- $(n \in \mathbb{N})$ $\frac{d}{dx}[x^n] = \lim\limits_{h\to 0}\frac{(x+h)^n - x^n}{h}$ by definition

$$= \lim_{h\to 0}\frac{\left[\binom{n}{0}x^n h^0 + \binom{n}{1}x^{n-1}h^1 + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n}x^0 h^n\right] - x^n}{h} \quad \text{binomial expansion}$$

$$= \lim_{h\to 0}\frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + x^0 h^n}{h}$$

$$= \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1}\right]$$

$$= nx^{n-1} + 0 + \cdots + 0 = nx^{n-1}.$$

- $\frac{d}{dx}[\ln x] = \lim\limits_{h\to 0}\frac{\ln(x+h) - \ln x}{h}$ by definition

$$= \lim_{h\to 0}\left[\frac{1}{x}\frac{x}{h}\ln\left(1 + \frac{h}{x}\right)\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{x}\ln\left(\left(1 + \frac{h}{x}\right)^{\frac{x}{h}}\right)\right]$$

$$= \frac{1}{x}\lim_{h\to 0}\left[\ln\left(\left(1 + \frac{h}{x}\right)^{\frac{x}{h}}\right)\right]$$

$$= \frac{1}{x}\ln\left(\lim_{h\to 0}\left[\left(1 + \frac{h}{x}\right)^{\frac{x}{h}}\right]\right)$$ 因為 $\lim\limits_{\triangle\to 0}\left[(1+\triangle)^{\frac{1}{\triangle}}\right] = e$ 且 $\ln()$ 在 e 連續

$$= \frac{1}{x}\ln e = \frac{1}{x}.$$

- 令 $y = e^x$, 即: $\ln y = x$, $\frac{d}{dx}[e^x] = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

  又根據 $\frac{dx}{dy} = \frac{d}{dy}[\ln y] = \frac{1}{y}$, 所以 $\frac{d}{dx}[e^x] = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{1}{y}} = y = e^x$.

- $\frac{d}{dx}[\cos x] = \lim\limits_{h\to 0}\left[\frac{\cos(x+h) - \cos x}{h}\right]$ by definition

$$= \lim_{h\to 0}\left[\frac{\cos x\cdot\cos h - \sin x\cdot\sin h - \cos x}{h}\right]$$ 合分角公式

$$= \lim_{h\to 0}\left[\cos x\cdot\frac{\cos h - 1}{h} - \sin x\cdot\frac{\sin h}{h}\right]$$

$$= \cos x\cdot\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x\cdot\lim_{h\to 0}\frac{\sin h}{h}$$

$$= \cos x\cdot 0 - \sin x\cdot 1 = -\sin x.$$

- $\frac{d}{dx}[\sin x] = \lim\limits_{h\to 0}\left[\frac{\sin(x+h) - \sin x}{h}\right]$ by definition

$$= \lim_{h\to 0}\left[\frac{\sin x\cdot\cos h + \cos x\cdot\sin h - \sin x}{h}\right]$$ 合分角公式

$$= \lim_{h\to 0}\left[\sin x\cdot\frac{\cos h - 1}{h} + \cos x\cdot\frac{\sin h}{h}\right]$$

$$= \sin x\cdot\lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x\cdot\lim_{h\to 0}\frac{\sin h}{h}$$

$$= \sin x\cdot 0 + \cos x\cdot 1 = \cos x.$$