Hyperbolic Functions and Cable/Chain Curve

定義 $\begin{cases} \sinh x := \frac{e^x - e^{-x}}{2} \\ \cosh x := \frac{e^x + e^{-x}}{2} \end{cases}$, 分別唸做 *hyperbolic sine*, *hyperbolic cosine*.

很明顯地, (1) sinh 、cosh 不是週期函數, sinh 爲奇, cosh 爲偶。

(2) $\Rightarrow \sinh x$ 嚴格遞增, $\cosh x$ 右半部 嚴格遞增。

(3) 當 x 很大時, $\sinh x$ 和 $\cosh x$ 都很接近 $\frac{e^x}{2}$, $\sinh x$ 比 $\frac{e^x}{2}$ 小, $\cosh x$ 比 $\frac{e^x}{2}$ 大, $\frac{\sinh x}{\cosh x} \xrightarrow{x\to\infty} 1^-$。

$$\begin{cases} \frac{d}{dx}\sinh x = \frac{e^x + e^{-x}}{2} = \cosh x, \\ \frac{d}{dx}\cosh x = \frac{e^x - e^{-x}}{2} = \sinh x, \end{cases}$$

還有, $\begin{cases} \sinh^2 x = \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{\cosh(2x) - 1}{2}, \\ \cosh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{\cosh(2x) + 1}{2}, \end{cases}$ 即 $\begin{aligned} \cosh 2x &= \cosh^2 x + \sinh^2 x \\ &= 2\cosh^2 x - 1 \\ &= 2\sinh^2 x + 1 \end{aligned}$

又可得 $\cosh^2 x - \sinh^2 x = 1$. $\sinh x \cosh x = \frac{e^{2x} - e^{-2x}}{4} = \frac{1}{2}\sinh 2x$, 即 $\sinh 2x = 2\sinh x \cosh x$.

由於 sinh 與 cosh 的彼此的關係 有點類似 sin 與 cos, 因而得名。另外, 由於其他的三角函數 tan, cot, sec, csc 都是以 sin 和 cos 定義的, 我們也以類似方式來產生其它 hyperbolic 函數:

$$\begin{cases} \tanh x := \frac{\sinh x}{\cosh x}, \\ \coth x := \frac{\cosh x}{\sinh x}, \end{cases} \quad \begin{cases} \mathrm{sech} x := \frac{1}{\cosh x}, \\ \mathrm{csch} x := \frac{1}{\sinh x}. \end{cases}$$

$$\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x + e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - e^{-2x}}{e^{2x} + 2 + e^{-2x}} = \frac{\sinh 2x}{\cosh 2x + 1} \\ &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x - e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} - 2 + e^{-2x}}{e^{2x} - e^{-2x}} = \frac{\cosh 2x - 1}{\sinh 2x} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}\tanh x &= \frac{d}{dx}\big[\sinh x\big] \cdot (\cosh x)^{-1} + \sinh x \cdot \frac{d}{dx}\Big[(\cosh x)^{-1}\Big] \\ &= 1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \mathrm{sech}^2 x \\ &= 1 - \tanh^2 x \end{aligned}$$

同時可又得到 $1 - \tanh^2 x = \mathrm{sech}^2 x$.

三角函數	hyperbolic 函數
$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$	$\sinh^2 x = \frac{\cosh(2x) - 1}{2}$
$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$	$\cosh^2 x = \frac{\cosh(2x) + 1}{2}$
$\cos^2\theta + \sin^2\theta = 1$	$\cosh^2 x - \sinh^2 x = 1$
$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \end{aligned}$	$\begin{aligned} \cosh 2x &= \cosh^2 x + \sinh^2 x \\ &= 2\cosh^2 x - 1 \\ &= 2\sinh^2 x + 1 \end{aligned}$
$\begin{aligned} \tan\theta &= \frac{\sin 2\theta}{1 + \cos 2\theta} \\ &= \frac{1 - \cos 2\theta}{\sin 2\theta} \end{aligned}$	$\begin{aligned} \tanh x &= \frac{\sinh 2x}{\cosh 2x + 1} \\ &= \frac{\cosh 2x - 1}{\sinh 2x} \end{aligned}$

其他 hyperbolic 函數的做法類似, 課本也有, 自己去導。

Derivatives of Inverse Hyperbolic Functions

sinh 爲 1-1, cosh 右半邊是 1-1, tanh 是 1-1, sech 右半邊是 1-1, $\cdots$
$y = \sinh x \iff x = \sinh^{-1} y$, $\frac{d}{dy}[\sinh^{-1} y] = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\cosh x} = \frac{1}{\sqrt{1+\sinh^2 x}} = \frac{1}{\sqrt{1+y^2}}$。另外,

$$\left\{\begin{array}{l} \int \frac{dt}{\sqrt{1+t^2}} = (\text{令 } t = \tan\theta) \cdots = \ln(t + \sqrt{1+t^2}) + c_1 \\ \int \frac{dt}{\sqrt{1+t^2}} = \sinh^{-1} t + c_2 \end{array}\right\} \Rightarrow \sinh^{-1} t = \ln(t + \sqrt{1+t^2})$$

$$y = \sinh x, \Leftrightarrow \sqrt{1+y^2} = \cosh x \,(\because \cosh x > 0), \Leftrightarrow y + \sqrt{1+y^2} = e^x, \Leftrightarrow x = \ln(y + \sqrt{1+y^2})$$

$$\left\{\begin{array}{l} y = \cosh x \\ x > 0 \end{array}\right. \iff \left\{\begin{array}{l} x = \cosh^{-1} y \\ y > 1 \end{array}\right. \Rightarrow \frac{d}{dy}[\cosh^{-1} y] = \frac{1}{\sinh x} = \frac{1}{\sqrt{\cosh^2 x - 1}} = \frac{1}{\sqrt{y^2-1}}$$

$$\left\{\begin{array}{l} \int \frac{dt}{\sqrt{t^2-1}} = (\text{令 } t = \sec\theta) \cdots = \ln(t + \sqrt{t^2-1}) + c_1 \\ \int \frac{dt}{\sqrt{t^2-1}} = \cosh^{-1} t + c_2 \end{array}\right. \Rightarrow \cosh^{-1} t = \ln(t + \sqrt{t^2-1}),\ t \geq 1$$

$$y = \cosh x, x \geq 0 \Rightarrow \sqrt{y^2-1} = \sinh x \text{ since } x \geq 0, \Leftrightarrow y + \sqrt{y^2-1} = e^x, \Leftrightarrow x = \ln(y + \sqrt{y^2-1})$$

前面第 (3) 就說了, 奇函數 $\tanh x \xrightarrow{x \to \infty} 1^-$, $\therefore y = \tanh x \iff \begin{array}{l} x = \tanh^{-1} y, \\ -1 < y < 1. \end{array}$

$\frac{d}{dy}\tanh^{-1} y = \frac{1}{\operatorname{sech}^2 x} = \frac{1}{1-\tanh^2 x} = \frac{1}{1-y^2}$,

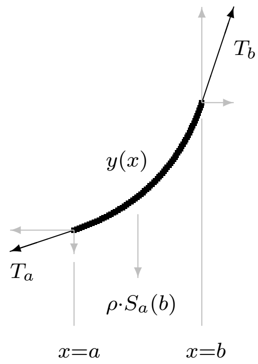
$$\left\{\begin{array}{l} \int \frac{dt}{1-t^2} = \frac{1}{2}(\int \frac{dt}{1-t} + \int \frac{dt}{1+t}) = \frac{1}{2} \ln\left|\frac{1+t}{1-t}\right| + c_1 \ \forall t \neq \pm 1. \\ \int \frac{dt}{1-t^2} = \tanh^{-1} t + c_2 \text{ if } |t| < 1. \end{array}\right. \Rightarrow \tanh^{-1} t = \frac{1}{2} \ln(\frac{1+t}{1-t}) \text{ if } |t| < 1$$

$$y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \Rightarrow -1 < y < 1, \left\{\begin{array}{l} 1 + y = \frac{2e^x}{e^x + e^{-x}} \\ 1 - y = \frac{2e^{-x}}{e^x + e^{-x}} \end{array}\right. \Rightarrow \frac{1+y}{1-y} = e^{2x} \Rightarrow x = \frac{1}{2} \ln(\frac{1+y}{1-y})$$

關於其他 inverse hyperbolic 的做法類似, 課本也有, 自己去導。

Hanging Cable (chain curve/catenary/懸鏈線)

已知懸鏈之線密度爲 ρ 且密度均勻, 假設 懸鏈的形狀爲 函數 $y(x)$,
在 $x = a$ 的張力爲 T_a,
在 $x = b$ 的張力爲 T_b,



固定 a, 以 b 爲變量,

則 從 $x = a$ 到 $x = b$ 的 鏈長爲 $S_a(b) = \int_a^b \sqrt{1 + y'(x)^2}\, dx$,
重量爲 密度×鏈長 $= \rho \cdot S_a(b)$.

由力平衡可得:

$$\begin{cases} -T_a \frac{dx}{ds}\Big|_{x=a} & +T_b \frac{dx}{ds}\Big|_{x=b} & & = 0, & \text{——}(1) \\ -T_a \frac{dy}{ds}\Big|_{x=a} & +T_b \frac{dy}{ds}\Big|_{x=b} & -\rho \cdot S_a(b) & = 0, & \text{——}(2) \end{cases}$$

以 b 爲變量, 在 a 處的 $\begin{matrix} A_1 := T_a \frac{dx}{ds}\Big|_{x=a} \\ A_2 := T_a \frac{dy}{ds}\Big|_{x=a} \end{matrix}$ 皆爲常數。將 (1) 的 $T_b = A_1 \frac{ds}{dx}\Big|_{x=b}$ 代入 (2) 得到

$$-A_2 + A_1 \underbrace{\frac{ds}{dx}\Big|_{x=b} \frac{dy}{ds}\Big|_{x=b}}_{\frac{dy}{dx}\big|_{x=b}} - \rho \cdot S_a(b) = 0,$$

$$\text{即: } -A_2 + A_1 y'(b) - \rho \cdot \int_a^b \sqrt{1 + y'(x)^2}\, dx = 0, \text{—— 爲 } b \text{ 的方程式。}$$

$$\xrightarrow{\frac{d}{db}} A_1\, y''(b) - \rho \sqrt{1 + y'(b)^2} = 0,$$

令 $u = y'(b)$, 則

$$\begin{aligned} & A_1 \tfrac{du}{db} = \rho \sqrt{1 + u^2}, \\ \iff\quad & \tfrac{du}{\sqrt{1+u^2}} = \tfrac{\rho}{A_1} db, \\ \iff\quad & \int \tfrac{du}{\sqrt{1+u^2}} = \tfrac{\rho}{A_1} \int db, \\ \iff\quad & \sinh^{-1} u = \tfrac{\rho}{A_1} b + c_1, \\ \iff\quad & u = \sinh(\tfrac{\rho}{A_1} b + c_1), \\ \iff\quad & y(b) = \int u\, db = \tfrac{A_1}{\rho} \cosh(\tfrac{\rho}{A_1} b + c_1) + c_2, \end{aligned}$$

所以 懸鏈的形狀爲 cosh 的圖形。