時 拆/加項。
2, L'Hôpital Rule 不一定能解決所有不定式的 limit 問題, 有時還是得分析。

例如, 解 $\lim\limits_{x\to+\infty}\frac{e^x}{x^x}$。(我們可以預期 $x^x$ 遠比 $e^x$ 增長得快) 因爲 對任何 $x > e$, $0 < \frac{e^x}{x^x} < 1$ 並且遞減, 所以 $\lim\limits_{x\to+\infty}\frac{e^x}{x^x}$ 必然存在, 令它爲 $L$ 好了。

則 $L = \lim\limits_{x\to+\infty}\frac{e^x}{x^x}$ ($\frac{\infty}{\infty}$ form)

$\overset{\mathcal{L}}{=} \lim\limits_{x\to+\infty}\frac{e^x}{x^x(1+\ln x)}$, 雖然仍舊是 $\frac{\infty}{\infty}$ 型, 可以再用 l'Hôpital rule, 但求導更複雜了, 況且, 你不能沒完沒了地用 l'Hôpital rule 下去, 問題還是沒解決!

$= \lim\limits_{x\to+\infty}\frac{e^x}{x^x}\cdot\lim\limits_{x\to+\infty}\frac{1}{1+\ln x}$

$= L\cdot 0 = 0.$

這裡用到以前學過的極限基本觀念: **當** $x\to a$ **時**, $f(x)\nearrow$(**或**$\searrow$) **並且** $f(x)$ **有上界**(**或下界**),
**則** $\lim\limits_{x\to a} f(x)$ **必然存在。**

重解: $\lim\limits_{x\to+\infty}\frac{e^x}{x^x} = \lim\limits_{x\to+\infty}(\frac{e}{x})^x = \lim\limits_{x\to+\infty} e^{x\ln\frac{e}{x}} = e^{\lim\limits_{x\to+\infty}\underbrace{x\ln\frac{e}{x}}_{\infty\cdot(-\infty)}} = 0$, 只有用到基本的: $\boldsymbol{g^f = e^{\ln(g^f)} = e^{f\ln g}}$.

另外一例: $\lim\limits_{x\to0^+}\frac{e^x}{x^x} = \lim\limits_{x\to0^+}(\frac{e}{x})^x = \lim\limits_{x\to0^+} e^{x\ln(\frac{e}{x})} = e^{\lim\limits_{x\to0^+}\underbrace{x\ln(\frac{e}{x})}_{0\cdot\infty}} = e^{\lim\limits_{x\to0^+}\underbrace{\frac{1-\ln x}{1/x}}_{\frac{\infty}{\infty}}} \overset{\mathcal{L}}{=} e^{\lim\limits_{x\to0^+}\frac{-1/x}{-1/x^2}} = e^{\lim\limits_{x\to0^+}x} = 1$, 只小用了一下 l'Hôpital rule, 還是用到基本的: $\boldsymbol{g^f = e^{\ln(g^f)} = e^{f\ln g}}$.