Cubic Spline

Given data points $(x_0, y_0), \cdots, (x_n, y_n)$, define a corresponding interpolating function $S(x)$ satisfying the following conditions:

- $S_j(x) := S(x)|_{[x_j, x_{j+1}]} \in \Pi_3 \ \forall j = 0, \ldots, n-1$;
- $S(x), S'(x), S''(x)$ are all continuous;

Let $h_j := x_{j+1} - x_j$, and $M_j := S''(x_j)$ be the 2nd derivative evaluated at x_j. Then $S_j''(x) = M_j + \frac{M_{j+1}-M_j}{x_{j+1}-x_j}(x-x_j) = M_j + \frac{M_{j+1}-M_j}{h_j}(x-x_j)$ by point-slope formula, $\Rightarrow$ the continuity of S'' is automatically satisfied.

By the continuity of S, we can derive $S_j(x)$ formula in the form $a_j + b_j(x-x_j) + c_j(x-x_j)^2 + d_j(x-x_j)^3$, in terms of M_* and h_*:

$$\stackrel{\iint}{\Longrightarrow} S_j(x) = a_j + b_j(x-x_j) + \underbrace{\frac{M_j}{2}}_{c_j}(x-x_j)^2 + \underbrace{\frac{M_{j+1}-M_j}{6h_j}}_{d_j}(x-x_j)^3 \text{ by } S_j''(x) \text{ above.} \quad (1)$$

$S_j(x_j) = y_j \Longrightarrow a_j = y_j$,

$S_j(x_{j+1}) = y_{j+1} \Longrightarrow y_{j+1} = y_j + b_j \cdot h_j + \frac{M_j}{2}h_j^2 + \frac{M_{j+1}-M_j}{6}h_j^2$,

$b_j = \Big((y_{j+1}-y_j) - (\frac{2M_j+M_{j+1}}{6})h_j^2\Big)/h_j = \frac{y_{j+1}-y_j}{h_j} - \frac{2M_j+M_{j+1}}{6}h_j$.

By the continuity of S', we can derive a relationship among M_{j-1}, M_j, and M_{j+1}:

From (1) $\stackrel{\frac{d}{dx}}{\Longrightarrow} S_j'(x) = (\frac{y_{j+1}-y_j}{h_j}) - (\frac{2M_j+M_{j+1}}{6})h_j + M_j(x-x_j) + (\frac{M_{j+1}-M_j}{2h_j})(x-x_j)^2$.

$S'(x_j^-) = S'(x_j^+)$, i.e. $S_{j-1}'(x_j) = S_j'(x_j),\ j = 1, \cdots, n-1$,

$\Longleftrightarrow (\frac{y_j-y_{j-1}}{h_{j-1}}) - (\frac{2M_{j-1}+M_j}{6})h_{j-1} + M_{j-1}h_{j-1} + (\frac{M_j-M_{j-1}}{2})h_{j-1} = (\frac{y_{j+1}-y_j}{h_j}) - (\frac{2M_j+M_{j+1}}{6})h_j$,

$$\Longleftrightarrow (\frac{h_{j-1}}{6})M_{j-1} + (\frac{h_{j-1}+h_j}{6})M_j + (\frac{h_j}{6})M_{j+1} = (\frac{y_{j+1}-y_j}{h_j}) - (\frac{y_j-y_{j-1}}{h_{j-1}}),\ j = 1, \cdots, n-1 \quad (2)$$

(2), plus two additional boundary conditions (eg. free B.C.($S''(x_0) = S''(x_n) = 0$), or, clamped B.C.(assign $S'(x_0)$ and $S'(x_n)$), etc.), becomes a square system($n+1$ eqn's and $n+1$ M_j's). Then we can solve for $M_0, \cdots, M_n$ easily.

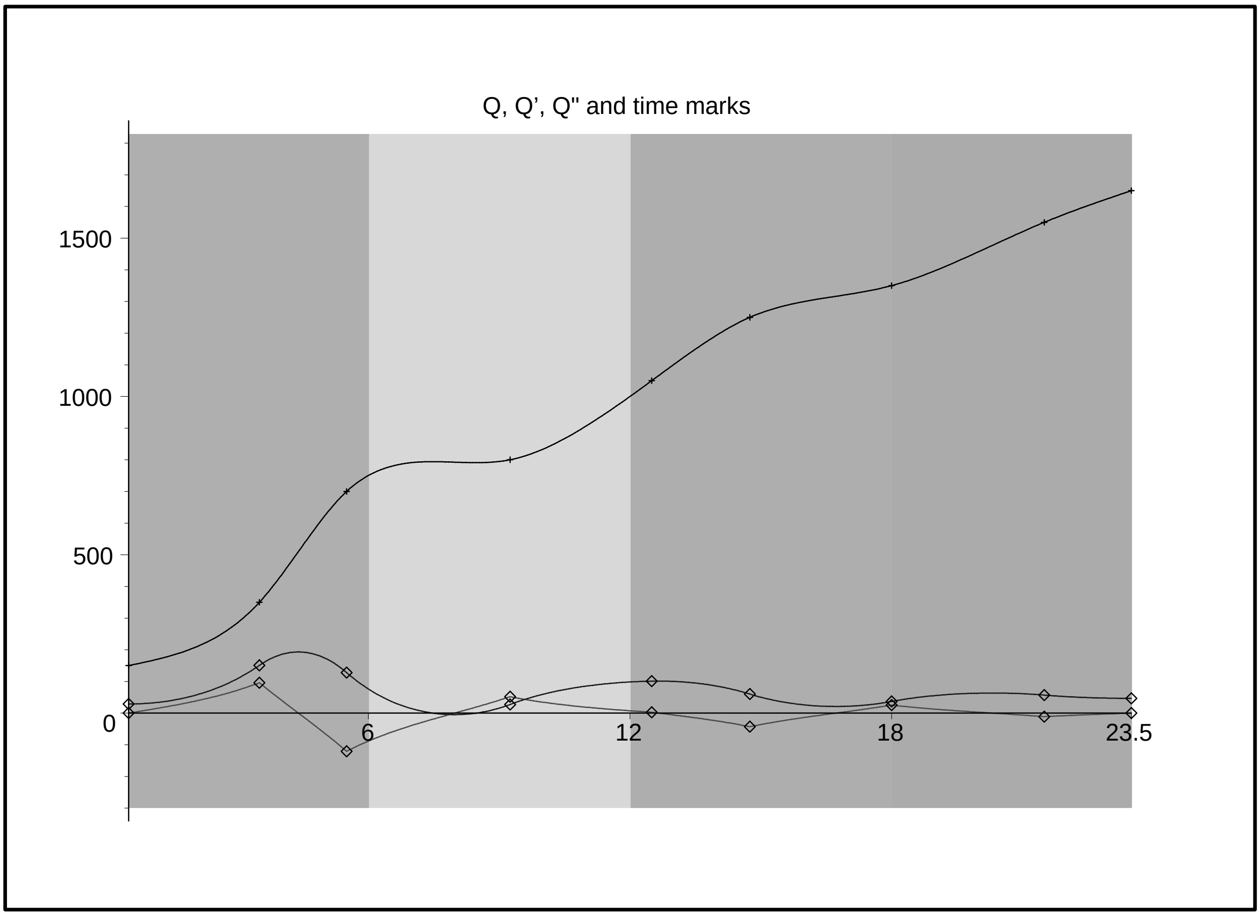
$$\text{with free B.C. :} \begin{bmatrix} \mathbf{1} & 0 & 0 & & & \\ & * & * & * & & \\ & & \ddots & \ddots & \ddots & \\ & & & * & * & * \\ & & & & 0 & 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ * \\ \vdots \\ * \\ \mathbf{1} \end{bmatrix},$$

$$\text{with clamped B.C. :} \begin{bmatrix} * & * & 0 & & & \\ & * & * & * & & \\ & & \ddots & \ddots & \ddots & \\ & & & * & * & * \\ & & & & 0 & * & * \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \end{bmatrix}.$$

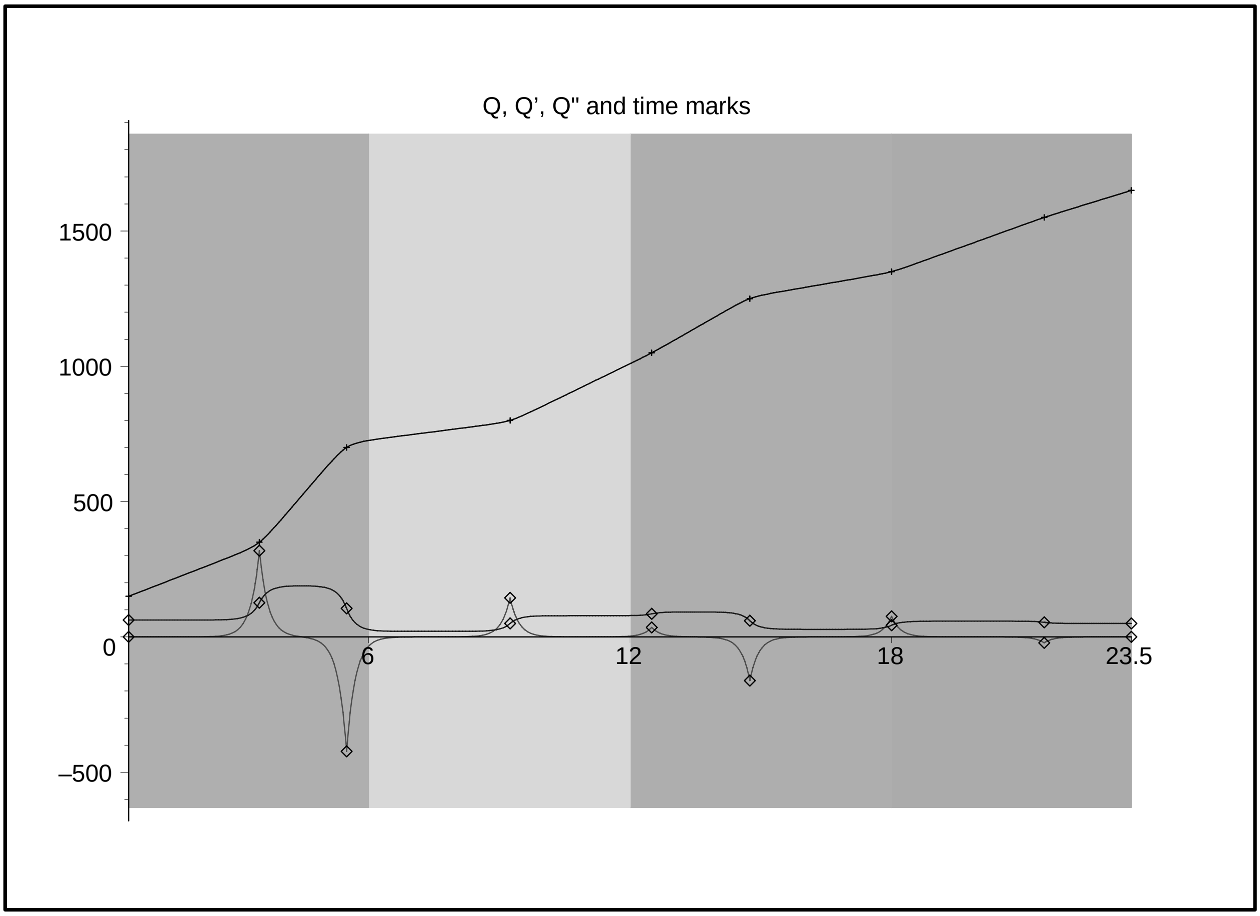
Tension Spline (Fun!) : A spline that satisfies the ordinary differential equation

$$S^{(4)}(x) - \tau^2 S''(x) = 0$$

on each $[x_j, x_{j+1}]$ with continuous $S^{(4)}(x)$. Obviously, for $\tau \approx 0$, $S(x)$ is almost a 3-spline. For large τ, the differential equation is almost like $S''(x) \approx 0$, therefore the solution $S(x)$ is almost a 1-spline but with degree 4 smoothness.



tension parameter = 0.5



tension parameter = 5