FORCED WAVES FOR AN EPIDEMIC MODEL OF WEST-NILE VIRUS WITH CLIMATE CHANGE EFFECT

JONG-SHENQ GUO, WONHYUNG CHOI, TOSHIKO OGIWARA, AND CHIN-CHIN WU

ABSTRACT. This paper deals with the existence of forced waves for an epidemic model of West-Nile virus in a shifting environment. Here a forced wave is a traveling wave with wave speed the same as the environmental shifting speed. The forced waves we constructed have the property that the waves tend to the positive endemic state of the epidemic model as the time tends to infinity. The derivation of these forced waves relies on a careful construction of a suitable lower solution with the help of Schauder's fixed point theorem.

1. INTRODUCTION

A vector-borne disease is an infectious disease such that the pathogen is transmitted from an organism such as mosquitoes, ticks and fleas. West Nile virus (WNv) is one of the representative vector-borne infectious diseases with mosquito as a vector. The primary host of WNv is bird, and the virus has a 'bird-mosquito-bird' transmission cycle. So, disease dynamics of WNv depends on the interaction between birds and mosquitoes and on their movements.

In this paper, we are interested in the following simplified model of WNv introduced by Lewis et al. [17]:

$$\begin{cases} \frac{dI_b}{dt}(x,t) = d_1 \frac{d^2 I_b}{dx^2}(x,t) + \alpha_b \beta_b \frac{N_b - I_b(x,t)}{N_b} I_m(x,t) - \gamma_b I_b(x,t),\ x \in \mathbb{R},\ t > 0, \\ \frac{dI_m}{dt}(x,t) = d_2 \frac{d^2 I_m}{dx^2}(x,t) + \alpha_m \beta_b \frac{A_m - I_m(x,t)}{N_b} I_b(x,t) - c_m I_m(x,t),\ x \in \mathbb{R},\ t > 0, \end{cases} \tag{1.1}$$

in which I_b is the population of infectious birds, I_m is the population of infected mosquitoes, d_1, d_2 are the diffusion coefficients of birds and mosquitoes, respectively, α_b and α_m are the transmission probabilities of WNv per bite to birds and mosquitoes, respectively, β_b is the biting rate of mosquitoes on birds, γ_b is the recovery rate of birds from WNv, N_b is the total population of birds, c_m is the death rate of mosquitoes, and A_m is the total population of adult (female) mosquitoes.

In fact, the simplified model (1.1) is derived from the model proposed in [21] by adopting the following hypothesis:

Date: September 28, 2024. Corresponding author: C.-C. Wu.

This work was supported in part by the National Science and Technology Council of Taiwan under the grants 112-2115-M-032-001 (JSG) and 112-2115-M-005-001-MY2 (CCW). The second author was supported by the National Research Foundation of Korea under the grant NRF-2021R1A6A3A01086879 and the third author was partially supported by JSPS KAKENHI Grant Number JP22K03418. We thank the anonymous referee for the careful reading with some valuable comments.

2000 Mathematics Subject Classification. 35K45, 35K57, 34B40, 92D25.

Key words and phrases. Epidemic model, West-Nile virus, climate change, forced wave.

(A1) there is no death of birds by WNv;
(A2) recovered birds become immediately susceptible;
(A3) exposed mosquitoes are immediately infective.

Furthermore, in [17], the adult mosquitoes population A_m and the total population of birds N_b are assumed to be constants.

On the other hand, the climate change can affect the dynamics of vector-borne disease because the change alters the habitat of vectors. A temperature is one of the important factors of habitat change, and especially the survival of mosquitoes crucially depends on the temperature. As global warming makes their habitat move northwards, climate change increases the risk of exposure to WNv. Therefore, considering climate change effect is essential when we study WNv model. This motivates us to consider the model (1.1) with non-constant adult mosquitoes in time and space.

In this paper, we consider a widely used modelling for the climate change effect and assume that the total population of mosquitoes is expressed as $A_m = \theta(x - st)$ with a shifting speed $s > 0$ such that

(h1) θ is a continuous function with $\theta(+\infty) = 0 \le \theta \le 1 = \theta(-\infty)$ in $\mathbb{R}$.

This means that the maximal carrying capacity of mosquitoes is normalized to be 1, i.e., $\theta(-\infty) = 1$, which can be realized by taking a suitable unit of mosquitoes. Moreover, the mosquitoes become saturated at the level 1 as $t \to \infty$ or $x \to -\infty$.

For notational convenience, we take a suitable unit of birds so that $N_b = 1$ and set

$$\alpha = \alpha_b\beta_b,\ \gamma = \gamma_b,\ \delta = \alpha_m\beta_b,\ c = c_m.$$

Then (1.1) becomes

$$I_{b,t} = d_1 I_{b,xx} - \gamma I_b + \alpha[1 - I_b]I_m,\ x \in \mathbb{R},\ t > 0, \tag{1.2}$$

$$I_{m,t} = d_2 I_{m,xx} + \delta[\theta(x - st) - I_m]I_b - cI_m,\ x \in \mathbb{R},\ t > 0. \tag{1.3}$$

Note that, under the condition

$$\Delta := \alpha\delta - \gamma c > 0, \tag{1.4}$$

there is a unique stable positive co-existence state

$$E^* = (\phi_*, \psi_*),\ \phi_* := \frac{\Delta}{(\alpha+\gamma)\delta} \in (0,1),\ \psi_* := \frac{\Delta}{\alpha(c+\delta)} \in (0,1),$$

of the diffusion-free limiting system

$$\begin{cases} I_b'(t) = -\gamma I_b(t) + \alpha I_m(t) - \alpha I_b(t) I_m(t),\ t > 0, \\ I_m'(t) = \delta I_b(t) - c I_m(t) - \delta I_m(t) I_b(t),\ t > 0. \end{cases}$$

The state E^* is called the endemic state.

We are concerned with the spreading dynamics of the infected birds and mosquitoes. In particular, we are interested in traveling wave solutions of system (1.2)-(1.3) in the form

$$(I_b, I_m)(x,t) = (\phi, \psi)(z),\ z := x - st, \tag{1.5}$$

for some functions (wave profiles) $\{\phi, \psi\}$, where the wave speed s is the same as the environmental shifting speed. This type of traveling wave is called a forced wave. Therefore, to find a forced wave of (1.2)-(1.3) in the form (1.5) is equivalent to finding a solution (ϕ, ψ) of

$$\begin{cases} d_1\phi''(z) + s\phi'(z) - \gamma\phi(z) + \alpha\psi(z) - \alpha\phi(z)\psi(z) = 0, \ z \in \mathbb{R}, \\ d_2\psi''(z) + s\psi'(z) + \delta\theta(z)\phi(z) - c\psi(z) - \delta\phi(z)\psi(z) = 0, \ z \in \mathbb{R}. \end{cases} \tag{1.6}$$

We are interested in forced waves satisfying

$$(\phi, \psi)(\infty) = (0, 0), \quad (\phi, \psi)(-\infty) = (\phi_*, \psi_*). \tag{1.7}$$

Biologically, this means that with the climate change effect the West Nile virus spreads through the entire habitat eventually. For the study of forced waves, we refer the reader to, e.g., [1, 2, 3, 4, 6, 7, 8, 10, 13, 14, 19, 20, 23].

The problem with constant adult mosquitoes, i.e., $A_m \equiv 1$, was studied in [17]. Set

$$s_* := \inf_{\mu>0} \frac{\lambda(\mu)}{\mu}, \tag{1.8}$$

where $\lambda(\mu)$ is the Perron-Frobenius eigenvalue of the matrix

$$S(\mu) = \begin{bmatrix} d_1\mu^2 - \gamma & \alpha \\ \delta & d_2\mu^2 - c \end{bmatrix}.$$

In particular, it is proved in [17] that a traveling wave connecting the disease free state $(0, 0)$ to the endemic state E^* of system (1.2)-(1.3) with $\theta = 1$ exists if and only if its wave speed exceeds the positive constant s_* (the minimal speed). Notice that system (1.2)-(1.3) with $\theta = 1$ is a homogeneous cooperative system. However, we do not know whether system (1.2)-(1.3) is a cooperative system, since mathematically we were unable to verify that $I_m \leq \theta$ (although it is true biologically).

The main theorem of this paper reads

Theorem 1.1. *Let $s > 0$ and let condition* (h1) *and* (1.4) *be enforced. Then system* (1.2)-(1.3) *has a forced wave in the form* (1.5) *for any $s < s_*$ such that $(\phi, \psi)(\infty) = (0, 0)$. Moreover, if we further assume*

$$\alpha > \gamma, \ \delta > c, \tag{1.9}$$

then we have $(\phi, \psi)(-\infty) = (\phi_*, \psi_*)$.

As mentioned earlier, there are already many interesting works on the existence of forced waves in various reaction-diffusion systems. It seems that the problem of finding solutions of (1.6) and (1.7) is an easy consequence of the well-known results and methods in the literature. However, there are certain difficulties to be overcome in dealing with system (1.6). Note that a solution of system (1.6) is a stationary solution of the cooperative system

$$\begin{cases} \phi_t = d_1\phi_{zz} + s\phi_z - \gamma\phi + \alpha\psi - \alpha\phi\psi, \ z \in \mathbb{R}, \ t > 0, \\ \psi_t = d_2\psi_{zz} + s\psi_z + \delta\theta\phi - c\psi - \delta\phi\psi, \ z \in \mathbb{R}, \ t > 0, \end{cases}$$

in the space

$$\mathcal{S} := \{(\phi, \psi) \in C^0(\mathbb{R}) \times C^0(\mathbb{R}) \mid 0 \leq \phi \leq 1, \ 0 \leq \psi \leq \theta \ \text{ in } \mathbb{R}\}.$$

Biologically, one should have $\psi \le \theta$. However, we were unable to derive rigorously such an a priori bound. On the other hand, although $[0,\phi_*]\times[0,\psi_*]$ is an invariant set of system (1.6), the set $[0,\phi_*]\times[0,\min\{\psi_*,\theta\}]$ is not necessary invariant. Therefore, the classical monotone iteration method cannot be applied.

On the other hand, the method of upper-lower solutions with the help of Schauder's fixed point theorem has been proved very successful in deriving traveling wave solutions (in particular, forced waves) in many non-cooperative systems. See, e.g., [5, 12] and the references cited therein. The success of this method relies on the suitable definition and the careful construction of upper-lower solutions. Moreover, in order to apply the Schauder's fixed point theorem, one need to verify that the corresponding integral operator to the original ODE system maps the set defined by the pair of upper-lower solutions into itself. Note that neither $\psi_* \le \theta$ nor $\theta \le \psi_*$ in $\mathbb{R}$. Our strategy is to construct a lower-solution $(\underline{\phi},\underline{\psi})$ of (1.6) (to be defined later) so that $\underline{\phi} \le \phi_*$ and $\underline{\psi} \le \theta$ in $\mathbb{R}$. With this valuable information, we are able to obtain an invariant domain of the corresponding integral operator so that Schauder's fixed point theorem can be applied to derive the existence of a solution of (1.6).

The outline of this paper is as follows. In §2, we first introduce the definition of upper-lower solutions of system (1.6). Then we provide a derivation of the existence of solution to (1.6) under certain conditions on the lower solution. Motivated by [11], the construction of a suitable lower solution is given in §3. Finally, in §4, we finish the proof of the existence of forced waves as stated in Theorem 1.1.

2. Preliminaries

To find a solution of (1.6)-(1.7), we first introduce the following definition of upper-lower solutions.

Definition 2.1. Continuous functions $(\overline{\phi},\overline{\psi})$ and $(\underline{\phi},\underline{\psi})$ are called a pair of upper- and lower-solutions of (1.6) if $\overline{\phi} \ge \underline{\phi}, \overline{\psi} \ge \underline{\psi}$, and the following inequalities

$$\mathcal{U}_1(z) := d_1\overline{\phi}''(z) + s\overline{\phi}'(z) - \gamma\overline{\phi}(z) + \alpha\overline{\psi}(z) - \alpha\overline{\phi}(z)\overline{\psi}(z) \le 0, \tag{2.1}$$

$$\mathcal{U}_2(z) := d_2\overline{\psi}''(z) + s\overline{\psi}'(z) + \delta\theta(z)\overline{\phi}(z) - c\overline{\psi}(z) - \delta\overline{\phi}(z)\overline{\psi}(z) \le 0, \tag{2.2}$$

$$\mathcal{L}_1(z) := d_1\underline{\phi}''(z) + s\underline{\phi}'(z) - \gamma\underline{\phi}(z) + \alpha\underline{\psi}(z) - \alpha\underline{\phi}(z)\underline{\psi}(z) \ge 0, \tag{2.3}$$

$$\mathcal{L}_2(z) := d_2\underline{\psi}''(z) + s\underline{\psi}'(z) + \delta\theta(z)\underline{\phi}(z) - c\underline{\psi}(z) - \delta\underline{\phi}(z)\underline{\psi}(z) \ge 0, \tag{2.4}$$

hold for all $z \in \mathbb{R}\setminus E$ for some finite subset E of $\mathbb{R}$.

Due to $\theta \le 1$, it is easy to see that $(\overline{\phi},\overline{\psi})(z) \equiv (\phi_*,\psi_*)$ is an upper solution of (1.6).

Next, suppose that there exists a lower solution $(\underline{\phi},\underline{\psi})$ of (1.6) such that

$$0 \le \underline{\phi} \le \phi_*,\ \ 0 \le \underline{\psi} \le \theta, \tag{2.5}$$

and for each $z_j \in E$ it holds

$$\lim_{z\nearrow z_j^-} \underline{\phi}'(z) \le \lim_{z\searrow z_j^+} \underline{\phi}'(z),\ \ \lim_{z\nearrow z_j^-} \underline{\psi}'(z) \le \lim_{z\searrow z_j^+} \underline{\psi}'(z). \tag{2.6}$$

Then we define

$$\Gamma = \{(\phi,\psi)\in C^0(\mathbb{R})\times C^0(\mathbb{R}) \mid \underline{\phi}\le\phi\le\phi_*,\ \underline{\psi}\le\psi\le\psi_*\}.$$

Also, we introduce

$$\begin{cases} F_1(\phi,\psi)(z) = \rho\phi(z)+\alpha\psi(z)-\alpha\phi(z)\psi(z)-\gamma\phi(z),\\ F_2(\phi,\psi)(z) = \rho\psi(z)+\delta\theta(z)\phi(z)-\delta\phi(z)\psi(z)-c\psi(z),\end{cases}$$

where ρ is a constant such that $\rho>\max\{\alpha+\gamma,\delta+c\}$.

Now we may rewrite (1.6) as

$$\begin{cases} d_1\phi''(z)+s\phi'(z)-\rho\phi(z)+F_1(\phi,\psi)(z)=0,\ z\in\mathbb{R},\\ d_2\psi''(z)+s\psi'(z)-\rho\psi(z)+F_2(\phi,\psi)(z)=0,\ z\in\mathbb{R},\end{cases} \tag{2.7}$$

and define an integral operator $Q=(Q_1,Q_2)$ by

$$Q_1(\phi,\psi)(z) := \frac{1}{d_1(\lambda_+-\lambda_-)}\Big[\int_{-\infty}^{z} e^{\lambda_-(z-y)}+\int_{z}^{+\infty} e^{\lambda_+(z-y)}\Big]F_1(\phi,\psi)(y)dy,$$
$$Q_2(\phi,\psi)(z) := \frac{1}{d_2(\mu_+-\mu_-)}\Big[\int_{-\infty}^{z} e^{\mu_-(z-y)}+\int_{z}^{+\infty} e^{\mu_+(z-y)}\Big]F_2(\phi,\psi)(y)dy,$$

where $\lambda_\pm$ and $\mu_\pm$ are defined by

$$\lambda_\pm=\frac{-s\pm\sqrt{s^2+4\rho d_1}}{2d_1},\ \ \mu_\pm=\frac{-s\pm\sqrt{s^2+4\rho d_2}}{2d_2}.$$

Note that $Q(\phi,\psi)$ satisfies

$$\begin{cases} d_1[Q_1(\phi,\psi)]''(z)+s[Q_1(\phi,\psi)]'(z)-\rho[Q_1(\phi,\psi)](z)+F_1(\phi,\psi)(z)=0,\ z\in\mathbb{R},\\ d_2[Q_2(\phi,\psi)]''(z)+s[Q_2(\phi,\psi)]'(z)-\rho[Q_2(\phi,\psi)](z)+F_2(\phi,\psi)(z)=0,\ z\in\mathbb{R}.\end{cases}$$

Thus, a fixed-point (ϕ,ψ) of the operator Q renders a solution of (1.6).

Note that

$$F_1(\phi,\psi)(z)\ge F_1(\underline{\phi},\underline{\psi})(z),\ F_2(\phi,\psi)(z)\ge F_2(\underline{\phi},\underline{\psi})(z),\ \forall\, z\in\mathbb{R}, \tag{2.8}$$

if $(\phi,\psi)\in\Gamma$. Indeed, this follows from $\phi\le\phi_*<1$, $\underline{\psi}\le\theta\le 1$ and the identities

$$F_1(\phi,\psi)(z)-F_1(\underline{\phi},\underline{\psi})(z)=(\rho-\gamma-\alpha\underline{\psi})(\phi-\underline{\phi})+\alpha(1-\phi)(\psi-\underline{\psi}),$$
$$F_2(\phi,\psi)(z)-F_2(\underline{\phi},\underline{\psi})(z)=(\rho-c-\delta\phi)(\psi-\underline{\psi})+\delta(\theta-\underline{\psi})(\phi-\underline{\phi}).$$

It follows from (2.8) that $Q_1(\phi,\psi)\ge Q_1(\underline{\phi},\underline{\psi})$ and $Q_2(\phi,\psi)\ge Q_2(\underline{\phi},\underline{\psi})$. Hence, by the definition of the lower-solution $(\underline{\phi},\underline{\psi})$ along with the property (2.6), we obtain

$$Q_1(\phi,\psi)\ge Q_1(\underline{\phi},\underline{\psi})\ge\underline{\phi},\ Q_2(\phi,\psi)\ge Q_2(\underline{\phi},\underline{\psi})\ge\underline{\psi},\ \ \text{if } (\phi,\psi)\in\Gamma. \tag{2.9}$$

On the other hand, since

$$\frac{\partial F_1}{\partial\phi}(\phi,\psi)=(\rho-\gamma-\alpha\psi)\ge 0,\ \frac{\partial F_1}{\partial\psi}(\phi,\psi)=\alpha(1-\phi)\ge 0,$$

we get $F_1(\phi,\psi) \le F_1(\phi_*,\psi_*)$ for all $(\phi,\psi) \in \Gamma$. Since $F_1(\phi_*,\psi_*) = \rho\phi_*$, we obtain that $Q_1(\phi,\psi) \le \phi_*$ for all $(\phi,\psi) \in \Gamma$. Finally, since $\theta < 1$ and $\psi \le \psi_* < 1$, we have

$$\begin{aligned} F_2(\phi,\psi) &\le \rho\psi + \delta\phi - \delta\phi\psi - c\psi = (\rho - c)\psi + \delta(1-\psi)\phi \\ &\le (\rho - c)\psi + \delta(1-\psi)\phi_* = (\rho - c - \delta\phi_*)\psi + \delta\phi_* \\ &\le (\rho - c - \delta\phi_*)\psi_* + \delta\phi_* = \rho\psi_*. \end{aligned}$$

It follows that $Q_2(\phi,\psi) \le \psi_*$. Hence we have proved that the operator Q maps Γ into Γ.

Then, by an application of the Schauder's fixed point theorem (cf., e.g., [18, 22, 15]), we obtain that Q has a fixed-point $(\phi,\psi) \in \Gamma$. Therefore, we conclude that system (1.6) has a solution $(\phi,\psi) \in \Gamma$, provided that we can find a lower solution $(\underline{\phi},\underline{\psi})$ of (1.6) such that conditions (2.5) and (2.6) hold.

3. Construction of a lower solution

This section is devoted to the construction of a suitable lower solution to (1.6).

First, by (1.4), we can choose $\varepsilon_0 > 0$ sufficiently small such that $\alpha\delta(1-\varepsilon_0)(1-2\varepsilon_0) > \gamma c$. Also, recall from (1.8) that

$$s_* := \inf_{\mu>0} \frac{\lambda(\mu)}{\mu},$$

where $\lambda(\mu)$ is the Perron-Frobenius eigenvalue of the matrix

$$S(\mu) = \begin{bmatrix} d_1\mu^2 - \gamma & \alpha \\ \delta & d_2\mu^2 - c \end{bmatrix}.$$

For $0 < \varepsilon \le \varepsilon_0$, $s > 0$ and $L > 0$, let $\Lambda = \Lambda_L := \Lambda_L(s,\varepsilon)$ be the principal eigenvalue of

$$\begin{cases} -d_1\phi_{xx} - s\phi_x + \gamma\phi - \alpha(1-\varepsilon)\psi = \Lambda\phi, \ x \in (-L,L), \\ -d_2\psi_{xx} - s\psi_x + c\psi - \delta(1-2\varepsilon)\phi = \Lambda\psi, \ x \in (-L,L), \\ \phi(\pm L) = \psi(\pm L) = 0. \end{cases} \tag{3.1}$$

Then, for a given $s \in (0,s_*)$, recall from [9, Proposition 3.5] (see also [11]) that $\Lambda_L(s,\varepsilon) < 0$ for a large L and a small ε. We also choose a fixed eigenfunction (ϕ_L,ψ_L) of (3.1) corresponding to the principal eigenvalue $\Lambda_L(s,\varepsilon)$ such that $0 < \phi_L, \psi_L \le 1$ in $(-L,L)$.

Next, let $0 < s < s_*$. Since $\theta(-\infty) = 1$, there exists $z_* \in \mathbb{R}$ such that

$$\theta(z) \ge 1 - \varepsilon, \ \forall\, z \le z_*. \tag{3.2}$$

For any $z_0 \le z_*$, we define

$$(\Phi,\Psi)(z) := (\varepsilon\phi_L(z - z_0 + L), \varepsilon\psi_L(z - z_0 + L)), \ z \in (z_0 - 2L, z_0).$$

Then (Φ,Ψ) satisfies

$$\begin{aligned} & d_1\Phi''(z) + s\Phi'(z) - \gamma\Phi(z) + \alpha\Psi(z) - \alpha\Phi(z)\Psi(z) \\ \ge\ & \varepsilon\{d_1\phi_L'' + s\phi_L' - \gamma\phi_L + \alpha(1-\varepsilon)\psi_L\}(z - z_0 + L) \ge 0, \ z \in (z_0 - 2L, z_0), \quad (3.3) \\ & d_2\Psi''(z) + s\Psi'(z) + \delta\theta(z)\Phi(z) - c\Psi(z) - \delta\Phi(z)\Psi(z) \\ \ge\ & \varepsilon\{d_2\psi_L'' + s\psi_L' + \delta(1-2\varepsilon)\phi_L - c\psi_L\}(z - z_0 + L) \ge 0, \ z \in (z_0 - 2L, z_0), \quad (3.4) \end{aligned}$$

using (3.1), $0 \le \Phi, \Psi \le \varepsilon$, $\Lambda_L(s,\varepsilon) < 0$ and (3.2). Denote the extension of (ϕ_L, ψ_L) to be zero outside of $(-L, L)$ by $(\hat{\phi}, \hat{\psi})$. Then $(\varepsilon\hat{\phi}, \varepsilon\hat{\psi})(z - z_0 + L)$ is a lower solution of (1.6) in $\mathbb{R}$ such that (2.5) and (2.6) hold for any $z_0 \le z_*$. Therefore, the function

$$(\underline{\phi}, \underline{\psi})(z) := \left(\varepsilon \sup_{z_0 \le z_*} \{\hat{\phi}(z - z_0 + L)\}, \varepsilon \sup_{z_0 \le z_*} \{\hat{\psi}(z - z_0 + L)\}\right)$$

is a lower solution of (1.6) such that (2.5) and (2.6) hold. Moreover, we have

$$\lim_{z \to -\infty} \underline{\phi}(z) = \varepsilon \max_{z \in [-L,L]} \phi_L(z) > 0, \ \lim_{z \to -\infty} \underline{\psi}(z) = \varepsilon \max_{z \in [-L,L]} \psi_L(z) > 0. \tag{3.5}$$

4. Existence of forced waves

First, the existence a positive solution (ϕ, ψ) of (1.6) follows from the Schauder's fixed point theorem by using the pair of upper-lower solutions constructed in the previous sections.

Next, we derive the wave tail limits of this wave profile (ϕ, ψ) as follows.

Proposition 4.1. *It holds* $(\phi, \psi)(\infty) = (0, 0)$.

Proof. Although the proof is standard (cf. [6, 7]), we provide the details for the reader's convenience due to some cares need to be taken. Let

$$\phi^+ := \limsup_{z \to +\infty} \phi(z), \ \psi^+ := \limsup_{z \to +\infty} \psi(z).$$

Since $\underline{\phi}, \underline{\psi} \ge 0$, we have $\phi^+, \psi^+ \ge 0$. For contradiction, we suppose that either $\psi^+ > 0$ or $\phi^+ > 0$.

Suppose first that $\psi^+ > 0$. If ψ is oscillatory near $z = +\infty$, there is a sequence of maximal points $\{z_n\}$ of ψ such that $\psi(z_n) \to \psi^+$ and $z_n \to \infty$ as $n \to \infty$. Since $\theta(z_n) \to 0$ as $n \to \infty$, $\psi'(z_n) = 0$ and $\psi''(z_n) \le 0$ for all n, we obtain

$$0 = \lim_{n \to \infty} [d_2 \psi''(z_n) + s\psi'(z_n) + \delta\theta(z_n)\phi(z_n) - c\psi(z_n) - \delta\phi(z_n)\psi(z_n)] \le -c\psi^+ < 0,$$

a contradiction. On the other hand, when ψ is monotone ultimately at $z = +\infty$, we can find a sequence $\{z_n\}$ tending to $+\infty$ such that $\psi'(z_n) \to 0$. Since $\psi(\infty) = \psi^+ > 0$ and $\theta(\infty) = 0$, we can find a sufficiently large integer m such that

$$\theta(z) \le \psi(z), \ \forall\, z \ge z_m.$$

By an integration of ψ-equation in (1.6) from z_m to z_n with $n > m$, we obtain that

$$d_2[\psi'(z_m) - \psi'(z_n)] + s[\psi(z_m) - \psi(z_n)] = \int_{z_m}^{z_n} [\delta\theta(z)\phi(z) - c\psi(z) - \delta\phi(z)\psi(z)]dz. \tag{4.1}$$

Since $\psi(z_n) \to \psi^+$ as $n \to \infty$, the left-hand side of (4.1) is uniformly bounded for all n. However, the right-hand side of (4.1) tends to $-\infty$ as $n \to \infty$, a contradiction. Hence we have proved that $\psi^+ = 0$.

Now, suppose that $\phi^+ > 0$. If ϕ is oscillatory near $z = +\infty$, then there is a sequence of maximal points $\{z_n\}$ of ϕ such that $\phi(z_n) \to \phi^+$ and $z_n \to \infty$ as $n \to \infty$. Since $\psi(z_n) \to 0$ as $n \to \infty$, $\phi'(z_n) = 0$ and $\phi''(z_n) \le 0$ for all n, we have a contradiction as that for ψ in above.

Suppose that ϕ is monotone ultimately at $z = +\infty$. Then we can find a sequence $\{z_n\}$ such that $\phi'(z_n) \to 0$ and $z_n \to \infty$ as $n \to \infty$. Since $\psi(\infty) = 0$ and $\phi(\infty) = \phi^+ > 0$, there is a sufficiently large integer m such that

$$\alpha\psi(z) \leq \gamma\phi(z)/2, \ \forall\, z \geq z_m.$$

Integrating the ϕ-equation of (1.6) from z_m to z_n, we have

$$d_1[\phi'(z_m) - \phi'(z_n)] + s[\phi(z_m) - \phi(z_n)] = \int_{z_m}^{z_n} [-\gamma\phi(z) + \alpha\psi(z) - \alpha\phi(z)\psi(z)]dz. \tag{4.2}$$

Again, as before, the left-hand side of (4.2) is uniformly bounded for all n and the right-hand side of (4.2) tends to $-\infty$ as $n \to \infty$, a contradiction. Therefore, we must have $\phi^+ = 0$ and so the proposition is proved. □

To derive $(\phi, \psi)(-\infty) = E^* = (\phi_*, \psi_*)$, we use the idea of shrinking rectangles (see, e.g., [16, 5, 12]).

Proposition 4.2. *It holds $(\phi, \psi)(-\infty) = (\phi_*, \psi_*)$, if (1.9) is enforced.*

Proof. First, set

$$\begin{cases} p^- := \liminf_{z\to-\infty} \phi(z), \ p^+ := \limsup_{z\to-\infty} \phi(z), \\ q^- := \liminf_{z\to-\infty} \psi(z), \ q^+ := \limsup_{z\to-\infty} \psi(z), \\ \kappa_0 := \min\{\underline{\phi}(-\infty), \underline{\psi}(-\infty), \phi_*, \psi_*\}. \end{cases}$$

Note that $\kappa_0 > 0$ due to (3.5). From the construction of upper-lower solutions, we have

$$\kappa_0 \leq p^- \leq p^+ \leq \phi_*, \ \kappa_0 \leq q^- \leq q^+ \leq \psi_*. \tag{4.3}$$

Define the set

$$A := \{\nu \in [0, 1) \mid p^- > m(\nu), \, q^- > M(\nu)\},$$

where

$$m(\nu) := (1 - \nu)\kappa + \nu\phi_*, \ M(\nu) := (1 - \nu)\kappa + \nu\psi_*, \ \nu \in [0, 1],$$

in which $\kappa \in (0, \kappa_0)$ is constant such that

$$\kappa < \min\{1 - \gamma/\alpha, 1 - c/\delta, \phi_*\psi_*/(\phi_* + \psi_*)\}. \tag{4.4}$$

It is well-defined due to (1.9). Note that $m(\nu) \nearrow \phi_*$ and $M(\nu) \nearrow \psi_*$ as $\nu \nearrow 1$. Also, $0 \in A$, thanks to (4.3). Then the quantity $\nu_0 := \sup A$ is well-defined and $\nu_0 \leq 1$.

Next, we claim that $\nu_0 = 1$. Otherwise, if $\nu_0 < 1$, then we have

$$p^- \geq m(\nu_0), \ q^- \geq M(\nu_0), \tag{4.5}$$

by passing to the limit. Due to the continuities of m and M in ν, we must have either $p^- = m(\nu_0)$ or $q^- = M(\nu_0)$. Assume that $p^- = m(\nu_0)$. Then, along a sequence $\{z_n\}$ tending to $-\infty$ such that $\phi(z_n) \to m(\nu_0)$ as $n \to \infty$, we have

$$\begin{aligned}
&\liminf_{n\to\infty}\{-\gamma\phi(z_n)+\alpha\psi(z_n)[1-\phi(z_n)]\} \geq -\gamma m(\nu_0)+\alpha M(\nu_0)[1-m(\nu_0)] \\
= \ & \kappa(1-\nu_0)\{\alpha[1-(1-\nu_0)\kappa]-\gamma\} - \alpha\nu_0(1-\nu_0)\kappa\phi_* \\
&\quad +\nu_0(-\gamma\phi_*+\alpha\psi_*) - \alpha\nu_0(1-\nu_0)\kappa\psi_* - \alpha\nu_0^2\phi_*\psi_* \\
= \ & \kappa(1-\nu_0)\{\alpha[1-(1-\nu_0)\kappa]-\gamma\} + \nu_0(1-\nu_0)\alpha\{\phi_*\psi_* - \kappa(\phi_*+\psi_*)\} > 0,
\end{aligned}$$

using (4.5), $-\gamma\phi_* + \alpha\psi_* = \alpha\phi_*\psi_*$, (4.4) and $\nu_0 \in (0, 1)$. This leads to a contradiction by following the method used in [5, 12]. We do not repeat it here.

Similarly, when $q^- = M(\nu_0)$, along a sequence $\{z_n\}$ tending to $-\infty$ such that $\psi(z_n) \to M(\nu_0)$ as $n \to \infty$, we also have

$$\begin{aligned}
&\liminf_{n\to\infty}\{\delta\phi(z_n)[\theta(z_n)-\psi(z_n)] - c\psi(z_n)\} \geq \delta m(\nu_0)[1-M(\nu_0)] - cM(\nu_0) \\
= \ & \kappa(1-\nu_0)\{\delta[1-(1-\nu_0)\kappa]-c\} + \nu_0(1-\nu_0)\delta\{\phi_*\psi_* - \kappa(\phi_*+\psi_*)\} > 0,
\end{aligned}$$

using (4.5), $\theta(-\infty) = 1$, $\delta\phi_* - c\psi_* = \delta\phi_*\psi_*$, (4.4) and $\nu_0 \in (0, 1)$. This again leads to a contradiction. We conclude that $\nu_0 = 1$ and so $\phi^- = \phi_*$ and $\psi^- = \psi_*$. Hence the proposition is proved. □

Summarizing the above results, we thereby complete the proof of Theorem 1.1.

References

[1] H. Berestycki, L. Desvillettes, O. Diekmann, *Can climate change lead to gap formation?*, Ecological Complexity, 20 (2014), 264-270.

[2] H. Berestycki, J. Fang, *Forced waves of the Fisher-KPP equation in a shifting environment*, J. Differential Equations, 264 (2018), 2157-2183.

[3] H. Berestycki, L. Rossi, *Reaction-diffusion equations for population dynamics with forced speed, I - the case of the whole space*, Discrete Contin. Dyn. Syst., 21 (2008), 41-67.

[4] H. Berestycki, L. Rossi, *Reaction-diffusion equations for population dynamics with forced speed, II - cylindrical type domains*, Discrete Contin. Dyn. Syst., 25 (2009), 19-61.

[5] Y.-Y. Chen, J.-S. Guo, C.-H. Yao, *Traveling wave solutions for a continuous and discrete diffusive predator–prey model*, J. Math. Anal. Appl., 445(1) (2017), 212-239.

[6] W. Choi, T. Giletti, J.-S. Guo, *Persistence of species in a predator-prey system with climate change and either nonlocal or local dispersal*, J. Differential Equations 302 (2021), 807-853.

[7] W. Choi, J.-S. Guo, *Forced waves of a three species predator-prey system in a shifting environment*, J. Math. Anal. Appl., 514 (2022), 126283.

[8] F.-D. Dong, B. Li, W.-T. Li, *Forced waves in a Lotka-Volterra competition-diffusion model with a shifting habitat*, J. Differential Equations, 276 (2021), 433-459.

[9] A. Ducrot, T. Giletti, J.-S. Guo, M. Shimojo, *Asymptotic spreading speeds for a predator-prey system with two predators and one prey*, Nonlinearity, 34 (2021), 669-704.

[10] J. Fang, Y. Lou, J. Wu, *Can pathogen spread keep pace with its host invasion?*, SIAM J. Appl. Math., 76 (2016), 1633-1657.

[11] L. Girardin, *Non-cooperative Fisher-KPP systems: traveling waves and long-time behavior*, Nonlinearity, 31 (2018), 108-164.

[12] J.-S. Guo, K.-I. Nakamura, T. Ogiwara, C.-C. Wu, *Traveling wave solutions for a predator-prey system with two predators and one prey*, Nonlinear Analysis: Real World Applications, 54 (2020), 103111.

[13] J.-S. Guo, A. Poh, C.-C. Wu, *Forced waves of saturation type for Fisher-KPP equation in a shifting environment*, Applied Mathematics Letters, 140 (2023), Art. 108573.

[14] H. Hu, X. Zou, *Existence of an extinction wave in the Fisher equation with a shifting habitat*, Proc. Amer. Math. Soc., 145 (2017), 4763-4771.

[15] J. Huang, X. Zou, *Existence of traveling wave fronts of delayed reaction-diffusion systems without monotonicity*, Discrete Contin. Dyn. Syst., 9 (2003), 925-936.

[16] Y.L. Huang and G. Lin, *Traveling wave solutions in a diffusive system with two preys and one predator*, J. Math. Anal. Appl., 418 (2014), 163-184.

[17] M. Lewis, J. Renclawowicz, P. Driessche, *Traveling waves and spread rates for a West Nile virus mode*, Bull. Math. Biol., 68 (2006), 3-23

[18] S. Ma, *Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem*, J. Differential Equations 171 (2001), 294-314.

[19] J.-B. Wang, C. Wu, *Forced waves and gap formations for a Lotka-Volterra competition model with nonlocal dispersal and shifting habitats*, Nonlinear Analysis: Real World Applications, 58 (2021), 103208.

[20] J.-B. Wang, X.-Q. Zhao, *Uniqueness and global stability of forced waves in a shifting environment*, Proc. Amer. Math. Soc., 147 (2019), 1467-1481.

[21] M. J. Wonham, T. de-Camino-Beck, M. Lewis, *An epidemiological model for West Nile virus: invasion analysis and control applications*, Proc. Royal Soc. London. Series B: Biological Sciences, (2004) 271(1538), 501-507.

[22] J. Wu, X. Zou, Traveling wave fronts of reaction-diffusion systems with delay, J. Dynam. Differential Equations 13 (2001), 651-687.

[23] Y. Yang, C. Wu, Z. Li, *Forced waves and their asymptotics in a Lotka-Volterra cooperative model under climate change*, Appl. Math. Comput., 353 (2019), 254-264.

Department of Mathematics, Tamkang University, Tamsui, New Taipei City 251301, Taiwan
Email address: jsguo@mail.tku.edu.tw

Institute of Natural Science, Korea University, 2511, Sejong-ro, Sejong, 30019, Republic of Korea
Email address: whchoi@korea.ac.kr

Department of Mathematics, Josai University, Sakado 350-0295, Japan
Email address: toshiko@josai.ac.jp

Department of Applied Mathematics, National Chung Hsing University, Taichung 402, Taiwan
Email address: chin@email.nchu.edu.tw