同 l'Hôpital's rule 一樣, 各種積分技巧 都是用來轉換問題的。如果沒有本質性的變化或更爲簡單, 通常不會產生效果的。試著用不同方式做, 只能透過不斷練習吸取經驗。

- 求: 與 曲線族 $\mathscr{A}: y = ax^2,\ a \in \mathbb{R}$ (拋物線族) 正交的 曲線族 $\mathscr{B}$ 。

  $\mathscr{A}: \frac{y}{x^2} = a \,, \overset{d}{\Longrightarrow} \mathscr{A}: \frac{x^2\,dy - 2xy\,dx}{x^4} = 0$ 意即 $\mathscr{A}$ 族的 $(dx, dy) \perp (-2y, x)$, 則 $\mathscr{B}$ 族的 $(dx, dy) \parallel (-2y, x)$, 即: $\mathscr{B}: dx/dy = -2y/x \Rightarrow x\,dx + 2y\,dy = 0, \overset{\int}{\Longrightarrow} \mathscr{B}: x^2 + 2y^2 = b,\ b \geq 0$ (橫縱軸比例 $\sqrt{2} : 1$ 橢圓族)。

- 求: 與 曲線族 $\mathscr{A}: x^2 + 4y^2 = a,\ a \in \mathbb{R}$ (橫縱軸比例 $2 : 1$ 橢圓族) 正交的 曲線族 $\mathscr{B}$ 。

  $\overset{d}{\Longrightarrow} \mathscr{A}: x\,dx + 4y\,dy = 0$ 意即 $\mathscr{A}$ 族的 $(dx, dy) \perp (x, 4y)$, 則 $\mathscr{B}$ 族的 $(dx, dy) \parallel (x, 4y)$, 即: $\mathscr{B}: dx/dy = x/4y \Rightarrow 4\frac{dx}{x} = \frac{dy}{y}, \overset{\int}{\Longrightarrow} \mathscr{B}: y = bx^4,\ b \in \mathbb{R}$。

  解一階微分方程的問題 其實 還是得用到 不定積分的技巧。不會積分就休想解題。

- $$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx = -\int \frac{d(\cos x)}{\cos x} \quad \left(\text{令 } s{=}\cos x \text{ 僅僅是簡寫, 說是三角代換有點勉強}\right)\\ &= -\ln|\cos x| = \ln|\sec x|\end{aligned}$$

  所以跟它 “co” 的也可一樣畫葫蘆:

  $$\begin{aligned}\int \cot x\,dx &= \int \frac{\cos x}{\sin x}\,dx = \int \frac{d(\sin x)}{\sin x} \quad \left(\text{令 } s{=}\sin x \text{ 僅僅是簡寫, 說是三角代換有點勉強}\right)\\ &= \ln|\sin x| = -\ln|\csc x|\end{aligned}$$

- $$\begin{aligned}\int \sec x\,dx &= \int \frac{\cos x}{\cos^2 x}\,dx \text{ , 自己多生一個 } \cos x\\ &= \int \frac{d(\sin x)}{1-\sin^2 x} \text{ , 令 } s = \sin x \text{ 簡寫省事}\\ &= \frac{1}{2}\int \left(\frac{1}{1-s} + \frac{1}{1+s}\right) ds\\ &= \frac{1}{2}\ln\left|\frac{1+s}{1-s}\right| = \frac{1}{2}\ln\left|\frac{1+s}{1-s}\cdot\frac{1+s}{1+s}\right| = \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{\cos^2 x}\right| \quad = \ln|\sec x + \tan x|\end{aligned}$$

- $\int e^{\sqrt{x}}\,dx$, 令 $u = \sqrt{x}$, 則 $u^2 = x \Rightarrow 2u\,du = dx$,

  $\overset{\text{sub}}{=} \int 2e^u u\,du = 2\int u\,d(e^u)$

  $\overset{\text{ibp}}{=} 2\left(u\,e^u - \int e^u\,du\right) = 2(u\,e^u - e^u) = 2e^{\sqrt{x}}(\sqrt{x} - 1)$

- $\int e^{\sqrt{x}}\,dx$, 令 $v = e^{\sqrt{x}}$, 則 $(\ln v)^2 = x \Rightarrow \frac{2\ln v}{v}\,dv = dx$,

  $\overset{\text{sub}}{=} \int v\frac{2\ln v}{v}\,dv = 2\int \ln v\,dv$

  $\overset{\text{ibp}}{=} 2\left(\ln v\, v - \int v\,\frac{dv}{v}\right) = 2(v\,\ln v - v) = 2e^{\sqrt{x}}(\sqrt{x} - 1)$

- 這個 不定積分 沒有 封閉型式 (助教之前問的):

  $$\begin{aligned}\int \frac{x}{\ln x}\,dx &= \frac{1}{2}\int \frac{d(x^2)}{\ln x}\\ &= \frac{1}{2}\left(\frac{x^2}{\ln x} + \int \frac{x}{(\ln x)^2}\,dx\right) \swarrow \text{ 新的積分式比原來還多除 } \ln x\end{aligned}$$

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$$\int \frac{x}{\ln x}\,dx = \int \frac{x^2}{\ln x\cdot x}\,dx \text{ , 自己多生一個 } x$$
$$= \int x^2\,d(\ln(\ln x))$$
$$\stackrel{\text{ibp}}{=} x^2\ln(\ln x) - 2\int x\ln(\ln x)\,dx \text{ , 令 } u=\ln x$$
$$= e^{2u}\ln u - 2\int e^{2u}\ln u\,du$$

無論如何用 integration by parts 處理 $\int e^v \ln v\,dv$, e^v 與 $\ln v$ 總是同時出現; 硬是用 substitution 弄掉 log 或 exp, 最終仍等價於 求 $\int e^{e^t}\,dt$, $\int \ln(\ln t)\,dt$, $\int \frac{dt}{\ln t}$, ... 沒戲唱

$$\int \frac{x}{\ln x}\,dx = \int \frac{e^u}{u}e^u\,du \text{ , 令 } u=\ln x$$
$$\stackrel{\text{sub}}{=} \int e^{2u}\,d(\ln u)$$
$$\stackrel{\text{ibp}}{=} e^{2u}\ln u - 2\int e^{2u}\ln u\,du \text{ , 又跟前面一樣。}$$

- 12/27(一) 有人問我 $\lim_{x\to 0}\int_x^{2x}\frac{\sin t}{t}\,dt$ 怎麼算? 這純粹是觀念題。

  (大約感覺) $\int \frac{\sin t}{t}\,dt$ 令為 $F(t)$ (沒法積), 但 $\lim_{t\to 0}\frac{\sin t}{t}$ 早知道是 1, 也就是說, 當 $0 \lessapprox x \ll 1, \frac{\sin t}{t} \le 1$ 在 $[x,2x]$ 上的積分值 $\le 1\cdot x \xrightarrow{x\to 0} 0$ 。

  (嚴格一點) 由 FTC 及 MVT, 存在 ξ 介於 x 與 $2x$ 之間, 使得 $\int_x^{2x}\frac{\sin t}{t}\,dt \stackrel{\text{FTC}}{=} F(2x)-F(x) \stackrel{\text{MVT}}{=} \frac{\sin(\xi)}{\xi}x$, 而 $x\to 0$ 導致 $\xi\to 0$, 所以 $\lim_{x\to 0}\int_x^{2x}\frac{\sin t}{t}\,dt = \lim_{\xi\to 0}\frac{\sin\xi}{\xi}\lim_{x\to 0}x = 1\cdot 0 = 0$ 。

- (a) $\int x\sqrt{3x+1}\,dx \stackrel{\text{sub}}{=} \int \frac{u^2-1}{3}\cdot u\cdot\frac{2u}{3}\,du$

$$= \frac{2}{9}\int (u^4-u^2)\,du = \frac{2}{9}\left(\frac{u^5}{5}-\frac{u^3}{3}\right)$$
$$= \frac{2}{9}(3x+1)^{\frac{3}{2}}\left(\frac{3x+1}{5}-\frac{1}{3}\right)$$
$$= \frac{2}{135}(3x+1)^{\frac{3}{2}}(9x-2)$$

$$\int x\sqrt{3x+1}\,dx = \frac{2}{9}\int x\,d\left((3x+1)^{\frac{3}{2}}\right)$$
$$\stackrel{\text{ibp}}{=} \frac{2}{9}\left(x(3x+1)^{\frac{3}{2}} - \int (3x+1)^{\frac{3}{2}}\,dx\right)$$
$$= \frac{2}{9}\left(x(3x+1)^{\frac{3}{2}} - \frac{2}{15}(3x+1)^{\frac{5}{2}}\right)$$
$$= \frac{2}{9}(3x+1)^{\frac{3}{2}}\left(x-\frac{2}{15}(3x+1)\right)$$
$$= \frac{2}{135}(3x+1)^{\frac{3}{2}}(15x-6x-2)$$
$$= \frac{2}{135}(3x+1)^{\frac{3}{2}}(9x-2)$$

  哪個比較快?

  (b) $\int e^x\sqrt{3e^x+1}\,dx \cong$ (a)

- 期中考我記得考了一題: $\cosh x$ 的反函數是 $\ln(y+\sqrt{y^2-1})$, $\sinh x$ 的反函數是 $\ln(y+\sqrt{y^2+1})$ 。那現在我們「間接」來做這個問題:

$$\frac{d}{dx}\cosh x = \sinh x$$
$$\frac{d}{dx}\ln(y+\sqrt{y^2-1}) = \frac{1+\frac{y}{\sqrt{y^2-1}}}{y+\sqrt{y^2-1}} = \frac{1}{\sqrt{y^2-1}}$$
$$\frac{d}{dx}\ln(y+\sqrt{y^2+1}) = \frac{1+\frac{y}{\sqrt{y^2+1}}}{y+\sqrt{y^2+1}} = \frac{1}{\sqrt{y^2+1}}$$
$$\int \frac{dy}{\sqrt{y^2-1}} = \int \frac{\sinh x\,dx}{\sinh x} = x = \cosh^{-1}y + c_1$$
$$\int \frac{dy}{\sqrt{y^2-1}} = \int \frac{\sec x\tan x\,dx}{\tan x} = \int \sec x\,dx = \ln|\sec x+\tan x| + c_2 = \ln|y+\sqrt{y^2-1}| + c_2$$
$$\therefore \cosh^{-1}y = \ln|y+\sqrt{y^2-1}| + c \Rightarrow c = 0 \text{ 。}$$

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- 助教問: $\int \frac{dx}{x^4+1}$ 怎麼做? 下面的作法沒經驗不太容易想得出來:

$x^4+1=x^4+\mathbf{2x^2}+1-\mathbf{2x^2}=(x^2+1)^2-(\sqrt{2}x)^2=(x^2+\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$,

$\therefore \frac{1}{x^4+1}=\frac{1}{2\sqrt{2}}\left(\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right)=\frac{1}{2\sqrt{2}}\left(\frac{1}{2}\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}+\frac{1}{\sqrt{2}}\frac{1}{x^2+\sqrt{2}x+1}-\frac{1}{2}\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}-\frac{1}{\sqrt{2}}\frac{1}{x^2-\sqrt{2}x+1}\right)$,

$\frac{1}{x^2+\sqrt{2}x+1}=\frac{2}{2x^2+2\sqrt{2}x+2}=\frac{2}{(\sqrt{2}x+1)^2+1}$, $\frac{1}{x^2-\sqrt{2}x+1}=\frac{2}{2x^2-2\sqrt{2}x+2}=\frac{2}{(\sqrt{2}x-1)^2+1}$,

$$\begin{aligned}\therefore \int \frac{dx}{x^4+1} &= \frac{1}{2\sqrt{2}}\left(\frac{1}{2}\int \frac{(2x+\sqrt{2})\,dx}{x^2+\sqrt{2}x+1} + \int \frac{\sqrt{2}\,dx}{(\sqrt{2}x+1)^2+1} - \frac{1}{2}\int \frac{(2x-\sqrt{2})\,dx}{x^2-\sqrt{2}x+1} - \int \frac{\sqrt{2}\,dx}{(\sqrt{2}x-1)^2+1}\right)\\ &= \frac{1}{2\sqrt{2}}\left(\frac{1}{2}\ln\big(x^2+\sqrt{2}x+1\big) + \arctan\big(\sqrt{2}x+1\big) - \frac{1}{2}\ln\big(x^2-\sqrt{2}x+1\big) - \arctan\big(\sqrt{2}x-1\big)\right)\end{aligned}$$

另外有一個算是制式的作法:

將 $-1=\cos(\pi+2n\pi)+i\sin(\pi+2n\pi)$ 的四次方根求出

$$\begin{aligned} b&=\cos\tfrac{3\pi}{4}+i\sin\tfrac{3\pi}{4}, \quad a=\cos\tfrac{\pi}{4}+i\sin\tfrac{\pi}{4}=\tfrac{1+i}{\sqrt{2}},\\ c&=\cos\tfrac{5\pi}{4}+i\sin\tfrac{5\pi}{4}, \quad d=\cos\tfrac{7\pi}{4}+i\sin\tfrac{7\pi}{4},\end{aligned}$$

其中 $c=\overline{b}=-a,\ d=\overline{a}=-b$。

$$\begin{aligned}\therefore \frac{1}{x^4+1} &= \frac{1}{(x-a)(x-b)(x-c)(x-d)}\\ &= \frac{1}{(x-a)(x+\overline{a})(x+a)(x-\overline{a})} = \frac{1}{(x+a)(x+\overline{a})\cdot(x-a)(x-\overline{a})}\\ &= \frac{1}{4}\left(\frac{(a+\overline{a})x+2a\overline{a}}{x^2+(a+\overline{a})x+a\overline{a}} - \frac{(a+\overline{a})x-2a\overline{a}}{x^2-(a+\overline{a})x+a\overline{a}}\right)\\ &= \frac{1}{4}\left(\frac{\sqrt{2}x+2}{x^2+\sqrt{2}x+1} - \frac{\sqrt{2}x-2}{x^2-\sqrt{2}x+1}\right) = \cdots \text{(其餘同上)}\end{aligned}$$

上面神來一筆的作法 從這兒也可以看得出來。

- $\displaystyle\int_0^1 e^x \ln x\,dx$ 積分值存在嗎?

當 $x\in[0,1]$, $0\geq e^x\cdot\ln x\geq e\cdot\ln x$ 且 $e^x\ln x \xrightarrow{x\to\infty} -\infty$, $\therefore \displaystyle\int_0^1 e^x\ln x\,dx$ 是瑕積分, 且

$0\geq \displaystyle\int_0^1 e^x\ln x\,dx \geq \int_0^1 e\ln x\,dx = e\Big[x\ln x - x\Big]_{0^+}^1 = -e$, $\therefore \displaystyle\int_0^1 e^x\ln x\,dx$ 存在。

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