Existence of Trangulation: Prove that every n-dimensional convex polytope is the union of finitely many simplices, no two simplices of which have an interior in common.

(*pf.*) **Claim 1:** If $F = \text{conv}\{p_1, \cdots, p_n\}$ is an $(n-1)$-simplex and $H := \text{aff}(F)$, then for any $p \notin H$, the pyramid $\text{conv}(F \cup \{p\})$ is an n-simplex.

(*pf.*) Since $\{p_1, \cdots, p_n\}$ affinely independent, it suffices to show that

$$\lambda_1(p_1 - p) + \cdots + \lambda_n(p_n - p) = 0 \Leftrightarrow \lambda_1 = \cdots = \lambda_n = 0 :$$

$$\begin{aligned}
&\lambda_1(p_1 - p) + \cdots + \lambda_n(p_n - p) = 0 \\
&\Leftrightarrow \lambda_1(p_1 - p) + \lambda_2(p_2 - p_1 + p_1 - p) + \cdots + \lambda_n(p_n - p_1 + p_1 - p) = 0 \\
&\Leftrightarrow (\lambda_1 + \cdots + \lambda_n)(p_1 - p) + [\lambda_2(p_2 - p_1) + \cdots + \lambda_n(p_n - p_1)] = 0 \quad (1)
\end{aligned}$$

Since $p \notin H$, let $q := P_H(p) \neq p$ be the nearest point of p to H, then $p - q \perp H$ and inner product (1) with $p - q$:

$$\begin{aligned}
&(\lambda_1 + \cdots + \lambda_n)\overbrace{\langle p_1 - p, p - q\rangle}^{\neq 0} = 0 \\
&\Leftrightarrow \lambda_1 + \cdots + \lambda_n = 0 \\
&\Leftrightarrow [\lambda_2(p_2 - p_1) + \cdots + \lambda_n(p_n - p_1)] = 0 \\
&\Leftrightarrow \lambda_2 = \cdots = \lambda_n = 0 \text{ since } \{p_1, \cdots, p_n\} \text{ affinely independent,} \\
&\Leftrightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.
\end{aligned}$$

□

Claim 2: Let P be an n-dimensional convex polytope, $q \notin P$, $\begin{array}{l} H_1 := \text{aff } F_1 \\ H_2 := \text{aff } F_2 \end{array}$ be $(n-1)$-dimensional, and H_1, H_2 both separate q, P. If F_1, F_2 are distinct facets of P, then the interiors of pyramids $\begin{array}{l} S_1 := \text{conv}(F_1 \cup \{q\}) \\ S_2 := \text{conv}(F_2 \cup \{q\}) \end{array}$ do not intersect.

(*pf.*) $\begin{array}{l} \text{int } S_1 = \{\lambda_1 f_1 + (1-\lambda_1)q : f_1 \in \text{relint } F_1, \lambda_1 \in (0,1)\} \\ \text{int } S_2 = \{\lambda_2 f_2 + (1-\lambda_2)q : f_2 \in \text{relint } F_2, \lambda_2 \in (0,1)\} \end{array}$.

If $\text{int } S_1 \cap \text{int } S_2 \neq \emptyset$, then $\exists f_1 \in F_1, f_2 \in F_2, \lambda_1, \lambda_2 \in (0,1) \ni \lambda_1 f_1 + (1-\lambda_1)q = \lambda_2 f_2 + (1-\lambda_2)q$, i.e. $\lambda_1(f_1 - q) = \lambda_2(f_2 - q)$. This means that $f_1 - q$ are $f_2 - q$ pointing in the same direction, i.e. q, f_1, f_2 are colinear. If $\lambda_1 = \lambda_2$, then $f_1 = f_2$ and F_1, F_2 are the same facet, a contradiction; With lose of generality, say $\lambda_1 > \lambda_2$, i.e. f_1 lies between q and f_2, and this contradicts H_2 separating q, P. Hence, $\text{int } S_1 \cap \text{int } S_2 = \emptyset$.
Similarly, the above is still true if F_1, F_2 are facets of $P_1, P_2 \subset P$ respectively with $\text{int } P_1 \cup \text{int } P_2 = \emptyset$. □

Constructive Proof — Induction on $m = \#(K) \geq n + 1$, $K = \{p_1, \cdots, p_m\}$ **:**
$m = n + 1$ **:** Trivial. $\text{conv } K$ itself is an n-simplex, and we choose to do nothing.

$\boldsymbol{m = n + 2}$ **:** $\operatorname{conv} K$ is n-dimensional, $\exists p \in K \ni S := \operatorname{conv}(K \setminus \{p\})$ is an n-simplex. If $p \in S$, then $\operatorname{conv} K = S$, already an n-simplex; If $p \notin S$, since all facet of S are $(n-1)$-simplices,

$$\Big\{\operatorname{conv}(F \cup \{p\}) : F \text{ is a facet of } S \text{ and aff } F \text{ separates } p \text{ and } S\Big\}$$

are the set of new n-simplices with no common interior by claim 1 and 2.
Suppose the argument is true for $m = n+1, \cdots, k$.
$\boldsymbol{m = k + 1}$ **:** Let $K' = \{p_1, \cdots, p_k\}$ and $Q := \operatorname{conv} K'$ has simplicial subdivision: $Q = S_1 \cup \cdots \cup S_r$, $\operatorname{int} S_i \cap \operatorname{int} S_j = \emptyset$ for $i \neq j$. Let $\{p\} = K \setminus K'$. If $p \in Q$, then $\operatorname{conv} K = Q = S_1 \cup \cdots \cup S_r$ and choose to do nothing; If $p \notin Q$, then

$$\Big\{\operatorname{conv}(F \cup \{p\}) : F \text{ is a facet of } Q \text{ and aff } F \text{ separates } p \text{ and } S\Big\}$$

are the set of new n-pyramids (not necessarily simplices) with no common interior by Claim 1 and 2. Suppose F is a facet of Q whose affine hull separates p and Q. Then F consists of facets of some S_i's, $(n-1)$-simplices $T_1, \cdots, T_r$, Therefore a pyramid $\operatorname{conv}(F \cup \{p\})$ are the union of n-simplices $\operatorname{conv}(T_1 \cup \{p\}), \cdots, \operatorname{conv}(T_k \cup \{p\})$ by Claim 1, and $\operatorname{int}(\operatorname{conv}(T_1 \cup \{p\}))$'s do not intersect by Claim 2.

Clearly, for any $q \in \operatorname{conv} K \setminus Q = \operatorname{conv}(K' \cup \{p\}) \setminus Q$, $\exists\, q'$ on some facet F of $Q \ni q' = \overrightarrow{pq} \cap F$. Hence, q' is in some $(n-1)$-simplex $T \subset F$ and $q \in [p, q'] \subset \operatorname{conv}(T \cup \{p\})$, in a new n-simplex. □