FORCED WAVES OF A DELAYED DIFFUSIVE ENDEMIC MODEL WITH SHIFTING TRANSMISSION RATES

WONHYUNG CHOI, JONG-SHENQ GUO, AND CHIN-CHIN WU

Abstract. In this paper, we investigate the forced waves of a delayed diffusive endemic model with a shifting transmission rate. Here a forced wave is a traveling wave with wave speed the same as the environmental shifting speed. By constructing a new pair of upper-lower solutions, we prove the existence of forced waves for any negative shifting speed which corresponds the deceased of the disease, regardless of the magnitude of the delay. Moreover, we also derive the existence of forced waves with small shifting speeds without delay which signify the disease spread. A non-existence of forced wave when the limiting reproduction number is less than one is also proven.

1. Introduction

In epidemiology, taking the vital dynamics (births and deaths) into account, the classical delayed diffusive endemic model is written as

$$\begin{cases} S_t(x,t) = d_1 S_{xx}(x,t) + \Lambda - \mu S(x,t) - \frac{\beta S(x,t) I(x,t-\tau)}{1+\alpha I(x,t-\tau)},\ x \in \mathbb{R},\ t>0, \\ I_t(x,t) = d_2 I_{xx}(x,t) + \frac{\beta S(x,t) I(x,t-\tau)}{1+\alpha I(x,t-\tau)} - \sigma I(x,t) - \gamma I(x,t),\ x \in \mathbb{R},\ t>0, \\ R_t(x,t) = d_3 R_{xx}(x,t) + \gamma I(x,t) - \hat{\mu} R(x,t),\ x \in \mathbb{R},\ t>0, \end{cases} \tag{1.1}$$

where $S(x,t), I(x,t), R(x,t)$ represent the population densities of the susceptible, infective, removed individuals at position x and time t, respectively. The coefficients $d_i, \Lambda, \mu, \beta, \alpha, \sigma, \gamma, \hat{\mu}$ are all positive constants, in which d_i is the spatial motility of each group, $i = 1, 2, 3$, Λ is the entering flux of susceptible individuals, $\mu, \sigma, \hat{\mu}$ denote the death rates of susceptible, infective and removed populations, respectively, γ is the recovery rate of the infective populations, β is the infective transmission rate and α measures the saturation level ([8, 24]) in the Holling type II incidence function. The constant $\tau \geq 0$ is the latency of the infection in vectors.

One of the main concerns of epidemic models is whether the disease can spread. Among many different approaches, the existence of traveling waves connecting the disease-free state to the endemic state has been extensively studied. In particular, we refer the reader to the work [23] for a very nice survey of the literature on (1.1) with/without delay, with/without vitae dynamics and other types of incidence functions. See also [19].

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On the other hand, it is commonly assumed that the transmission rate β is a constant. Little is done for the inhomogeneous transmission case. However, it is natural that the transmission rate depends on both time and spatial position. In regarding of this aspect, we refer the reader to a very recent work [20]. For a vector-borne disease model [20, system (25)], the biting rate of the vectors (mosquitoes) and the total number of mosquitoes are assumed to be periodic in time and monotone in the moving coordinate with a given shifting speed caused by the climate change.

The main purpose of this paper is to investigate the endemic model (1.1) by taking into account the shifting effect on the transmission rates. Since the R-equation in (1.1) is decoupled from the first two equations, we only need to consider the first two equations in (1.1). Furthermore, by a suitable scaling (cf. [23]), we may assume that $\Lambda = \mu$ so that the first two equations in (1.1) with β replaced by $\beta(x-st)$ are re-written as

$$\begin{cases} S_t(x,t) = d_1 S_{xx}(x,t) + \mu - \mu S(x,t) - \frac{\beta(x-st)S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)},\ x\in\mathbb{R},\ t>0, \\ I_t(x,t) = d_2 I_{xx}(x,t) + \frac{\beta(x-st)S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)} - \kappa I(x,t),\ x\in\mathbb{R},\ t>0, \end{cases} \tag{1.2}$$

where $\kappa := \sigma + \gamma$ and s is a nonzero constant which stands for the shifting speed of the environment.

In this paper, we shall always assume that $\beta = \beta(\xi)$, $\xi := x - st$, satisfies the following properties:

(A1) β is a continuous function in $\mathbb{R}$ such that $\beta(\infty) = 0 \le \beta(\xi) \le \beta_0 = \beta(-\infty)$ for all $\xi \in \mathbb{R}$ for some positive constant β_0.

(A2) $\beta(\xi) \le e^{-\theta\xi}$ for all $\xi \ge K_1$ for positive constants θ and K_1.

In view of the sign of the shifting speed s, there are two cases for the transmission to be advantageous ($s>0$) or disadvantageous ($s<0$). The representative advantageous examples are mosquito-borne diseases such as dengue fever, malaria, Zika virus, and West Nile virus. Contrarily, rising temperatures could make the habitat suitability of some tick species decrease, which affects the related diseases [12]. This is the disadvantageous case corresponding to $s<0$.

Note that there is always the disease-free state $(1,0)$ of (1.2). We define the (limiting) basic reproduction number $\mathcal{R}_0 := \beta_0/\kappa$, which is an important threshold value in the epidemic model determining whether the disease spreads or not. When $\mathcal{R}_0 > 1$, i.e., $\beta_0 > \kappa$, there is a unique positive constant equilibrium (S^*, I^*) of the following limiting ODE system

$$\begin{cases} \frac{dS}{dt} = \mu - \mu S(t) - \frac{\beta_0 S(t) I(t-\tau)}{1+\alpha I(t-\tau)}, \\ \frac{dI}{dt} = \frac{\beta_0 S(t) I(t-\tau)}{1+\alpha I(t-\tau)} - \kappa I(t), \end{cases}$$

where

$$(S^*, I^*) = \Big(\frac{\alpha\mu+\kappa}{\alpha\mu+\beta_0}, \frac{\mu(\beta_0-\kappa)}{(\beta_0+\alpha\mu)\kappa}\Big).$$

Note that

$$\frac{\beta_0 S^*}{1+\alpha I^*} = \kappa, \quad \mu - \mu S^* - \frac{\beta_0 S^* I^*}{1+\alpha I^*} = 0. \tag{1.3}$$

We call (S^*, I^*) the endemic state of (1.2). Throughout this paper, a vector-valued function is positive (nonnegative, resp.) we mean that it is positive (nonnegative, resp.) componentwise.

We are interested in the traveling wave of (1.2) in the form

$$(S, I)(x, t) = (\phi_1, \phi_2)(\xi),\ \xi := x - st,$$

connecting the endemic state and the disease-free state. Since the wave speed s is the same as the environmental shifting speed, we call it a *forced wave* of (1.2). Therefore, (ϕ_1, ϕ_2) satisfies

$$\begin{cases} d_1\phi_1''(\xi) + s\phi_1'(\xi) + \mu - \mu\phi_1(\xi) - \frac{\beta(\xi)\phi_1(\xi)\phi_2(\xi+s\tau)}{1+\alpha\phi_2(\xi+s\tau)} = 0,\ \xi \in \mathbb{R}, \\ d_2\phi_2''(\xi) + s\phi_2'(\xi) + \frac{\beta(\xi)\phi_1(\xi)\phi_2(\xi+s\tau)}{1+\alpha\phi_2(\xi+s\tau)} - \kappa\phi_2(\xi) = 0,\ \xi \in \mathbb{R}, \end{cases} \tag{1.4}$$

and the boundary conditions

$$(\phi_1, \phi_2)(+\infty) = (1, 0),\ \ (\phi_1, \phi_2)(-\infty) = (S^*, I^*). \tag{1.5}$$

Note that (1.5) signifies a bistable connection. When $s > 0$, it corresponds to the disease spreads. But, the disease is deceased when $s < 0$.

For the basic reproduction number $\mathcal{R}_0 > 1$, our first existence result reads

Theorem 1.1. *Given $\tau \geq 0$. Suppose that $\mathcal{R}_0 > 1$ and $\beta(\cdot)$ satisfies* (A1) *and* (A2). *If $\alpha > \beta_0/[\mu(\mathcal{R}_0 - 1)]$, then* (1.4) *admits a positive solution (ϕ_1, ϕ_2) such that $(\phi_1, \phi_2)(+\infty) = (1, 0)$ for any $s < 0$.*

It follows from Theorem 1.1 that, due to $(\phi_1, \phi_2)(+\infty) = (1, 0)$, any solution (S, I) of (1.2) corresponding to a forced wave with $s < 0$ satisfying $I(x, t) \to 0$ as $t \to \infty$ for all $x \in \mathbb{R}$, regardless of the magnitude of latency period τ, as long as the saturation parameter $\alpha > \beta_0/[\mu(\mathcal{R}_0 - 1)]$. This is quite natural, the disease must decease eventually due to the diminishing of the transmission rate for large times.

We also remark that, for a nonnegative nonconstant bounded solution (ϕ_1, ϕ_2) of (1.4), we have $0 < \phi_1 < 1$ and $\phi_2 > 0$ in $\mathbb{R}$. Indeed, if $\phi_1(\xi_0) = 0$, then $\phi_1'(\xi_0) = 0 \leq \phi_1''(\xi_0)$ which is impossible by the ϕ_1-equation in (1.4). Similarly, any (local) maximal point ξ_0 of ϕ_1 must have value $\phi_1(\xi_0) \leq 1$. Hence $0 < \phi_1 \leq 1$ in $\mathbb{R}$. Moreover, since ϕ_1 satisfies

$$d_1\phi_1''(\xi) + s\phi_1'(\xi) + \mu - \mu\phi_1(\xi) \geq 0,\ \xi \in \mathbb{R},$$

the strong maximum principle gives that $\phi_1 < 1$ in $\mathbb{R}$. The same reasoning also implies that $\phi_2 > 0$ in $\mathbb{R}$.

Let $\phi^* = \mu/(\mu + \beta_0/\alpha)$. When the condition $\alpha > \beta_0/[\mu(\mathcal{R}_0 - 1)]$ is enforced, the quantity $s_* := 2\sqrt{d_2(\beta_0\phi^* - \kappa)}$ is well-defined and positive. Then we have the following existence result for advantageous forced waves.

Theorem 1.2. *Assume $\tau = 0$. Suppose that $\mathcal{R}_0 > 1$ and $\beta(\cdot)$ satisfies* (A1) *and* (A2). *If $\alpha > \beta_0/[\mu(\mathcal{R}_0 - 1)]$, then* (1.4) *admits a positive solution (ϕ_1, ϕ_2) such that $(\phi_1, \phi_2)(+\infty) = (1, 0)$ for any $s \in (0, s_*)$.*

The forced waves of (1.2) obtained in Theorems 1.1 and 1.2 have the right-hand tail limit $(\phi_1, \phi_2)(+\infty) = (1, 0)$. If the saturation parameter is larger than a certain value, then we are able to derive the left-hand tail limit as follows.

Theorem 1.3. *Let (ϕ_1, ϕ_2) be a positive solution of (1.4) obtained in Theorem 1.1 or Theorem 1.2. Then there exists $\alpha^* > \beta_0/[\mu(\mathcal{R}_0 - 1)]$ such that $(\phi_1, \phi_2)(-\infty) = (S^*, I^*)$, if $\alpha > \alpha^*$.*

Lastly, when the basic reproduction number $\mathcal{R}_0 < 1$, we obtain the following non-existence of forced waves.

Theorem 1.4. *Let $s \neq 0$. If $\mathcal{R}_0 < 1$, then there is no nonnegative nonconstant bounded solution of (1.4).*

There are a vast literature on the study of forced waves. For the works on forced waves for the classical diffusion, we refer the reader to, e.g., [1, 2, 3, 4, 5, 6, 9, 10, 11, 14, 15, 18, 21, 26, 27, 28, 29]. For the works on nonlocal dispersal, we only refer the reader to the references listed in a recent work [16]. Note that the shifting effect is imposed on the intrinsic growth term in the above-mentioned works. On the other hand, the model studied in [20] is a cooperative system so that the classical monotone iteration method can be applied. However, our system is non-cooperative and so certain difficulties arise in this study. Although the method of generalized upper-lower solutions with the help of Schauder's fixed point theorem is a very powerful method to derive the existence of forced waves for non-cooperative systems, to find a suitable pair of upper-lower solutions is not always available. This is actually a nontrivial task in many examples.

In this paper, we investigate the existence and non-existence of positive solutions to (1.4), and obtain the asymptotic boundary values for wave tails. Our main contribution of this work is the construction of a pair of upper-lower solutions, in particular, the lower solution for the case $s > 0$ is new and nontrivial. Moreover, the monotonicity condition on the transmission rate β is not imposed here. Unfortunately, we were unable to find a suitable lower solution for the advantageous forced waves when a time delay is taken into account. Also, we are not sure whether forced waves exist for $s \geq s_*$. We leave these two questions as open problems for future studies.

The rest of this paper is organized as follows. In §2, some preliminaries are given. In particular, a non-existence of forced waves for $\mathcal{R}_0 < 1$ and the right-hand tail limit $(\phi_1, \phi_2)(+\infty) = (1, 0)$ for any solution of (1.4) are proven. The existence of forced waves, Theorems 1.1 and 1.2, are proved in §3. Then, under a more stronger restriction on the saturation parameter, by adopting the method of contracting rectangles (cf., e.g., [22, 7, 17]) a proof for the left-hand tail limit of forced waves is given in §4. Finally, we give a brief discussion in §5. This includes some numerical simulations in order to provide some hints of two above-mentioned open questions.

2. Preliminaries

First, we introduce the following notion of (generalized) upper-lower solutions of (1.4).

Definition 2.1. Continuous functions $(\overline{\phi}_1, \overline{\phi}_2)$ and $(\underline{\phi}_1, \underline{\phi}_2)$ are called a pair of upper and lower solutions of (1.4) if $\overline{\phi}_1 \geq \underline{\phi}_1$, $\overline{\phi}_2 \geq \underline{\phi}_2$, and the following inequalities

$$\mathcal{U}_1(\xi) := d_1\overline{\phi}_1''(\xi) + s\overline{\phi}_1'(\xi) + \mu - \mu\overline{\phi}_1(\xi) - \frac{\beta(\xi)\overline{\phi}_1(\xi)\underline{\phi}_2(\xi+s\tau)}{1+\alpha\underline{\phi}_2(\xi+s\tau)} \leq 0, \tag{2.1}$$

$$\mathcal{U}_2(\xi) := d_2\overline{\phi}_2''(\xi) + s\overline{\phi}_2'(\xi) + \frac{\beta(\xi)\overline{\phi}_1(\xi)\overline{\phi}_2(\xi+s\tau)}{1+\alpha\overline{\phi}_2(\xi+s\tau)} - \kappa\overline{\phi}_2(\xi) \leq 0, \tag{2.2}$$

$$\mathcal{L}_1(\xi) := d_1\underline{\phi}_1''(\xi) + s\underline{\phi}_1'(\xi) + \mu - \mu\underline{\phi}_1(\xi) - \frac{\beta(\xi)\underline{\phi}_1(\xi)\overline{\phi}_2(\xi+s\tau)}{1+\alpha\overline{\phi}_2(\xi+s\tau)} \geq 0, \tag{2.3}$$

$$\mathcal{L}_2(\xi) := d_2\underline{\phi}_2''(\xi) + s\underline{\phi}_2'(\xi) + \frac{\beta(\xi)\underline{\phi}_1(\xi)\underline{\phi}_2(\xi+s\tau)}{1+\alpha\underline{\phi}_2(\xi+s\tau)} - \kappa\underline{\phi}_2(\xi) \geq 0, \tag{2.4}$$

hold for all $\xi \in \mathbb{R} \setminus E$ for some finite subset E of $\mathbb{R}$.

Then we have the following lemma for the existence of solution to (1.4).

Lemma 2.1. *Let $s \neq 0$ be given. Let $(\overline{\phi}_1, \overline{\phi}_2)$ and $(\underline{\phi}_1, \underline{\phi}_2)$ be a pair of upper and lower solutions of* (1.4) *satisfying*

$$\begin{cases} \overline{\phi}_1'(\xi-) \geq \overline{\phi}_1'(\xi+), \quad \underline{\phi}_1'(\xi-) \leq \underline{\phi}_1'(\xi+), \ \forall\, \xi \in E, \\ \overline{\phi}_2'(\xi-) \geq \overline{\phi}_2'(\xi+), \quad \underline{\phi}_2'(\xi-) \leq \underline{\phi}_2'(\xi+), \ \forall\, \xi \in E. \end{cases} \tag{2.5}$$

Then (1.4) *admits a solution (ϕ_1, ϕ_2) such that $\underline{\phi}_1(\xi) \leq \phi_1(\xi) \leq \overline{\phi}_1(\xi)$ and $\underline{\phi}_2(\xi) \leq \phi_2(\xi) \leq \overline{\phi}_2(\xi)$ for all $\xi \in \mathbb{R}$.*

Proof. The proof of this lemma can be carried out in almost the same manner as that in [23]. For the reader's convenience, we present an outline of the proof as follows.

First, we define the set

$$\Gamma = \{(\phi_1, \phi_1) \in C^0(\mathbb{R}) \times C^0(\mathbb{R}) : \underline{\phi}_1 \leq \phi_1 \leq \overline{\phi}_1, \underline{\phi}_2 \leq \phi_2 \leq \overline{\phi}_2\}.$$

For (ϕ_1, ϕ_2), consider

$$F_1(\phi_1,\phi_2)(\xi) = \eta\phi_1(\xi) + \mu - \mu\phi_1(\xi) - \frac{\beta(\xi)\phi_1(\xi)\phi_2(\xi+s\tau)}{1+\alpha\phi_2(\xi+s\tau)}, \ \xi \in \mathbb{R}$$

$$F_2(\phi_1,\phi_2)(\xi) = \eta\phi_2(\xi) + \frac{\beta(\xi)\phi_1(\xi)\phi_2(\xi+s\tau)}{1+\alpha\phi_2(\xi+s\tau)} - \kappa\phi_2(\xi), \ \xi \in \mathbb{R}$$

where η is a large constant such that F_i is monotone increasing in ϕ_i for $i = 1, 2$. Define an integral operator $P = (P_1, P_2)$ by

$$P_1(\phi_1,\phi_2)(\xi) := \frac{1}{d_1(\lambda_1^+ - \lambda_1^-)}\Big[\int_{-\infty}^{\xi} e^{\lambda_1^-(\xi-y)} + \int_{\xi}^{+\infty} e^{\lambda_1^+(\xi-y)}\Big] F_1(\phi_1,\phi_2)(y)dy,$$

$$P_2(\phi_1,\phi_2)(\xi) := \frac{1}{d_2(\lambda_2^+ - \lambda_2^-)}\Big[\int_{-\infty}^{\xi} e^{\lambda_2^-(\xi-y)} + \int_{\xi}^{+\infty} e^{\lambda_2^+(\xi-y)}\Big] F_2(\phi_1,\phi_2)(y)dy,$$

where $\lambda_1^\pm$ and $\lambda_2^\pm$ are defined by

$$\lambda_1^\pm = \frac{-s \pm \sqrt{s^2+4\eta d_1}}{2d_1}, \quad \lambda_2^\pm = \frac{-s \pm \sqrt{s^2+4\eta d_2}}{2d_2}.$$

Note that $P(\phi_1,\phi_2)$ satisfies

$$\begin{cases} d_1[P_1(\phi_1,\phi_2)]''(\xi) + s[P_1(\phi_1,\phi_2)]'(\xi) - \eta[P_1(\phi_1,\phi_2)](\xi) + F_1(\phi_1,\phi_2)(\xi) = 0, \ z \in \mathbb{R}, \\ d_2[P_2(\phi_1,\phi_2)]''(\xi) + s[P_2(\phi_1,\phi_2)]'(\xi) - \eta[P_2(\phi_1,\phi_2)](\xi) + F_2(\phi_1,\phi_2)(\xi) = 0, \ z \in \mathbb{R}. \end{cases}$$

By a standard process using (2.1)-(2.4) and (2.5), we can show that P maps Γ into Γ, and $P:\Gamma\to\Gamma$ is completely continuous with respect to the norm $|\cdot|_\nu$, where

$$|(\phi_1,\phi_2)|_\nu = \sup_{\xi\in\mathbb{R}}\{\max(|\phi_1(\xi)|,|\phi_2(\xi)|)e^{-\nu|\xi|}\}, \quad (\phi_1,\phi_2)\in\Gamma,$$

with $\nu < \min\{-\lambda_1^-, -\lambda_2^-\}$. Then, by the Schauder's fixed point theorem, we obtain that P has a fixed-point $(\phi_1,\phi_2)\in\Gamma$, and thus the system (1.4) has a solution $(\phi_1,\phi_2)\in\Gamma$. ☐

Next, we provide a universal property for any nonnegative solution (ϕ_1,ϕ_2) of (1.4).

Proposition 2.2. *Suppose that $\beta(\infty)=0$. Let (ϕ_1,ϕ_2) be a nonnegative bounded solution of (1.4) for a given $s\neq 0$. Then $(\phi_1,\phi_2)(\infty)=(1,0)$.*

Proof. First, we prove that $\phi_2(\infty)=0$.

For contradiction, we suppose that $\phi_2^+ := \limsup_{\xi\to\infty}\phi_2(\xi) > 0$. If ϕ_2 is oscillatory near $\xi=+\infty$, we can choose a sequence of maximal points $\{\xi_n\}$ such that $\xi_n\to+\infty$ and $\phi_2(\xi_n)\to\phi_2^+$ as $n\to\infty$. Since ξ_n is a maximal point, $\phi_2'(\xi_n)=0$ and $d_2\phi_2''(\xi_n)\le 0$. Then, from $\beta(\infty)=0$, we have

$$0 = \limsup_{n\to\infty}\Big\{d_2\phi_2''(\xi_n) + s\phi_2'(\xi_n) + \frac{\beta(\xi_n)\phi_1(\xi_n)\phi_2(\xi_n+s\tau)}{1+\alpha\phi_2(\xi_n+s\tau)} - \kappa\phi_2(\xi_n)\Big\} \le -\kappa\phi_2^+ < 0,$$

a contradiction.

On the other hand, we assume that ϕ_2 is monotone ultimately at $\xi=+\infty$. Then $\lim_{\xi\to+\infty}\phi_2(\xi)=\phi_2^+>0$ and we can find a sequence $\{\xi_n\}$ such that $\xi_n\to\infty$ and $\phi_2'(\xi_n)\to 0$ as $n\to\infty$. Integrating the ϕ_2-equation in (1.4) from 0 to ξ_n, we obtain

$$d_2[\phi_2'(0)-\phi_2'(\xi_n)] + s[\phi_2(0)-\phi_2(\xi_n)] = \int_0^{\xi_n}\Big(\frac{\beta(y)\phi_1(y)\phi_2(y+s\tau)}{1+\alpha\phi_2(y+s\tau)} - \kappa\phi_2(y)\Big)dy. \tag{2.6}$$

Note that the left hand side of (2.6) is uniformly bounded with respect to n. Since $\beta(\infty)=0$, and ϕ_1 and ϕ_2 are bounded, we can choose $K\gg 1$ such that

$$\phi_2(y) \ge \frac{\phi_2^+}{2}, \quad \frac{\beta(y)\phi_1(y)\phi_2(y+s\tau)}{\kappa[1+\alpha\phi_2(y+s\tau)]} \le \frac{\phi_2^+}{4} \ \text{ for } y\ge K.$$

Then, we have

$$\frac{\beta(y)\phi_1(y)\phi_2(y+s\tau)}{1+\alpha\phi_2(y+s\tau)} - \kappa\phi_2(y) \le -\frac{\phi_2^+}{4}\kappa \ \text{ for } y\ge K,$$

and thus, the integral

$$\int_0^\infty\Big(\frac{\beta(y)\phi_1(y)\phi_2(y+s\tau)}{1+\alpha\phi_2(y+s\tau)} - \kappa\phi_2(y)\Big)dy$$

diverges. Hence, we have a contradiction. This proves that $\phi_2(\infty) = 0$.

Next, we show that $\phi_1(\infty) = 1$. Otherwise, suppose that $\liminf_{\xi\to\infty} \phi_1(\xi) < 1$. Then, using

$$\lim_{y\to\infty} \frac{\beta(y)\phi_1(y)\phi_2(y+s\tau)}{1+\alpha\phi_2(y+s\tau)} = 0,$$

due to $\beta(\infty) = \phi_2(\infty) = 0$, we reach a contradiction by a similar argument as that for $\phi_2(\infty) = 0$. This proves $\phi_1(\infty) = 1$ and the proposition follows. □

In the following, we show the non-existence of forced waves when $\mathcal{R}_0 < 1$.

Proof of Theorem 1.4. The proof is motivated by that of [23, Theorem 4.3]. Suppose that (1.4) has a nonnegative nonconstant bounded solution (ϕ_1, ϕ_2). Set $v(x,t) := \phi_2(x - st)$. Then v satisfies

$$v_t(x,t) \le d_2 v_{xx}(x,t) + \beta_0 v(x, t-\tau) - \kappa v(x,t),\ x \in \mathbb{R},\ t > 0,$$

using $\beta(\xi) \le \beta_0$ for all $\xi \in \mathbb{R}$, $\phi_1 < 1$ in $\mathbb{R}$ and $\phi_2 \ge 0$. Suppose that $\phi_2 \le L < \infty$ in $\mathbb{R}$. Then, by comparing with the ODE

$$V'(t) = \beta_0 V(t-\tau) - \kappa V(t),\ t > 0,\ V(t) = Le^{\lambda_0 t},\ t \in [-\tau, 0],$$

we obtain that

$$v(x,t) \le Le^{\lambda_0 t},\ x \in \mathbb{R},\ t > 0, \tag{2.7}$$

where $\lambda_0 < 0$ satisfies $\lambda_0 = \beta_0 e^{-\lambda_0\tau} - \kappa$, due to $\beta_0 < \kappa$. Now, given $\xi \in \mathbb{R}$. It follows from (2.7) that

$$\phi_2(\xi) = v(\xi + st, t) \le Le^{\lambda_0 t},\ \forall\, t > 0. \tag{2.8}$$

Letting $t \to \infty$ in (2.8), we obtain that $\phi_2(\xi) \le 0$ and so $\phi_2(\xi) = 0$. This proves that $\phi_2 \equiv 0$, which is a contradiction to ϕ_2 is nonconstant. Thus Theorem 1.4 is proved. □

3. Existence of forced waves

This section is devoted to the proofs of the existence of forced waves.

3.1. Forced waves for $s < 0$ for any $\tau \ge 0$.

Proof of Theorem 1.1. Fixed $\tau \ge 0$ and $\alpha > \beta_0/[\mu(\mathcal{R}_0 - 1)]$. Consider the following positive constants

$$\phi^* = \frac{\mu}{\mu + \beta_0/\alpha},\quad \psi^* = \frac{\beta_0 - \kappa}{\alpha\kappa}.$$

Note that $0 < \phi^* < 1$, $\psi^* > I^*$ and

$$\frac{\beta_0\psi^*}{1+\alpha\psi^*} - \kappa\psi^* = 0. \tag{3.1}$$

We also recall from (A2) that there exists θ and K_1 such that $\beta(\xi) \le e^{-\theta\xi}$ when $\xi > K_1$. Since $\alpha > \frac{\beta_0}{\mu(\mathcal{R}_0-1)}$ is equivalent to $\beta_0\phi^* > \kappa$, we can choose small positive constants ε and δ such that

$$\frac{(\beta_0 - \varepsilon)\phi^*}{1+\alpha\delta} - \kappa > 0. \tag{3.2}$$

Then we define

$$\overline{\phi}_1(\xi)=1,\ \overline{\phi}_2(\xi)=\psi^*,\ \underline{\phi}_1(\xi)=\max\Big\{1-\rho_1 e^{-\sigma\xi},\phi^*\Big\}, \tag{3.3}$$

$$\underline{\phi}_2(\xi)=\max\{\delta(1-\rho_2 e^{-\frac{s}{d_2}\xi}),0\}, \tag{3.4}$$

where $\sigma\in(0,\theta)$ is a fixed small positive constant satisfying $-d_1\sigma^2+s\sigma+\mu>0$, and ρ_1,ρ_2 are positive constants to be chosen later.

Now we show $(\overline{\phi}_1,\overline{\phi}_2)$ and $(\underline{\phi}_1,\underline{\phi}_2)$ satisfy (2.1)-(2.4). First, from $\beta\le\beta_0$ and (3.1), (2.1) and (2.2) immediately hold for all $\xi\in\mathbb{R}$.

For (2.3), there exists $\xi_1=\xi_1(\rho_1)\in\mathbb{R}$ such that $\underline{\phi}_1(\xi)=\phi^*$ for $\xi\le\xi_1$, and $\underline{\phi}_1(\xi)=1-\rho_1e^{-\sigma\xi}$ for $\xi>\xi_1$. Next we choose ρ_1 large enough so that $\xi_1>K_1>0$ and

$$\rho_1\ge\frac{1}{\alpha(-d_1\sigma^2+s\sigma+\mu)} \tag{3.5}$$

Note that

$$\mu-\mu\underline{\phi}_1(\xi)-\frac{\beta(\xi)\underline{\phi}_1(\xi)\overline{\phi}_2(\xi+s\tau)}{1+\alpha\overline{\phi}_2(\xi+s\tau)}\ge\mu-\mu\underline{\phi}_1(\xi)-\frac{\beta_0}{\alpha}\underline{\phi}_1(\xi)=0,\ \xi<\xi_1,$$

using $\underline{\phi}_1(\xi)=\phi^*=\frac{\mu}{\mu+\beta_0/\alpha}$ and $y/(1+\alpha y)\le 1/\alpha$ for $y\ge0$. Thus, (2.3) holds for $\xi<\xi_1$.

For $\xi>\xi_1$, $\underline{\phi}_1(\xi)=1-\rho_1e^{-\sigma\xi}$. Since $\underline{\phi}_1(\xi)\le1$ and $\beta(\xi)\le e^{-\theta\xi}$ for $\xi>\xi_1$, we compute

$$\begin{aligned}\mathcal{L}_1(\xi)&\ge\rho_1(-d_1\sigma^2+s\sigma+\mu)e^{-\sigma\xi}-\frac{1}{\alpha}e^{-\theta\xi}\\&\ge e^{-\sigma\xi}\Big[\rho_1(-d_1\sigma^2+s\sigma+\mu)-\frac{1}{\alpha}\Big]\ge0,\ \xi<\xi_1,\end{aligned}$$

by the choice of ρ_1 in (3.5) and $\sigma\in(0,\theta)$. Thus, (2.3) holds for $\xi\ne\xi_1$.

Finally, for (2.4), we let $\xi_2:=\frac{d_2}{s}\ln\rho_2<0$ for $\rho_2>1$. Note that $s<0$. Moreover, we can choose ρ_2 large so that $\beta(\xi)\ge\beta_0-\varepsilon$ for $\xi<\xi_2<0$. We only need to show that (2.4) holds for $\xi<\xi_2$. Note that $\underline{\phi}_1(\xi)=\phi^*$ for $\xi<\xi_2$ and $\underline{\phi}_2(\xi)\le\underline{\phi}_2(\xi+s\tau)\le\delta$ for all $\xi\in\mathbb{R}$. Then, by using the monotone increasing property of the function

$$\frac{y}{1+\alpha y}=\frac{1}{\alpha}\Big\{1-\frac{1}{1+\alpha y}\Big\},\ y\ge0,$$

we deduce that

$$\frac{\beta(\xi)\underline{\phi}_1(\xi)\underline{\phi}_2(\xi+s\tau)}{1+\alpha\underline{\phi}_2(\xi+s\tau)}\ge\frac{\phi^*(\beta_0-\varepsilon)\underline{\phi}_2(\xi)}{1+\alpha\underline{\phi}_2(\xi)}\ge\frac{\phi^*(\beta_0-\varepsilon)}{1+\alpha\delta}\underline{\phi}_2(\xi),\ \forall\xi<\xi_2.$$

Then we obtain from (3.2) that

$$\mathcal{L}_2(\xi)=\frac{\beta(\xi)\underline{\phi}_1(\xi)\underline{\phi}_2(\xi+s\tau)}{1+\alpha\underline{\phi}_2(\xi+s\tau)}-\kappa\underline{\phi}_2(\xi)\ge\underline{\phi}_2(\xi)\Big[\frac{\phi^*(\beta_0-\varepsilon)}{1+\alpha\delta}-\kappa\Big]\ge0,\ \forall\xi<\xi_2.$$

Clearly, $\mathcal{L}_2(\xi)\ge0$ for $\xi>\xi_2$. This implies that (2.4) holds for all $\xi\ne\xi_2$. We conclude that $(\overline{\phi}_1,\overline{\phi}_2)$ and $(\underline{\phi}_1,\underline{\phi}_2)$ are a pair of upper and lower solutions of (1.4) for any $s<0$, when $\mathcal{R}_0>1$ and $\alpha>\frac{\beta_0}{\mu(\mathcal{R}_0-1)}$. Therefore, the existence of a solution (ϕ_1,ϕ_2) of (1.4) follows by applying Lemma 2.1. Since both $\underline{\phi_1}$ and $\underline{\phi_2}$ are nonnegative and nonconstant, we see that both ϕ_1 and ϕ_2 are positive.

The right-hand tail limit of a wave profile follows immediately from Proposition 2.2 and Theorem 1.1 is proved. □

3.2. **Forced waves for $s > 0$ with $\tau = 0$.**

Proof of Theorem 1.2. Recall (3.3). Given a fixed $s \in (0, s_*)$. Then we can check that (2.1),(2.2) and (2.3) hold for any $s > 0$ as that in the proof for case $s < 0$.

For (2.4), we consider the function

$$\psi(z) := e^{-\frac{s}{2d_2}z}\cos(\omega z + \frac{\pi}{2}),$$

where ω is a positive constant to be determined later. Then it is easy to check that

$$d_2\psi''(z) + s\psi'(z) = -\left(\frac{s^2}{4d_2} + d_2\omega^2\right)\psi(z),\ z \in \mathbb{R}. \tag{3.6}$$

Moreover, $\psi(-\pi/\omega) = \psi(0) = 0$ and the maximum of ψ for $z \in (-\pi/\omega, 0)$ is given by

$$M := e^{-\frac{s}{2d_2}z_*}\frac{2d_2\omega}{\sqrt{4d_2^2\omega^2 + s^2}},$$

where z_* is the unique maximal point in $(-\pi/\omega, 0)$ defined by

$$\tan(\omega z_* + \frac{\pi}{2}) = \frac{-s}{2d_2\omega}.$$

Next, since $s < s_*$, we have

$$\frac{s^2}{4d_2} < \beta_0\phi^* - \kappa.$$

we can choose positive constants $\varepsilon, \delta, \omega$ small enough such that

$$\frac{(\beta_0 - \varepsilon)\phi^*}{1 + \alpha\delta} > \kappa + \frac{s^2}{4d_2} + d_2\omega^2. \tag{3.7}$$

With these constants, we now replace $\underline{\phi}_2$ in (3.4) by

$$\underline{\phi}_2(\xi) = \begin{cases} \delta,\ \xi \le \xi_2 + z_*, \\ \frac{\delta}{M}\psi(\xi - \xi_2),\ \xi \in (\xi_2 + z_*, \xi_2), \\ 0,\ \xi \ge \xi_2, \end{cases} \tag{3.8}$$

where ξ_2 is chosen so that $\beta(\xi) \ge \beta_0 - \varepsilon$ for all $\xi \le \xi_2$. It is important to remark that (2.5) holds for this $\underline{\phi}_2$ and $\underline{\phi}_2 \le \delta$ in $\mathbb{R}$. Then one can check that

$$\mathcal{L}_2(\xi) \ge \left\{-\left(\frac{s^2}{4d_2} + d_2\omega^2\right) + \frac{(\beta_0 - \varepsilon)\phi^*}{1 + \alpha\delta} - \kappa\right\}\underline{\phi}_2(\xi) \ge 0$$

for $\xi \in (\xi_2 + z_*, \xi_2)$, due to (3.7). It is clear that $\mathcal{L}_2(\xi) \ge 0$ for all $\xi < \xi_2 + z_*$ and $\xi > \xi_2$. Hence $\mathcal{L}_2(\xi) \ge 0$ for all $\xi \ne \xi_2 + z_*, \xi_2$. Therefore, we have verified that the functions defined in (3.3) and (3.8) are a pair of upper-lower solutions of (1.4) for $s \in (0, s_*)$ such that $\underline{\phi}_1 \le \overline{\phi}_1$, $\underline{\phi}_2 \le \overline{\phi}_2$ and condition (2.5) holds. Therefore, for $\alpha > \beta_0/[\mu(\mathcal{R}_0 - 1)]$ and $s \in (0, s_*)$, the existence of a positive solution (ϕ_1, ϕ_2) of (1.4) such that $(\phi_1, \phi_2)(+\infty) = (1, 0)$ follows by applying Lemma 2.1 and Proposition 2.2. Theorem 1.2 is thereby proved. □

4. Left-hand tail limit

In this section, we derive the left-hand tail limit of forced waves when the saturation parameter is large enough.

Proof of Theorem 1.3. First, from the construction of upper solution, $\phi_1(\xi) \le 1$ and $\phi_2(\xi) \le \psi^*$ for $\xi \in \mathbb{R}$. Also, by the construction of lower solution $(\underline{\phi}_1, \underline{\phi}_2)$, we have

$$\phi_1^- := \liminf_{\xi\to-\infty} \phi_1(\xi) \ge \phi^* > 0, \quad \phi_2^- := \liminf_{\xi\to-\infty} \phi_2(\xi) \ge \delta > 0.$$

Set

$$\phi_1^+ := \limsup_{\xi\to-\infty} \phi_1(\xi), \quad \phi_2^+ := \limsup_{\xi\to-\infty} \phi_2(\xi).$$

We consider the following functions for $\nu \in [0,1]$

$$\begin{aligned} &m_1(\nu) := \nu S^*, \quad M_1(\nu) := \nu S^* + (1-\nu)(1+A), \\ &m_2(\nu) := \nu I^* + (1-\nu)\left(-\frac{1}{\alpha}\right), \quad M_2(\nu) := \nu I^* + (1-\nu)(\psi^* + B), \end{aligned}$$

where $A = \frac{k\beta_0 S^*}{\alpha\mu(1+\alpha I^*)}$ and $B = \frac{kA(1+\alpha I^*)}{\alpha S^*} = \frac{k^2\beta_0}{\alpha^2\mu}$ for some constant k with $0 < k-1 \ll 1$. Note that

$$m_i(\nu) < \phi_i^- \le \phi_i^+ < M_i(\nu), \ i = 1,2, \tag{4.1}$$

holds for $\nu = 0$. Hence the quantity

$$\nu_0 := \sup\{\nu \in [0,1) : (4.1) \text{ holds}\}$$

is well-defined.

Note that $m_i(\nu)$ is an increasing function and $M_i(\nu)$ is a decreasing function of $\nu \in [0,1]$ for $i = 1,2$. Since $m_1(1) = M_1(1) = S^*$ and $m_2(1) = M_2(1) = I^*$, the proof is done if we can prove $\nu_0 = 1$. Thus, for contradiction, we assume that $\nu_0 < 1$. Then, by passing to the limit, we have

$$m_i(\nu_0) \le \phi_i^- \le \phi_i^+ \le M_i(\nu_0).$$

But, by the definition of ν_0 and the continuity of $m_i(\nu)$ and $M_i(\nu)$, at least one of the following equalities holds:

$$\phi_i^- = m_i(\nu_0), \ \phi_i^+ = M_i(\nu_0), \quad i = 1,2.$$

(i) Suppose that $\phi_1^- = m_1(\nu_0)$. If ϕ_1 is oscillatory at $-\infty$, then we can choose a sequence $\{\xi_n\}$ of minimal points of ϕ_1 such that $\xi_n \to -\infty$ and $\lim_{n\to\infty} \phi_1(\xi_n) = m_1(\nu_0)$. Note that

$$\limsup_{n\to\infty} \phi_2(\xi_n + s\tau) \le M_2(\nu_0).$$

Since $\phi_1'(\xi_n) = 0$, $\phi_1''(\xi_n) \ge 0$ for $n \in \mathbb{N}$ and using

$$1 + \alpha[\nu_0 I^* + (1-\nu_0)(\psi^* + B)] \ge \nu_0(1 + \alpha I^*),$$

we obtain from (1.3) that

$$\begin{aligned} 0 &= \liminf_{n\to\infty}\Big\{d_1\phi_1''(\xi_n)+s\phi_1'(\xi_n)+\mu-\mu\phi_1(\xi_n)-\frac{\beta(\xi_n)\phi_1(\xi_n)\phi_2(\xi_n+s\tau)}{1+\alpha\phi_2(\xi_n+s\tau)}\Big\} \\ &\geq \mu-\mu\nu_0S^*-\frac{\beta_0\nu_0S^*[\nu_0I^*+(1-\nu_0)(\psi^*+B)]}{1+\alpha[\nu_0I^*+(1-\nu_0)(\psi^*+B)]} \\ &\geq (1-\nu_0)\mu+\nu_0(\mu-\mu S^*)-\frac{\beta_0\nu_0^2S^*I^*}{\nu_0(1+\alpha I^*)}-\frac{\beta_0\nu_0(1-\nu_0)S^*(\psi^*+B)}{\nu_0(1+\alpha I^*)} \\ &= (1-\nu_0)\Big[\mu-\kappa(\psi^*+B)\Big]:=\omega_1. \end{aligned}$$

Since ψ^*+B is a decreasing function of α such that it tends to zero as $\alpha\to\infty$, we can choose α large enough such that $\omega_1>0$. Hence, we have a contradiction.

Next, we assume that ϕ_1 is eventually monotone. Then there exists a sequence $\{\xi_n\}$ such that $\xi_n\to-\infty$ as $n\to\infty$, $\lim_{n\to\infty}\phi_1'(\xi_n)=0$ and $\lim_{n\to\infty}\phi_1(\xi_n)=m_1(\nu_0)$. Similarly to the above, we have

$$\liminf_{n\to\infty}\Big\{\mu-\mu\phi_1(\xi_n)-\frac{\beta(\xi_n)\phi_1(\xi_n)\phi_2(\xi_n+s\tau)}{1+\alpha\phi_2(\xi_n+s\tau)}\Big\}>0.$$

By integrating the ϕ_1-equation of (1.4) from 0 to ξ_n, we have

$$d_1\phi_1'(\xi_n)-\phi_1'(0)+s(\phi_1(\xi_n)-\phi_1(0))=-\int_0^{\xi_n}\Big[\mu-\mu\phi_1(\xi)-\frac{\beta(\xi)\phi_1(\xi)\phi_2(\xi+s\tau)}{1+\alpha\phi_2(\xi+s\tau)}\Big]d\xi. \quad (4.2)$$

Since the left-hand side of (4.2) is bounded uniformly for all n, but the right-hand side of (4.2) goes to $-\infty$ as $n\to\infty$, it is a contradiction. Hence, $\phi_1^-=m_1(\nu_0)$ cannot happen.

We can treat the other cases similarly using the following inequalities:

(ii) $\phi_1^+=M_1(\nu_0)$: it follows from $\liminf_{n\to\infty}\phi_2(\xi_n+s\tau)\geq m_2(\nu_0)$ that

$$\begin{aligned} &\limsup_{n\to\infty}\Big\{\mu-\mu\phi_1(\xi_n)-\frac{\beta(\xi_n)\phi_1(\xi_n)\phi_2(\xi_n+s\tau)}{1+\alpha\phi_2(\xi_n+s\tau)}\Big\} \\ \leq\ & \mu-\mu[\nu_0S^*+(1-\nu_0)(1+A)]-\frac{\beta_0\nu_0S^*[\nu_0I^*-(1-\nu_0)/\alpha]}{\nu_0(1+\alpha I^*)} \\ =\ & (1-\nu_0)\Big[-\mu A+\frac{\beta_0S^*}{\alpha(1+\alpha I^*)}\Big]<0, \end{aligned}$$

by (1.3) and the choice of A.

(iii) $\phi_2^-=m_2(\nu_0)$: in this case, we may assume that $\phi_1^->m_1(\nu_0)$. Otherwise, it can be reduced to case (i). Then

$$\begin{aligned} \liminf_{n\to\infty}\Big\{\frac{\beta(\xi_n)\phi_1(\xi_n)\phi_2(\xi_n+s\tau)}{1+\alpha\phi_2(\xi_n+s\tau)}-\kappa\phi_2(\xi_n)\Big\} &> m_2(\nu_0)\Big\{\frac{\beta_0m_1(\nu_0)}{1+\alpha m_2(\nu_0)}-\kappa\Big\} \\ &= m_2(\nu_0)\Big\{\frac{\beta_0S^*}{1+\alpha I^*}-\kappa\Big\}=0, \end{aligned}$$

using (1.3).

(iv) $\phi_2^+ = M_2(\nu_0)$: Note that a direct calculation gives $1-S^* = \alpha(S^*\psi^* - I^*)$. Then, using (1.3) and by the choice of B, we compute

$$\begin{aligned}
&\limsup_{n\to\infty}\Big\{\frac{\beta(\xi_n)\phi_1(\xi_n)\phi_2(\xi_n+s\tau)}{1+\alpha\phi_2(\xi_n+s\tau)} - \kappa\phi_2(\xi_n)\Big\}\\
\le\ & \beta_0 M_2(\nu_0)\Big\{\frac{\nu_0 S^* + (1-\nu_0)(1+A)}{1+\alpha[\nu_0 I^* + (1-\nu_0)(\psi^*+B)]} - \frac{S^*}{1+\alpha I^*}\Big\}\\
=\ & \frac{\beta_0 M_2(\nu_0)(1-\nu_0)}{(1+\alpha I^*)\{1+\alpha[\nu_0 I^* + (1-\nu_0)(\psi^*+B)]\}}\Big\{A(1+\alpha I^*) - \alpha S^* B\Big\} < 0.
\end{aligned}$$

Therefore, Theorem 1.3 is proved. $\square$

5. Discussion

In this section, we give some concluding remarks for this work. First, for the Cauchy problem of system (1.2) with the initial condition

$$S(x,0) = S_0(x),\ x\in\mathbb{R}, \quad I(x,t) = I_0(x,t),\ x\in\mathbb{R},\ t\in[-\tau,0], \tag{5.1}$$

where τ is a nonnegative constant, the existence of solutions in a certain function space can be derived by using, e.g., semigroup theory approach (cf. [25, 13]). In particular, our forced wave solutions derived in Theorems 1.1 and 1.2 along with Theorem 1.3 provide a class of solutions of (1.2) with the boundary condition

$$S(-\infty,t) = (S^*, I^*),\ S(\infty,t) = (1,0),\ t>0.$$

Since the main purpose of this paper is to address the existence vs nonexistence of forced waves, we shall not go into more details in the issue on the existence of solutions for the Cauchy problem.

Instead, we present some results on the numerical simulations to give some information on the two open questions mentioned in the Introduction section. For our simulations, we choose the initial functions in (5.1): $S_0 = 1$ and $I_0(x,t) \equiv I_0(x)$ for $x\in\mathbb{R}$, $t\in[-\tau,0]$, where I_0 is chosen appropriately as follows to produce forced waves (if any). For the disadvantage case $s<0$, we choose the initial function I_0 by

$$I_0(x) = \begin{cases} I^* & \text{for } x<0,\\ I^*\exp\left(1-\frac{1}{1-(x/50)^2}\right) & \text{for } 0\le x\le 50,\\ 0 & \text{for } x>50,\end{cases}$$

while, a continuous function with compact support given by

$$I_0(x) = \begin{cases} I^*\exp\left(1-\frac{1}{1-|x/50|^2}\right) & \text{for } |x|\le 50,\\ 0 & \text{otherwise,}\end{cases}$$

is chosen for the advantageous case $s > 0$. Moreover, the following parameters and the function β are set in the simulations:

$$d_1 = d_2 = 1; \quad \mu = 1; \quad \alpha = 1.5; \quad \kappa = 0.5;$$
$$\beta(x - st) = \frac{1}{\pi} \arctan\left(-(x - st)\right) + 0.5.$$

In the setting of parameters, α satisfies $\alpha > \beta_0/[\mu(\mathcal{R}_0 - 1)] = 1$. Also, we have the (limiting) basic reproduction number $\mathcal{R}_0 = 2$ and the quantity $s_* \approx 0.6325$ in Theorem 1.2. In the following figures, the solid curve is the density of susceptible population and the dash curve corresponds to the density of infective population.

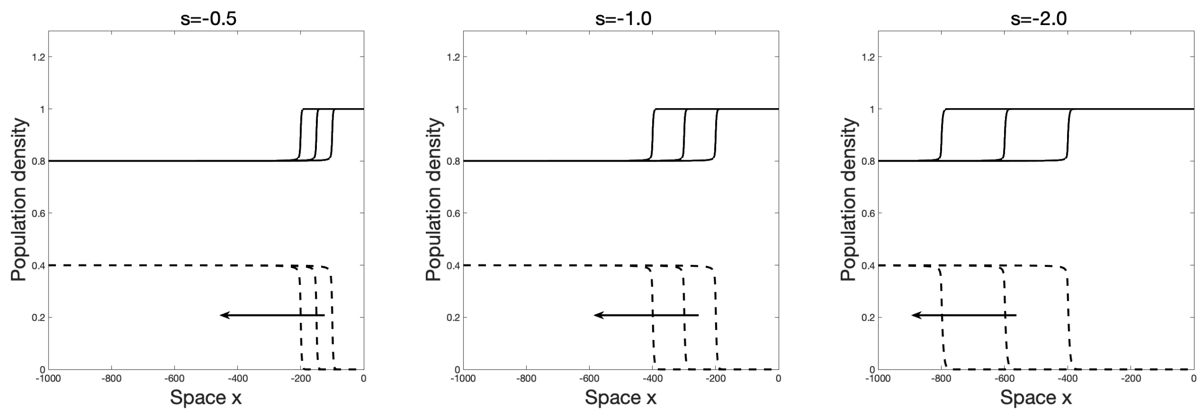


FIGURE 1. Numerical solutions for the disadvantageous case $s < 0$ with a time delay $\tau = 0.5$ (Theorem 1.1). The arrow indicates the wave moving direction.

In Figure 1, the graphs are the solutions at three different times, $T = 300, 400, 500$, for the shifting speeds $s = -0.5$, $s = -1.0$ and $s = -2.0$, respectively. The spatial distances with time length 100 are approximately $50.01, 100.01$, and 200.02 for $s = -0.5$, $s = -1.0$ and $s = -2.0$, respectively. Hence each solution moves with the shifting speed $|s|$ for the disadvantageous case $s < 0$. This can be seen as a numerical verification for the existence of forced waves stated in Theorem 1.1.

In Figure 2, the graphs are the solutions at three different times, $T = 400, 500, 600$, for the shifting speeds $s = 0.5$, $s = 1.0$ and $s = 2.0$, respectively. The spatial distances with time length 100 are approximately $50.0, 100.01$, and 140.4 for $s = 0.5$, $s = 1.0$ and $s = 2.0$, respectively. Therefore, we may interpret numerically the existence of forced waves for the shifting speeds $s = 0.5$ and $s = 1.0$, but not for $s = 2.0$. Notice that numerically forced waves exist for $s = 1.0$ in which $s > s_*(\approx 0.6325)$. We conjecture that for the existence of forced waves the critical upper bound for the shifting speeds is $s^* = 2\sqrt{d_2(\beta_0 - \kappa)} \approx 1.414$. However, due to some technical difficulties, we are unable to verify this conjecture rigorously in this paper.

Finally, it is interesting to see whether advantageous forced waves exist when a time delay is taken into account. Figure 3 represents the numerical solutions for the advantageous case

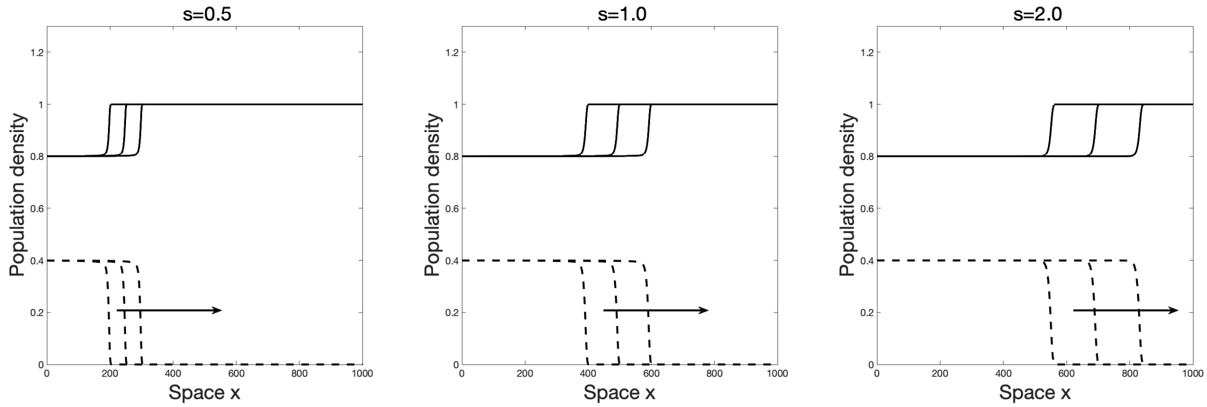
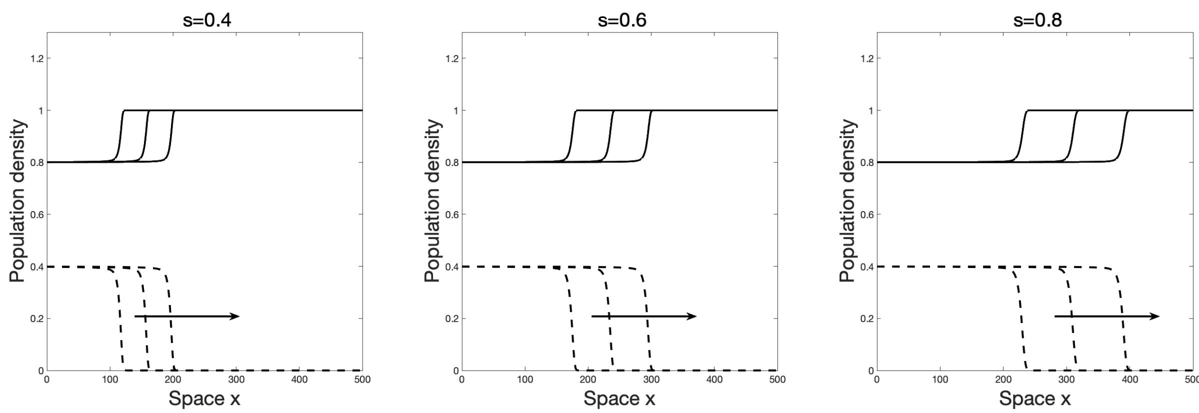
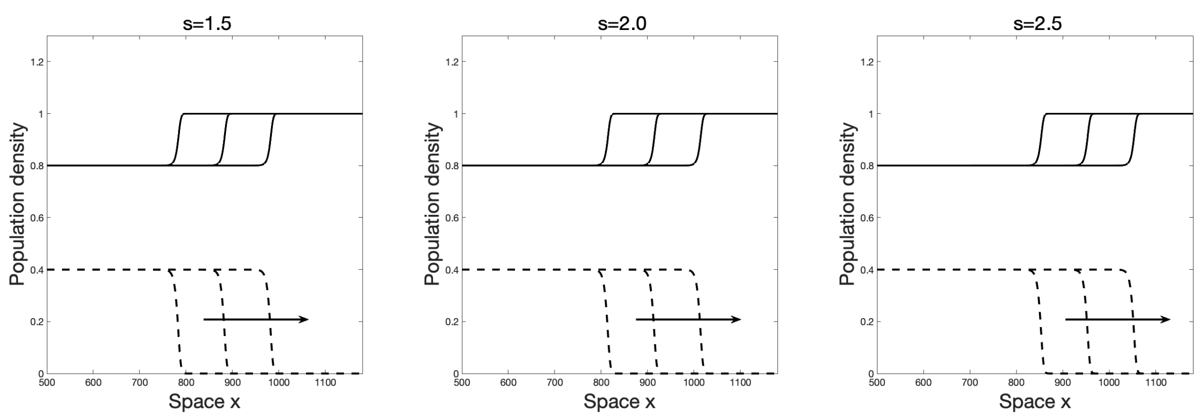


FIGURE 2. Numerical solutions for the advantageous case $s > 0$ without time delay ($\tau = 0$). The arrow indicates the moving direction of each solution.



(a) slow shifting speed with $T = 300, 400, 500$



(b) fast shifting speed with $T = 800, 900, 1000$

FIGURE 3. Numerical solutions for the advantageous case $s > 0$ with a time delay $\tau = 0.5$. The arrow indicates the moving direction of each solution.

$s > 0$ with a time delay $\tau = 0.5$. In Figure 3(a), the spatial distances with time length 100 are approximately $40, 60.01$, and 80.01 for $s = 0.4$, $s = 0.6$ and $s = 0.8$, respectively. This represents numerically that forced waves exist when the shifting speeds are small enough. On the other hand, in Figure 3(b), the spatial distances with time length 100 are approximately $99.4, 99.8$ and 99.6 for $s = 1.5$, $s = 2.0$ and $s = 2.5$, respectively. From our numerical simulations, we conjecture that for the existence of forced waves the critical upper bound for the shifting speeds is $s^*_\tau \approx 0.99$, where s^*_τ is the smallest $s > 0$ such that $\lambda^2 - s\lambda + \beta_0 e^{-\tau s\lambda} - \kappa = 0$ has a positive solution λ. However, this is a very difficult problem to be verified and we leave it as an open question.

References

[1] H. Berestycki, L. Desvillettes, O. Diekmann, *Can climate change lead to gap formation?* Ecological Complexity 20 (2014), 264-270.

[2] H. Berestycki, O. Diekmann, C.J. Nagelkerke, P.A. Zegeling, *Can a species keep pace with a shifting climate?* Bull. Math. Biol. 71 (2009), 399-429.

[3] H. Berestycki, J. Fang, *Forced waves of the Fisher-KPP equation in a shifting environment*, J. Diff. Equations 264 (2018), 2157-2183.

[4] H. Berestycki, L. Rossi, *Reaction-diffusion equations for population dynamics with forced speed, I - the case of the whole space*, Discrete Contin. Dyn. Syst. 21 (2008), 41-67.

[5] H. Berestycki, L. Rossi, *Reaction-diffusion equations for population dynamics with forced speed, II - cylindrical type domains*, Discrete Contin. Dyn. Syst. 25 (2009), 19-61.

[6] J. Bouhours, T. Giletti, *Spreading and vanishing for a monostable reaction-diffusion equation with forced speed*, J. Dynam. Diff. Equations,31 (2019), 247-286.

[7] Y.-Y. Chen, J.-S. Guo, C.-H. Yao, *Traveling wave solutions for a continuous and discrete diffusive predator-prey model*, J. Math. Anal. Appl. 445 (2017), 212-239.

[8] V. Capasso, G. Serio, *A generalization of the Kermack-Mckendrick deterministic epidemic model*, Math. Biosci., 42 (1978), 41–61.

[9] W. Choi, T. Giletti, J.-S. Guo, *Persistence of species in a predator-prey system with climate change and either nonlocal or local dispersal*, J. Diff. Equations 302 (2021), 807-853.

[10] W. Choi, J.-S. Guo, *Forced waves of a three species predator-prey system in a shifting environment*, J. Math. Anal. Appl. 514 (2022), Art. 126283.

[11] F.-D. Dong, B. Li, W.-T. Li, *Forced waves in a Lotka-Volterra competition-diffusion model with a shifting habitat*, J. Diff. Equations, 276 (2021), 433-459.

[12] A. Estrada-Peña, *Climate change decreases habitat suitability for some tick species (Acari: Ixodidae) in South Africa*, The Onderstepoort Journal of Veterinary Research, 70(2) (2003), 79-93.

[13] K. Ezzinbi, M. Adimy, *The Basic Theory of Abstract Semilinear Functional Differential Equations with Non-Dense Domain*, Chapter 9 in the book "Delay Differential Equations and Applications", edited by O. Arino, M.L. Hbid and E. Ait Dads, Springer, 2006.

[14] J. Fang, R. Peng, X.-Q. Zhao, *Propagation dynamics of a reaction-diffusion equation in a time-periodic shifting environment*, J. Math. Pures Appl. 147 (2021), 1-28.

[15] J. Fang, Y. Lou, J. Wu, *Can pathogen spread keep pace with its host invasion?*, SIAM J. Appl. Math. 76 (2016), 1633-1657.

[16] J.-S. Guo, F. Hamel, C.-C. Wu, *Forced waves for a three-species predator-prey system with nonlocal dispersal in a shifting environment*, J. Differential Equations, 345 (2023), 485-518.

[17] J.-S. Guo, K.-I. Nakamura, T. Ogiwara and C.-C. Wu, *Traveling wave solutions for a predator-prey system with two predators and one prey*, Nonlinear Analysis: Real World Applications, 54 (2020), Art. 103111.

[18] F. Hamel, *Reaction-diffusion problems in cylinders with no invariance by translation, Part II: Monotone perturbations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 14 (1997), 555-596.

[19] H.W. Hethcote, *The mathematics of infectious diseases*, SIAM Review, 42 (2000), 599–653.

[20] T. Hou, Y. Wang, X.-Q. Zhao, *Propagation dynamics of cooperative reaction-diffusion systems in a periodic shifting environment*, J. Differential Equations, 379 (2024), 468-496.

[21] H. Hu, X. Zou, *Existence of an extinction wave in the Fisher equation with a shifting habitat*, Proc. Amer. Math. Soc. 145 (2017), 4763-4771.

[22] Y.-L. Huang, G. Lin, Traveling wave solutions in a diffusive system with two preys and one predator, J. Math. Anal. Appl. 418 (2014), 163-184.

[23] Y. Li, W.-T. Li, G. Lin, *Traveling waves of a delayed diffusive SIR epidemic model*, Comm. Pure Appl. Anal., 14 (2015), 1001-1022.

[24] C. McCluskey, *Global stability for an SIR epidemic model with delay and nonlinear incidence*, Nonlinear Anal. Real Word Appl., 11 (2010), 3106–3109.

[25] C.C. Travis, G.F. Webb, *Existence and stability for partial functional differential equations*, Trans. Amer. Math. Soc. 200, (1974), 395-418.

[26] J.-B. Wang, C. Wu, *Forced waves and gap formations for a Lotka-Volterra competition model with nonlocal dispersal and shifting habitats*, Nonlinear Analysis: Real World Appl. 58 (2021), Art. 103208.

[27] J.-B. Wang, X.-Q. Zhao, *Uniqueness and global stability of forced waves in a shifting environment*, Proc. Amer. Math. Soc. 147 (2019), 1467-1481.

[28] C. Wu, Y. Wang, X. Zou, *Spatial-temporal dynamics of a Lotka-Volterra competition model with nonlocal dispersal under shifting environment*, J. Diff. Equations 267 (2019), 4890-4921.

[29] Y. Yang, C. Wu, Z. Li, *Forced waves and their asymptotics in a Lotka-Volterra cooperative model under climate change*, Appl. Math. Comput. 353 (2019), 254-264.

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