1.(15) For positive series: C onverges or D iverges; for alternating series: converges A bsolutely, converges C onditionly, or D iverges. No confusion.

| | | | | | |
|---|---|---|---|---|---|
| $a_1=\frac{1}{2},\ a_{n+1}=(a_n)^{n+1},\ \sum a_n$ | C | $\sum n^2e^{-n}$ | C | $\sum \frac{1}{n\sqrt[n]{n}}$ | D |
| $a_1=2,\ a_{n+1}=\frac{1+\arctan n}{n}a_n,\ \sum a_n$ | C | $\sum \frac{2^n n!n!}{(2n)!}$ | C | $\sum \sin\frac{1}{n}$ | D |
| $a_1=\frac{1}{3},\ a_{n+1}=\sqrt[n]{a_n},\ \sum a_n$ | D | $\sum \frac{\sqrt[n]{n}}{n^2}$ | C | $\sum \frac{3^n}{n^3 2^n}$ | D |
| $a_1=3,\ a_{n+1}=\frac{n}{n+1}a_n,\ \sum a_n$ | D | $\sum \frac{(\ln n)^2}{n^{3/2}}$ | C | $\sum(-1)^n n^2(2/3)^n$ | A |
| $\sum(-1)^n(\sqrt{n+1}-\sqrt{n})$ | C | $\sum(-1)^n\frac{1}{n\ln n}$ | C | $\sum(-1)^n\ln(1+\frac{1}{n})$ | C |

2.(4) Evaluate $\lim_{y\to0}\frac{\arctan y-\sin y}{y^3\cos y}$ by series.

原式 $=\lim_{y\to0}\frac{y-y^3/3+y^5/5-\cdots-y+y^3/6-y^5/24+\cdots}{y^3\cos y}$ (1+1)

$=\lim_{y\to0}\frac{\frac{1}{6}y^3+\frac{19}{5}y^5+*y^7\cdots}{y^3\cos y}$

$=\lim_{y\to0}\frac{\frac{1}{6}+*y^2+*y^4\cdots}{\cos y}$ (+1) $=\frac{1}{6}$ (+1)

3.(4) Expand $\frac{1}{x^2}$ at 1 (five non-zero terms).

$\frac{1}{x^2}=-\left(\frac{1}{x}\right)'=-\left(\frac{1}{1+(x-1)}\right)'$ (+2)

$=-\left(1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4-(x-1)^5+\cdots\right)'$ (+1)

$=1-2(x-1)+3(x-1)^2-4(x-1)^3+5(x-1)^4+\cdots$ (+1)

4.(4) Expand $\frac{2}{(1-x)^3}$ at 0 (five non-zero terms).

$\frac{2}{(1-x)^3}=\left(\frac{1}{1-x}\right)''$ (+2)

$=\left(1+x+x^2+x^3+x^4+x^5+x^6\cdots\right)''$ (+1)

$=2+6x+12x^2+20x^3+30x^4+\cdots$

5.(4) Expand $2^x$ at 1 (five non-zero terms).

$2^x=e^{\ln 2\cdot x}=2\cdot e^{\ln 2(x-1)}$ (+2)

$=2\left(1+\ln 2(x-1)+\frac{(\ln 2)^2}{2!}(x-1)^2+\frac{(\ln 2)^3}{2!}(x-1)^3+\cdots\right)$ (+1)

$=2+2\ln 2(x-1)+(\ln 2)^2(x-1)^2+\frac{(\ln 2)^3}{3}(x-1)^3+\frac{(\ln 2)^4}{12}(x-1)^4+\cdots$ (+1)

6.(9) Compute $\sum_{n=0}^{\infty}(-1)^n n^2(2/3)^n$ step by step: 1) find corrseponding power series, closed form, and its $D_{\text{conv}}$ 2) take some number in $D_{\text{conv}}$ into the closed form to obtain the infinite sum.

$\sum_0 x^n=\frac{1}{1-x}$, $x\in(-1,1)$ (+1)

$\overset{x\frac{d}{dx}}{\Rightarrow}$ (+1) $\sum_1 nx^n=\frac{x}{(1-x)^2}$ (+1), $x\in(-1,1)$

$\overset{x\frac{d}{dx}}{\Rightarrow}$ (+1) $\sum_1 n^2x^n=\frac{x+x^2}{(1-x)^3}$ (+2), $x\in(-1,1)$

令 $f(x)=\frac{x+x^2}{(1-x)^3}$。因爲 $\frac{-2}{3}\in(-1,1)$ (+1),

$\therefore\sum_{n=0}^{\infty}(-1)^n n^2(2/3)^n=f(\frac{-2}{3})=\frac{-6}{125}$ (+2)。

(Tidy up your work above. Anything written below does not count.)

# 3 另解: $\frac{1}{x^2}=\left(1+(x-1)\right)^{-2}$ (+1)

$=1+\binom{-2}{1}(x-1)^1+\binom{-2}{2}(x-1)^2+\binom{-2}{3}(x-1)^3+\binom{-2}{4}(x-1)^4+\cdots$ (+1)

$=1+(-2)(x-1)^1+(-2)\frac{-3}{2}(x-1)^2+(3)\frac{-4}{3}(x-1)^3+(-4)\frac{-5}{4}(x-1)^4+\cdots$ (+1)

$=1-2(x-1)+3(x-1)^2-4(x-1)^3+5(x-1)^4+\cdots$ (+1)

# 4 另解: $\frac{2}{(1-x)^3}=2\left(1+(-x)\right)^{-3}$ (+1)

$=2\left(1+\binom{-3}{1}(-x)^1+\binom{-3}{2}(-x)^2+\binom{-3}{3}(-x)^3+\binom{-3}{4}(-x)^4+\cdots\right)$ (+1)

$=2\left(1-(-3)x+(-3)\frac{-4}{2}x^2-(6)\frac{-5}{3}x^3+(-10)\frac{-6}{4}x^4+\cdots\right)$ (+1)

$=2+6x+12x^2+20x^3+30x^4+\cdots$ (+1)