《範例二》將 $A=\begin{bmatrix} 30 & 38 & -3 & -17 & 3 \\ -16 & -20 & 2 & 10 & -2 \\ -7 & -9 & 2 & 4 & -1 \\ 16 & 21 & -1 & -7 & 1 \\ 38 & 52 & -5 & -23 & 5 \end{bmatrix}$, 逐步化爲 Jordan form。

第一, 求特徵多項式 $\chi_A(t)=(t-2)^5$ 已經幫你算好了。
第二, 給出各個 (generalized) eigenspace (即 $\ker(A-2I)^i$) 的生成元

$$A-2I=\begin{bmatrix} 28 & 38 & -3 & -17 & 3 \\ -16 & -22 & 2 & 10 & -2 \\ -7 & -9 & 0 & 4 & -1 \\ 16 & 21 & -1 & -9 & 1 \\ 38 & 52 & -5 & -23 & 3 \end{bmatrix} ,\xrightarrow{r_2:=3r_2-7r_3,} \begin{bmatrix} 28 & 38 & -3 & -17 & 3 \\ 1 & -3 & 2 & 2 & 1 \\ -7 & -9 & 0 & 4 & -1 \\ 16 & 21 & -1 & -9 & 1 \\ 38 & 52 & -5 & -23 & 3 \end{bmatrix} ,\xrightarrow{\substack{r_1:=r_1-28r_2,\\ r_3:=r_3+7r_2,\\ r_4:=r_4-16r_2,\\ r_5:=r_5-38r_2,}} \begin{bmatrix} 0 & 122 & -171 & -73 & -25 \\ 1 & -3 & 6 & 2 & 1 \\ 0 & -30 & 42 & 18 & 6 \\ 0 & 69 & -97 & -41 & -15 \\ 0 & 166 & -233 & -99 & -35 \end{bmatrix},$$

$$\xrightarrow{r_3:=-14/6r_3-r_4,} \begin{bmatrix} 0 & 122 & -171 & -73 & -25 \\ 1 & -3 & 6 & 2 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 69 & -97 & -41 & -15 \\ 0 & 166 & -233 & -99 & -35 \end{bmatrix} ,\xrightarrow{\substack{r_1:=r_1-122r_3,\\ r_4:=r_4-69r_3,\\ r_5:=r_5-166r_3,}} \begin{bmatrix} 0 & 0 & -49 & 49 & -147 \\ 1 & -3 & 6 & 2 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -28 & 28 & -84 \\ 0 & 0 & -67 & 67 & -201 \end{bmatrix} ,\xrightarrow{\substack{r_1:=-r_1/49,\\ r_4:=-r_4/28,\\ r_5:=-r_5/67}} \begin{bmatrix} 0 & 0 & 1 & -1 & 3 \\ 1 & -3 & 6 & 2 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 & 3 \end{bmatrix},$$

$$\xrightarrow{\substack{r_4:=r_4-r_1,\\ r_5:=r_5-r_1,}} \begin{bmatrix} 0 & 0 & 1 & -1 & 3 \\ 1 & -3 & 6 & 2 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} ,\xrightarrow{\substack{r_3:=r_3+r_1,\\ r_2:=r_2-6r_1,\\ r_2:=r_2+3r_3,}} \begin{bmatrix} 0 & 0 & 1 & -1 & 3 \\ 1 & 0 & 0 & 2 & -5 \\ 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} ,\xrightarrow{(1\,2),\to(2\,3),} \begin{bmatrix} 1 & 0 & 0 & 2 & -5 \\ 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left(R: \begin{matrix} \to r_2:=3r_2-7r_3, & \to r_1:=r_1-28r_2, & \to r_3:=r_3+7r_2, & \to r_4:=r_4-16r_2, & \to r_5:=r_5-38r_2, & \to r_3:=-14/6r_3-r_4, \\ \to r_1:=r_1-122r_3, & \to r_4:=r_4-69r_3, & \to r_5:=r_5-166r_3, & \to r_1:=-r_1/49, & \to r_4:=-r_4/28, & \to r_5:=-r_5/67 \\ \to r_4:=r_4-r_1, & \to r_5:=r_5-r_1, & \to r_3:=r_3+r_1, & \to r_2:=r_2-6r_1, & \to r_2:=r_2+3r_3, & \to (1\,2),\to(2\,3) \end{matrix} \right)$$

$$\text{即: } R=\begin{bmatrix} \frac{3}{49} & \frac{114}{49} & 5 & \frac{219}{49} & 0 \\ \frac{-1}{49} & \frac{-87}{49} & -4 & \frac{-171}{49} & 0 \\ \frac{-1}{49} & \frac{-38}{49} & -4 & \frac{-122}{49} & 0 \\ \frac{1}{49} & \frac{5}{196} & 0 & \frac{-1}{98} & 0 \\ \frac{1}{49} & \frac{-2}{3283} & \frac{2}{67} & \frac{40}{3283} & \frac{-1}{67} \end{bmatrix}$$

令 $T:=A-2I$。接下來要解的, 全是 “ $Tx=y,\iff RTx=Ry$ ” 的形式,

$$\begin{aligned} \text{左邊 } RTx &= (x_1,x_2,x_3,0,0)^t + x_4 \underbrace{(2,-2,-1,0,0)^t}_{u_1} + x_5 \underbrace{(-5,4,3,0,0)^t}_{u_2} \\ &= (x_1,x_2,x_3,x_4,x_5)^t - x_4 \underbrace{(-2,2,1,1,0)^t}_{u_1'} - x_5 \underbrace{(5,-4,-3,0,1)^t}_{u_2'} \\ \text{即: 左邊 } &= (x_1,x_2,x_3,0,0)^t + x_4\,u_1 + x_5\,u_2 = x - x_4\,u_1' - x_5\,u_2' \end{aligned}$$

$$\begin{aligned} &u_1'=e_4-u_1, \quad Ru_1'=(14,-11,-8,0,0)^t, \quad R^2u_1'=\left(\tfrac{-3172}{49},\tfrac{2511}{49},\tfrac{1972}{49},\tfrac{1}{196},\tfrac{176}{3283}\right)^t, \\ &u_2'=e_5-u_2, \quad Ru_2'=(-24,19,15,0,0)^t, \quad R^2u_2'=\left(\tfrac{5769}{49},\tfrac{-4569}{49},\tfrac{-3638}{49},\tfrac{-1}{196},\tfrac{-176}{3283}\right)^t, \end{aligned}$$

$RTx=R0, \Leftrightarrow x-x_4\,u_1'-x_5\,u_2'=0, \Leftrightarrow x=x_4\,u_1'+x_5\,u_2', \Rightarrow \ker T=\operatorname{span}\{u_1',u_2'\}$。

這時候要特別注意, 因爲 $\ker T,\ker T^2,\ker T^3,\ker T^4,\ker T^5$ 的維數 有可能是 2,3,4,5,5 或 2,4,5,5,5 ,
$\ker T,\ker T^2,\ker T^3,\ker T^4,\ker T^5$ 的維數差 有可能是 2,1,1,1,0 或 2,2,1,0,0 。
所以 (generalized) eigenvector v_1,v_2,v_3 不能亂選 $Tv_1=0, Tv_2=v_1, Tv_3=v_2$

令 $v_1:=a\,u_1'+b\,u_2'$ $\left((a,b)\neq(0,0)\right)$ 。

若 $Tx=v_1 \Leftrightarrow RTx=Rv_1 \Leftrightarrow x-x_4\,u_1'-x_5\,u_2'=a\,R\,u_1'+b\,R\,u_2' \Leftrightarrow x=a\,R\,u_1'+b\,R\,u_2'+x_4\,u_1'+x_5\,u_2' \Leftrightarrow (x_1,x_2,x_3,x_4,x_5)^t=a(14,-11,-8,\underline{0,0})^t+b(-24,19,15,\underline{0,0})^t+x_4(-2,2,1,\underline{1,0})^t+x_5(5,-4,-3,\underline{0,1})^t$, 對

任意 a, b 都有解。$\Rightarrow \ker T^2 = \operatorname{span}\{R\,u_1',\ R\,u_2',\ u_1',\ u_2'\}$ 。

令 $v_2 = a\,R\,u_1' + b\,R\,u_2' + c\,u_1' + d\,u_2'$ $\big((a,b,c,d)\neq(0,0,0,0)\big)$。所以 a, b, c, d 隨便取, 便有 $v_1 = Tv_2$。

若 $Tx = v_2 \Leftrightarrow x - x_4\,u_1' - x_5\,u_2' = a\,R^2\,u_1' + b\,R^2\,u_2' + c\,R\,u_1' + d\,R\,u_2' \Leftrightarrow x = a\,R^2\,u_1' + b\,R^2\,u_2' + c\,R\,u_1' + d\,R\,u_2' + x_4\,u_1' + x_5\,u_2' \Leftrightarrow (x_1, x_2, x_3, x_4, x_5)^t = a\big(\tfrac{-3172}{49}, \tfrac{2511}{49}, \tfrac{1972}{49}, \underline{\tfrac{1}{196}, \tfrac{176}{3283}}\big)^t + b\big(\tfrac{5769}{49}, \tfrac{-4569}{49}, \tfrac{-3638}{49}, \underline{\tfrac{-1}{196}, \tfrac{-176}{3283}}\big)^t + c\big(14, -11, -8, \underline{0, 0}\big)^t + d\big(-24, 19, 15, \underline{0, 0}\big)^t + x_4\big(-2, 2, 1, \underline{1, 0}\big)^t + x_5\big(5, -4, -3, \underline{0, 1}\big)^t$ 有解, $\Leftrightarrow a = b$。$\Rightarrow \ker T^3 = \operatorname{span}\{R^2(u_1' + u_2'),\ R\,u_1',\ R\,u_2',\ u_1',\ u_2'\}$ 。

令 $v_3 = b\,R^2(u_1' + u_2') + c\,R\,u_1' + d\,R\,u_2' + e\,u_1' + f\,u_2'$ $\big((b,c,d,e,f)\neq(0,0,0,0,0)\big)$。只要 $a = b \neq 0$, c, d, e, f 隨便取, 便有 $v_1 = Tv_2 = T^2 v_3$。

$b\,R^2 u_1' + b\,R^2 u_2' = b\,R^2(u_1' + u_2') = (53, -42, -34, 0, 0)^t$

最長的一串是從 $v_3 \in \ker T^3$ 跳到 $v_2 \in \ker T^2$ 跳到 $v_1 \in \ker T$

$$\begin{aligned} v_3 &= b\,R^2(u_1' + u_2') + c\,R\,u_1' + d\,R\,u_2' + e\,u_1' + f\,u_2' \\ Tv_3 = v_2 &= b\,R\,u_1' + b\,R\,u_2' + c\,u_1' + d\,u_2' \\ Tv_2 = v_1 &= b\,u_1' + b\,u_2' \end{aligned}$$

取 $(b, c, d, e, f) = (1, 0, 2, 0, 0)$, 則 $v_3 = (5, -4, -4, 0, 0)^t$, $v_2 = (0, 0, 1, 0, 2)^t$, $v_1 = (3, -2, -2, 1, 1)^t$ 。

因爲維數差是 2,2,1, 所以另一串只能是 從 $v_5 \in \ker T^2$ 再跳到 $v_4 \in \ker T$

$$\begin{aligned} v_5 &= a\,R\,u_1' + b\,R\,u_2' + c\,u_1' + d\,u_2' \\ Tv_5 = v_4 &= a\,u_1' + b\,u_2' \end{aligned}$$

取 $(a, b, c, d) = (1, 0, 0, 0)$, 則 $v_5 = (14, -11, -8, 0, 0)^t$, $v_4 = (-2, 2, 1, 1, 0)^t$ 。

第三, 將 A 化爲 Jordan form

$$S = [\,v_1\,|\,v_2\,|\,v_3\,|\,v_4\,|\,v_5\,] = \left[\begin{array}{r|r|r|r|r} 3 & 0 & 5 & -2 & 14 \\ -2 & 0 & -4 & 2 & -11 \\ -2 & 1 & -4 & 1 & -8 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 \end{array}\right]$$

$$S^{-1} = \left[\begin{array}{r|r|r|r|r} -24 & -32 & 2 & 14 & -1 \\ 12 & 16 & -1 & -7 & 1 \\ 13 & 18 & -2 & -8 & 1 \\ 24 & 32 & -2 & -13 & 1 \\ 4 & 5 & 0 & -2 & 0 \end{array}\right]$$

$$J := S^{-1}AS = \left[\begin{array}{ccc|cc} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ \hline & & & 2 & 1 \\ & & & & 2 \end{array}\right]$$

選擇很多, 例如這個選擇 $\begin{array}{l} (b,c,d,e,f)=(1,0,2,-2,-2): \Rightarrow v_3 \xrightarrow{T} v_2 \xrightarrow{T} v_1 \\ (a,b,c,d)=(-2,-1,2,1): \Rightarrow v_5 \xrightarrow{T} v_4 \end{array}$ 的 $S = \left[\begin{array}{r|r|r|r|r} 3 & 0 & -1 & -1 & -3 \\ -2 & 0 & 0 & 0 & 3 \\ -2 & 1 & 0 & 1 & 0 \\ 1 & 0 & -2 & -2 & 2 \\ 1 & 2 & -2 & -1 & 1 \end{array}\right]$ 也很漂亮。