TRAVELING WAVE SOLUTIONS FOR A PREDATOR-PREY SYSTEM WITH TWO PREDATORS AND ONE PREY

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ABSTRACT. We study a predator-prey model with two alien predators and one aborigine prey in which the net growth rates of both predators are negative. We characterize the invading speed of these two predators by the minimal wave speed of traveling wave solutions connecting the predator-free state to the co-existence state. The proof of the existence of traveling waves is based on a standard method by constructing (generalized) upper-lower-solutions with the help of Schauder's fixed point theorem. However, in this three species model, we are able to construct some suitable pairs of upper-lower-solutions not only for the super-critical speeds but also for the critical speed. Moreover, a new form of shrinking rectangles is introduced to derive the right-hand tail limit of wave profile.

1. INTRODUCTION

Due to the diversity of ecology, it is very important to understand the interactions of multiple species. There have been a tremendous works done in the past years for one or two species ecological systems from both biological and mathematical points of view. Two typical examples are the competition systems and predator-prey models. However, the more species involved the more complicated dynamical behaviors are expected. Recently some three species models have attracted a lot of attention, including non-cooperative competition systems ([4, 20]), food chain models ([7, 6, 8, 19]) and predator-prey systems ([9, 15, 16, 3]).

In this paper, we are concerned with some 3-species predator-prey models. There can be one predator with two preys, or two predators with one prey. There are aborigine species living in a habitat and we would like to know what happen to the ecological system if we introduce some alien species into the habitat. Our aim is to

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determine the invading speed of the alien species such that these three species can live together. Mathematically, there are at least two approaches to answer this question, namely, to characterize the asymptotic spreading speed(s) and to find the minimal wave speed of traveling wave solutions connecting an appropriate constant state $\mathcal{O}$ to the co-existence state. The latter approach is taken in this paper.

To describe the interaction of 3 species in a predator-prey system in terms of traveling wave solutions, there are the following scenarios. In [9, 15, 16], it is assumed that all three species are alien species so that $\mathcal{O} = (0,0,0)$. In these works, it is also assumed that the growth rate of the predator is positive, which is equivalent to that the predator has other food resources than the prey so that the predator can survive without the prey. On the other hand, less is known when the (net) growth rate of predator is negative. In this case, the predator cannot survive without the prey. In [3], the (net) growth rate of the predator is assumed to be negative. Also, one alien predator and one alien prey are introduced into the habitat of an aborigine prey so that $\mathcal{O} = (0,0,1)$ is taken.

In this paper, we are interested in a predator-prey system with two alien predators and one aborigine prey in which the (net) growth rates of both predators are negative. More precisely, we study the following predator-prey system

$$u_t = d_1 u_{xx} + r_1 u(-1 - u - kv + aw),\ x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$v_t = d_2 v_{xx} + r_2 v(-1 - hu - v + aw),\ x \in \mathbb{R},\ t > 0, \tag{1.2}$$

$$w_t = d_3 w_{xx} + r_3 w(1 - bu - bv - w),\ x \in \mathbb{R},\ t > 0, \tag{1.3}$$

in which $u = u(x,t)$ and $v = v(x,t)$ are the densities of two predators and $w = w(x,t)$ is the density of the single prey. The parameters are all positive such that

$$a > 1, \quad 0 < h, k < 1, \quad 0 < b < \frac{1}{2(a-1)}. \tag{1.4}$$

In system (1.1)-(1.3), d_1, d_2, d_3 are diffusion coefficients of species u, v, w, respectively, the prey w follows the logistic growth with carrying capacity 1 and intrinsic growth rate r_3, both predation rates of predators u and v are equal to $r_3 b$ and their biomass conversion rates are assumed to be $r_1 a$ and $r_2 a$ (for simplicity). Moreover, the parameters r_1, r_2 denote the death rates of predators u, v, respectively, and h, k are the interspecific competition coefficients (so that two predators are weak competitors, by (1.4)). Note that, under condition (1.4), there is the unique co-existence (positive) state (u^*, v^*, w^*), where

$$w^* := \frac{(1-hk)+b(2-h-k)}{(1-hk)+ab(2-h-k)},\ v^* := \frac{1-h}{1-hk}(aw^*-1),\ u^* := \frac{1-k}{1-hk}(aw^*-1).$$

Also, there is an unstable predator-free state $(0,0,1)$. We are concerned with the invading speeds of two alien predators to the habitat of the aborigine prey.

A solution (u,v,w) of (1.1)-(1.3) is a traveling wave solution if

$$u(x,t)=\phi_1(x+st),\quad v(x,t)=\phi_2(x+st),\quad w(x,t)=\phi_3(x+st)$$

for some constant s, the *wave speed*, and some functions ϕ_i, $i=1,2,3$, the *wave profiles*. Then (s,ϕ_1,ϕ_2,ϕ_3) satisfies

$$d_1\phi_1''(z)-s\phi_1'(z)+r_1\phi_1(z)(-1-\phi_1-k\phi_2+a\phi_3)(z)=0,\ z\in\mathbb{R}, \tag{1.5}$$

$$d_2\phi_2''(z)-s\phi_2'(z)+r_2\phi_2(z)(-1-h\phi_1-\phi_2+a\phi_3)(z)=0,\ z\in\mathbb{R}, \tag{1.6}$$

$$d_3\phi_3''(z)-s\phi_3'(z)+r_3\phi_3(z)(1-b\phi_1-b\phi_2-\phi_3)(z)=0,\ z\in\mathbb{R}. \tag{1.7}$$

The main purpose of this paper is to study the minimal wave speed of traveling wave solutions of (1.1)-(1.3) connecting the predator-free state $(0,0,1)$ and the co-existence state (u^*, v^*, w^*). Hence we also imposed the following boundary condition

$$(\phi_1,\phi_2,\phi_3)(-\infty)=(0,0,1),\quad (\phi_1,\phi_2,\phi_3)(\infty)=(u^*,v^*,w^*). \tag{1.8}$$

Set $s^* := \max\{2\sqrt{r_1d_1(a-1)}, 2\sqrt{r_2d_2(a-1)}\}$. Our main result of this paper reads as follows. It determines the invading speed of these two alien predators.

Theorem 1.1. *Under condition* (1.4)*, there is a bounded positive solution* (ϕ_1,ϕ_2,ϕ_3) *of* (1.5)-(1.7) *such that the left-hand boundary condition* $(\phi_1,\phi_2,\phi_3)(-\infty)=(0,0,1)$ *holds, if* $s\ge s^*$. *Moreover, this solution satisfies the right-hand boundary condition* $(\phi_1,\phi_2,\phi_3)(\infty)=(u^*,v^*,w^*)$*, if we further assume that*

$$0<b<\frac{1}{2a}\min\{1-h,1-k\}. \tag{1.9}$$

On the other hand, there is no positive solution of (1.5)-(1.7) *with boundary condition* (1.8)*, if* $s<s^*$.

Since system (1.1)-(1.3) does not have the comparison principle, the classical monotone iteration method is not applicable for deriving the existence of traveling wave solutions. To treat non-monotone systems, the method of applying Schauder's fixed point theorem with the help of (generalized) upper-lower-solutions has been proved to be very successful. Since the works [17] and [18], there have been a lot of works done for various reaction-diffusion systems in past years. We refer the reader to, for examples, [10, 11, 13, 6, 12, 14, 5, 21] for 2-component systems and [9, 15, 20, 16, 3] for 3-component systems. For 2-species case, there is a nice paper by Zhang and Jin [21] which contains not only a quite complete collection of literature on the methods of deriving traveling wave solutions, but also some valuable biological interpretations of models. Moreover, the advection terms are also taken into consideration in [21].

The main contribution of this paper is the construction of suitable pairs of upper-lower-solutions for the 3-speices predator-prey model (1.1)-(1.3). Just as the construction of Lyapunov functional in the study of asymptotic behavior for evolution systems, if one can find a suitable Lyapunov functional then (plus some a priori estimates) the asymptotic behavior of solutions can be readily derived. However, the construction of upper-lower-solutions is by no means trivial in general, as it is well-known that a Lyapunov functional for a given evolution systems is not always available. In this paper, we are able to derive the existence of traveling wave solutions also for the critical (minimal) speed based on an idea from [5]. In fact, the critical speed case is left open in [15, 16, 3] and our method might be applicable for those models.

The traveling waves constructed in this paper are such that two predators propagate simultaneously. Ecologically, this means that these two alien predators invade the habitat of the aborigine prey with the same speed. This can be visualized by putting the constructed wave on the negative x-axis and reflecting it on the positive x-axis, so that an entire (in time) solution with two fronts is formed (formally). However, in practice, these two predators may have different invading speeds to the habitat of the prey. To find such traveling waves is a very interesting question. We leave it open in this paper. On the other hand, as we mentioned earlier, the invading phenomenon can also be studied by the (asymptotic) spreading speed(s). We provide here some heuristic observation as follows. Assume that $r_1d_1 > r_2d_2$. At the leading edge of the invading front, we have $w = 1$ (i.e., the prey is saturated). Without the predator v, the predator u should propagate with the speed $2\sqrt{d_1r_1(a-1)}$. In fact, this is the case if the predator v propagate with a slower speed. However, this fact is not easy to verify rigorously. We also leave this question as another open problem.

The rest of this paper is organized as follows. In section 2, we first give the definition of upper-lower-solutions for system (1.5)-(1.7). Then we provide a theorem for deriving the existence of solutions to (1.5)-(1.7) with a outline of its proof. In section 3, we construct some suitable upper-lower-solutions for system (1.5)-(1.7) for all speeds $s \geq s^*$. This proves the first part of the existence of traveling waves to system (1.5)-(1.7) stated in Theorem 1.1. Then, in section 4, using the idea of shrinking rectangles ([9, 5]) we derive the right-hand tail limit of wave profile obtained in section 3. It is worth to remark that our construction of two end points of these rectangles is different from that in [9, 5]. Finally, in section 5, we provide a proof for the non-existence part of Theorem 1.1. Hence the minimal wave speed is determined.

2. General theory

In this section, we shall provide a general framework for deriving the existence of traveling wave solutions.

We first introduce the following definition of upper-lower-solution to (1.5)-(1.7).

Definition 2.1. *Given $s > 0$. Continuous functions $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ defined on $\mathbb{R}$ are called a pair of (generalized) upper-lower-solutions of* (1.5)-(1.7) *if $\overline{\phi}_i''$, $\underline{\phi}_i''$, $\overline{\phi}_i'$, $\underline{\phi}_i'$, $i = 1, 2, 3$, are bounded functions such that the following inequalities hold:*

$$\mathcal{U}_1(z) := d_1\overline{\phi}_1''(z) - s\overline{\phi}_1'(z) + r_1\overline{\phi}_1(z)[-1 - \overline{\phi}_1(z) - k\underline{\phi}_2(z) + a\overline{\phi}_3(z)] \leq 0, \tag{2.1}$$

$$\mathcal{U}_2(z) := d_2\overline{\phi}_2''(z) - s\overline{\phi}_2'(z) + r_2\overline{\phi}_2(z)[-1 - h\underline{\phi}_1(z) - \overline{\phi}_2(z) + a\overline{\phi}_3(z)] \leq 0, \tag{2.2}$$

$$\mathcal{U}_3(z) := d_3\overline{\phi}_3''(z) - s\overline{\phi}_3'(z) + r_3\overline{\phi}_3(z)[1 - b\underline{\phi}_1(z) - b\underline{\phi}_2(z) - \overline{\phi}_3(z)] \leq 0, \tag{2.3}$$

$$\mathcal{L}_1(z) := d_1\underline{\phi}_1''(z) - s\underline{\phi}_1'(z) + r_1\underline{\phi}_1(z)[-1 - \underline{\phi}_1(z) - k\overline{\phi}_2(z) + a\underline{\phi}_3(z)] \geq 0, \tag{2.4}$$

$$\mathcal{L}_2(z) := d_2\underline{\phi}_2''(z) - s\underline{\phi}_2'(z) + r_2\underline{\phi}_2(z)[-1 - h\overline{\phi}_1(z) - \underline{\phi}_2(z) + a\underline{\phi}_3(z)] \geq 0, \tag{2.5}$$

$$\mathcal{L}_3(z) := d_3\underline{\phi}_3''(z) - s\underline{\phi}_3'(z) + r_3\underline{\phi}_3(z)[1 - b\overline{\phi}_1(z) - b\overline{\phi}_2(z) - \underline{\phi}_3(z)] \geq 0, \tag{2.6}$$

for $z \in \mathbb{R}\backslash E$ with some finite set $E = \{z_1, z_2, \ldots, z_m\}$.

Then a standard argument (cf., e.g., [17, 18, 9]) gives the following proposition for the existence of solution to system (1.5)-(1.7).

Proposition 2.2. *Given $s > 0$. Suppose that system* (1.5)-(1.7) *has a pair of upper-lower-solutions $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ such that*

$$\underline{\phi}_i(z) \leq \overline{\phi}_i(z),\ \forall\, z \in \mathbb{R},\ i = 1, 2, 3, \tag{2.7}$$

$$\lim_{z \to z_j^+} \overline{\phi}_i'(z) \leq \lim_{z \to z_j^-} \overline{\phi}_i'(z),\ \lim_{z \to z_j^-} \underline{\phi}_i'(z) \leq \lim_{z \to z_j^+} \underline{\phi}_i'(z),\ \forall\, z_j \in E,\ i = 1, 2, 3. \tag{2.8}$$

Then system (1.5)-(1.7) *has a solution (ϕ_1, ϕ_2, ϕ_3) such that $\underline{\phi}_i \leq \phi_i \leq \overline{\phi}_i$, $i = 1, 2, 3$.*

Since the proof is by now very standard, we safely omit it. In fact, there are the following major steps to derive the above existence theory.

First, we transform the differential system to an integral system with operator P so that the existence of a solution to the differential system is reduced to a fixed point of P. Indeed, take a constant κ such that

$$\kappa > \max\{r_1[1+(2+k)(a-1)], r_2[1+(2+h)(a-1)], r_3[1+2b(a-1)]\},$$

so that the function $f_i(y_1, y_2, y_3)$ is non-decreasing in y_i, $i = 1, 2, 3$, for $(y_1, y_2, y_3) \in [0, a-1] \times [0, a-1] \times [0, 1]$, where

$$\begin{aligned} f_1(y_1, y_2, y_3) &:= \kappa y_1 + r_1 y_1(-1 - y_1 - k y_2 + a y_3), \\ f_2(y_1, y_2, y_3) &:= \kappa y_2 + r_2 y_2(-1 - h y_1 - y_2 + a y_3), \\ f_3(y_1, y_2, y_3) &:= \kappa y_3 + r_3 y_3(1 - b y_1 - b y_2 - y_3). \end{aligned}$$

Then the operator $P := (P_1, P_2, P_3)$ is defined by

$$\begin{aligned} P_i(\phi_1, \phi_2, \phi_3)(z) \quad &:= \quad \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})} \Big\{ \int_{-\infty}^{z} e^{\lambda_{i1}(z-s)} f_i(\phi_1(s), \phi_2(s), \phi_3(s)) ds \\ &\qquad + \int_{z}^{\infty} e^{\lambda_{i2}(z-s)} f_i(\phi_1(s), \phi_2(s), \phi_3(s)) ds \Big\}, \ i = 1, 2, 3, \end{aligned}$$

for $z \in \mathbb{R}$, where $\lambda_{i1} < 0 < \lambda_{i2}$ are roots to

$$d_i \lambda^2 - s\lambda - \kappa = 0, \ i = 1, 2, 3.$$

Secondly, using the pair of upper-lower-solutions, we define the set

$$\Sigma := \{(\phi_1, \phi_2, \phi_3) \mid \underline{\phi}_i \le \phi_i \le \overline{\phi}_i, \ i = 1, 2, 3\}$$

and show that P maps Σ into itself. Indeed, this follows from the definition of upper-lower-solutions and the choice of P.

Thirdly, we show that the operator P is completely continuous with respect to a suitable complete weighted normed space on the nonempty bounded closed convex set Σ. Then we are done, since Schauder's fixed point theorem gives a fixed point of P in Σ. For more details, we refer the reader to [9].

3. Upper-lower-solutions

This section is devoted to the construction of suitable upper-lower-solutions.

For $s > s^*$, we define λ_i to be the smaller positive root of $g_i(\lambda) = 0$, where

$$g_i(\lambda) := d_i \lambda^2 - s\lambda + r_i(a-1), \tag{3.1}$$

for $i = 1, 2$. We also denote the larger positive root of $g_i(\lambda) = 0$ by $\hat{\lambda}_i$, $i = 1, 2$. Moreover, we let λ_3 be the unique positive root of

$$d_3\lambda^2 - s\lambda - r_3 = 0.$$

Now, we introduce the following continuous functions

$$\overline{\phi}_1(z) := \min\{\beta_1 e^{\lambda_1 z}, \beta_1\}, \quad \underline{\phi}_1(z) := \beta_1 \max\{e^{\lambda_1 z} - q_1 e^{\mu_1 \lambda_1 z}, 0\}, \tag{3.2}$$

$$\overline{\phi}_2(z) := \min\{\beta_2 e^{\lambda_2 z}, \beta_2\}, \quad \underline{\phi}_2(z) := \beta_2 \max\{e^{\lambda_2 z} - q_2 e^{\mu_2 \lambda_2 z}, 0\}, \tag{3.3}$$

$$\overline{\phi}_3(z) :\equiv 1, \quad \underline{\phi}_3(z) := \max\{1 - (e^{\lambda_3 z} + p e^{\alpha \lambda_3 z}), 1 - a\beta\}, \tag{3.4}$$

where parameters $\beta_1 = \beta_2 = a - 1$, $\beta \in [2b(a-1)/a, 1/a)$, and $\mu_i > 1$, $q_i > 1$, $i = 1, 2$, $p > 0$, $\alpha \in (0, 1)$ are positive constants to be determined. Note that the constant β is admissible due to the last condition in (1.4).

Then we check that the above functions is a pair of upper-lower-solutions.

First, for $z > 0$, $\overline{\phi}_1(z) = \beta_1$, since $\underline{\phi}_2 = 0$ and $\overline{\phi}_3 \equiv 1$ we have

$$\mathcal{U}_1(z) = r_1\beta_1(-1 - \beta_1 + a) = 0,$$

since $\beta_1 = a - 1$. For $z < 0$, $\overline{\phi}_1(z) = \beta_1 e^{\lambda_1 z}$ gives

$$\begin{aligned} \mathcal{U}_1(z) &= \beta_1 e^{\lambda_1 z}\{d_1\lambda_1^2 - s\lambda_1\} + r_1\beta_1 e^{\lambda_1 z}\{-1 - \beta_1 e^{\lambda_1 z} + a - k\underline{\phi}_2(z)\} \\ &= -r_1\beta_1 e^{\lambda_1 z}\{\beta_1 e^{\lambda_1 z} + k\underline{\phi}_2(z)\} \le 0, \end{aligned}$$

using $g_1(\lambda_1) = 0$. Hence $\mathcal{U}_1(z) \le 0$ for all $z \ne 0$.

Similarly, we can easily check that $\mathcal{U}_2(z) \le 0$ for all $z \ne 0$.

For (2.3), since $\overline{\phi}_3 \equiv 1$, we have

$$\mathcal{U}_3(z) = -r_3 b[\underline{\phi}_1(z) + \underline{\phi}_2(z)] \le 0$$

for all $z \in \mathbb{R}$.

Next, we claim that $\mathcal{L}_3(z) \ge 0$ for all $z \ne z_3$, where z_3 is defined by

$$e^{\lambda_3 z_3} + p e^{\alpha\lambda_3 z_3} = a\beta.$$

Note that $z_3 < 0$, since $a\beta < 1$. For $z > z_3$, $\underline{\phi}_3(z) = 1 - a\beta$. Then, using $\overline{\phi}_i(z) \le \beta_i$, $i = 1, 2$, we obtain

$$\mathcal{L}_3(z) \ge r_3(1 - a\beta)\{a\beta - b(\beta_1 + \beta_2)\} \ge 0,$$

since $\beta \ge 2b(a-1)/a$.

For $z < z_3 < 0$, we have

$$\underline{\phi}_3(z) = 1 - (e^{\lambda_3 z} + pe^{\alpha\lambda_3 z}),\ \overline{\phi}_i(z) = \beta_i e^{\lambda_i z},\ i = 1, 2.$$

Hence we have

$$\begin{aligned}
\mathcal{L}_3(z) &= -e^{\lambda_3 z}\{d_3\lambda_3^2 - s\lambda_3\} - pe^{\alpha\lambda_3 z}\{d_3(\alpha\lambda_3)^2 - s(\alpha\lambda_3)\} \\
&\quad + r_3\{1 - (e^{\lambda_3 z} + pe^{\alpha\lambda_3 z})\}\{(e^{\lambda_3 z} + pe^{\alpha\lambda_3 z}) - b\beta_1 e^{\lambda_1 z} - b\beta_2 e^{\lambda_2 z}\} \\
&\geq -e^{\lambda_3 z}\{d_3\lambda_3^2 - s\lambda_3 - r_3\} - pe^{\alpha\lambda_3 z}\{d_3(\alpha\lambda_3)^2 - s(\alpha\lambda_3) - r_3\} \\
&\quad - r_3\{(e^{\lambda_3 z} + pe^{\alpha\lambda_3 z})^2 + b\beta_1 e^{\lambda_1 z} + b\beta_2 e^{\lambda_2 z}\} \\
&\geq e^{\alpha\lambda_3 z}\Big(p\{-[d_3(\alpha\lambda_3)^2 - s(\alpha\lambda_3) - r_3]\} \\
&\quad - r_3\{a\beta e^{(1-\alpha)\lambda_3 z} + a\beta p + b\beta_1 e^{(\lambda_1 - \alpha\lambda_3)z} + b\beta_2 e^{(\lambda_2 - \alpha\lambda_3)z}\}\Big) \\
&:= e^{\alpha\lambda_3 z} l_3(z)
\end{aligned}$$

using $(e^{\lambda_3 z} + pe^{\alpha\lambda_3 z}) \leq a\beta$ for all $z < z_3$. Now, choosing a constant α such that

$$0 < \alpha < \min\{\lambda_1/\lambda_3, \lambda_2/\lambda_3, \alpha_0\}, \tag{3.5}$$

where α_0 is the positive constant such that

$$d_3(\alpha_0\lambda_3)^2 - s(\alpha_0\lambda_3) - (1 - a\beta)r_3 = 0.$$

Note that, by the definition of λ_3, $\alpha_0 < 1$. Hence

$$d_3(\alpha\lambda_3)^2 - s(\alpha\lambda_3) - (1 - a\beta)r_3 < 0.$$

On the other hand, due to $z_3 < 0$ and (3.5), we conclude that

$$l_3(z) \geq p\{-[d_3(\alpha\lambda_3)^2 - s(\alpha\lambda_3) - (1 - a\beta)r_3]\} - r_3(a\beta + b\beta_1 + b\beta_2) \geq 0$$

for all $z < z_3$, provided that

$$p > \frac{r_3(a\beta + b\beta_1 + b\beta_2)}{-[d_3(\alpha\lambda_3)^2 - s(\alpha\lambda_3) - (1 - a\beta)r_3]}. \tag{3.6}$$

Finally, set z_i, $i = 1, 2$, to be the unique point such that $Q_i(z_i) = 0$, where

$$Q_i(z) := e^{\lambda_i z} - q_i e^{\mu_i \lambda_i z}.$$

Note that $z_i < 0$ and $Q_i(z) > 0$ for $z < z_i$, due to $\mu_i > 1$ and $q_i > 1$, for $i = 1, 2$.

For $z > z_1$, $\underline{\phi}_1(z) = 0$ and so it is easy to see that $\mathcal{L}_1(z) = 0$. For $z < z_1$, we have

$$\underline{\phi}_1(z) = \beta_1(e^{\lambda_1 z} - q_1 e^{\mu_1 \lambda_1 z}),\ \overline{\phi}_2(z) = \beta_2 e^{\lambda_2 z},\ \underline{\phi}_3(z) \geq 1 - (e^{\lambda_3 z} + pe^{\alpha\lambda_3 z}).$$

Hence

$$\begin{aligned}
& \mathcal{L}_1(z) \\
\geq\ & \beta_1 e^{\lambda_1 z}\{d_1\lambda_1^2 - s\lambda_1\} - q_1\beta_1 e^{\mu_1\lambda_1 z}\{d_1(\mu_1\lambda_1)^2 - s(\mu_1\lambda_1)\} + r_1\beta_1(e^{\lambda_1 z} - q_1 e^{\mu_1\lambda_1 z})\cdot \\
& \{-1 - \beta_1(e^{\lambda_1 z} - q_1 e^{\mu_1\lambda_1 z}) - k\beta_2 e^{\lambda_2 z} + a - a(e^{\lambda_3 z} + pe^{\alpha\lambda_3 z})\} \\
=\ & \beta_1 e^{\lambda_1 z}\{d_1\lambda_1^2 - s\lambda_1 + r_1(a-1)\} - q_1\beta_1 e^{\mu_1\lambda_1 z}\{d_1(\mu_1\lambda_1)^2 - s(\mu_1\lambda_1) + r_1(a-1)\} \\
& -r_1\beta_1(e^{\lambda_1 z} - q_1 e^{\mu_1\lambda_1 z})\{\beta_1(e^{\lambda_1 z} - q_1 e^{\mu_1\lambda_1 z}) + k\beta_2 e^{\lambda_2 z} + a(e^{\lambda_3 z} + pe^{\alpha\lambda_3 z})\} \\
\geq\ & -q_1\beta_1 e^{\mu_1\lambda_1 z}\{d_1(\mu_1\lambda_1)^2 - s(\mu_1\lambda_1) + r_1(a-1)\} \\
& -r_1\beta_1\{\beta_1 e^{2\lambda_1 z} + k\beta_2 e^{(\lambda_1+\lambda_2)z} + a(e^{(\lambda_1+\lambda_3)z} + pe^{(\lambda_1+\alpha\lambda_3)z})\} \\
=\ & \beta_1 e^{\mu_1\lambda_1 z}\Big(q_1\{-[d_1(\mu_1\lambda_1)^2 - s(\mu_1\lambda_1) + r_1(a-1)]\} - r_1\cdot \\
& \{\beta_1 e^{(2\lambda_1-\mu_1\lambda_1)z} + k\beta_2 e^{(\lambda_1+\lambda_2-\mu_1\lambda_1)z} + a(e^{(\lambda_1+\lambda_3-\mu_1\lambda_1)z} + pe^{(\lambda_1+\alpha\lambda_3-\mu_1\lambda_1)z})\}\Big) \\
:=\ & \beta_1 e^{\mu_1\lambda_1 z} l_1(z).
\end{aligned}$$

Now, we choose a constant μ_1 such that

$$1 < \mu_1 < \min\{\hat{\lambda}_1/\lambda_1, 2, 1+\lambda_2/\lambda_1, 1+\alpha\lambda_3/\lambda_1\}. \tag{3.7}$$

Due to $z_1 < 0$ (using $q_1 > 1$), we deduce that

$$l_1(z) \geq q_1\{-[d_1(\mu_1\lambda_1)^2 - s(\mu_1\lambda_1) + r_1(a-1)]\} - r_1(\beta_1 + k\beta_2 + a + ap) \geq 0$$

for all $z < z_1$, provided that

$$q_1 > \max\left\{1, \frac{r_1(\beta_1 + k\beta_2 + a + ap)}{-[d_1(\mu_1\lambda_1)^2 - s(\mu_1\lambda_1) + r_1(a-1)]}\right\}. \tag{3.8}$$

Similarly, we can show that $\mathcal{L}_2(z) \geq 0$ for all $z \neq z_2$, provided that

$$1 < \mu_2 < \min\{\hat{\lambda}_2/\lambda_2, 2, 1+\lambda_1/\lambda_2, 1+\alpha\lambda_3/\lambda_2\}, \tag{3.9}$$

$$q_2 > \max\left\{1, \frac{r_2(h\beta_1 + \beta_2 + a + ap)}{-[d_2(\mu_2\lambda_2)^2 - s(\mu_2\lambda_2) + r_2(a-1)]}\right\}. \tag{3.10}$$

We summarize the above discussions as the following lemma.

Lemma 3.1. *For $s > \max\{2\sqrt{r_1 d_1(a-1)}, 2\sqrt{r_2 d_2(a-1)}\}$, the functions $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ defined by (3.2)-(3.4) are a pair of upper-lower-solutions of system (1.5)-(1.7), provided that parameters α, p, μ_1, q_1, μ_2, q_2 are chosen such that conditions (3.5)-(3.10) hold.*

Next, we consider the critical speed case when $s = s^*$. In the sequel, without loss of generality we may assume that $r_1 d_1 \geq r_2 d_2$ so that $s^* = 2\sqrt{r_1 d_1(a-1)}$.

We divide our discussions into two cases.

Case 1. $r_1 d_1 > r_2 d_2$**.** In this case, we have $s = 2\sqrt{r_1 d_1(a-1)}$ and $s > 2\sqrt{r_2 d_2(a-1)}$ when $s = s^*$. Hence $g_1(\lambda) = 0$ has a positive double root λ_1 and $g_2(\lambda) = 0$ has two positive roots $\lambda_2, \hat{\lambda}_2$ with $\lambda_2 < \hat{\lambda}_2$. Following [5], we replace the functions in (3.2) by

(3.11)
$$\overline{\phi}_1(z) := \begin{cases} h_1(-z)e^{\lambda_1 z}, & z \le z_{10}, \\ \beta_1, & z \ge z_{10}, \end{cases} \quad \underline{\phi}_1(z) := \begin{cases} [h_1(-z) - q_1(-z)^{1/2}]e^{\lambda_1 z}, & z \le z_{01}, \\ 0, & z \ge z_{01}, \end{cases}$$

where $h_1 := \beta_1 \lambda_1 e$, $z_{10} := -1/\lambda_1$, $z_{01} := -(q_1/h_1)^2$ and $q_1 > 1$ is a constant to be determined. Also, $s = 2d_1\lambda_1$, due to that λ_1 is a double root of $g_1(\lambda) = 0$. Similar to the above calculations, we can easily check that the functions $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ defined by (3.11), (3.3) and (3.4) are a pair of upper-lower-solutions of (1.5)-(1.7), provided that α, p, μ_2, q_2, q_1 satisfy conditions to be specified below.

For reader's convenience, we provide some necessary details as follows. First, it is easy to see that

$$\sup_{z \le 0} \{(-z)^\nu e^{\gamma z}\} = \left(\frac{\nu}{\gamma e}\right)^\nu$$

for any given positive constants ν and γ.

That $\mathcal{U}_1(z) \le 0$ for all $z \ne z_{10}$, $\mathcal{U}_2(z) \le 0$ for all $z \ne 0$ and $\mathcal{U}_3(z) \le 0$ for all $z \in \mathbb{R}$ are obtained by direct calculations.

For $\mathcal{L}_3$, as before, we have $\mathcal{L}_3(z) \ge 0$ for all $z > z_3$. For $z < z_3$, we first choose the constant α (instead of (3.5)) such that

$$0 < \alpha < \min\{\lambda_1/(2\lambda_3), \lambda_2/\lambda_3, \alpha_0\}. \tag{3.12}$$

Also, to make sure $z_3 < z_{10}$, we also choose $p > a\beta\sqrt{e}$. Then the term (appeared in $l_3(z)$ above) $b\beta_1 e^{(\lambda_1 - \alpha\lambda_3)z}$ becomes $bh_1(-z)e^{(\lambda_1-\alpha\lambda_3)z}$. So, due to $\alpha < \lambda_1/(2\lambda_3)$, we have the estimate

$$bh_1(-z)e^{(\lambda_1-\alpha\lambda_3)z} \le bh_1(-z)e^{\alpha\lambda_3 z} \le \frac{bh_1}{\alpha\lambda_3 e}, \ \forall\, z \le 0.$$

Then we choose p such that

$$p > \max\left\{a\beta\sqrt{e}, \frac{r_3[a\beta + bh_1/(\alpha\lambda_3 e) + b\beta_2]}{-[d_3(\alpha\lambda_3)^2 - s(\alpha\lambda_3) - (1-a\beta)r_3]}\right\} \tag{3.13}$$

to conclude that $\mathcal{L}_3(z) \ge 0$ for all $z < z_3$.

Similar argument as for $\mathcal{L}_3$ and using

$$h_1(-z)e^{(\lambda_1+\lambda_2-\mu_2\lambda_2)z} \le h_1(-z)e^{\mu_2\lambda_2 z} \le h_1/(\mu_2\lambda_2 e),$$

by choosing μ_2 satisfying

$$1 < \mu_2 < \min\{\hat{\lambda}_2/\lambda_2, 2, (1+\lambda_1/\lambda_2)/2, 1+\alpha\lambda_3/\lambda_2\} \tag{3.14}$$

and q_2 satisfying

$$q_2 > \max\left\{1, \frac{r_2(\beta_2 + hh_1/(\mu_2\lambda_2 e) + a + ap)}{-[d_2(\mu_2\lambda_2)^2 - s(\mu_2\lambda_2) + r_2(a-1)]}\right\}, \tag{3.15}$$

we obtain that $\mathcal{L}_2(z) \geq 0$ for all $z < z_2$ and so $\mathcal{L}_2(z) \geq 0$ for all $z \neq z_2$.

Finally, for $\mathcal{L}_1$, it is evident that $\mathcal{L}_1(z) \geq 0$ for all $z > z_{01}$. For $z < z_{01}$, after a simple computation, we end up with

$$\begin{aligned}
\mathcal{L}_1(z) \;\geq\;& (-z)^{-3/2} e^{\lambda_1 z}\Big\{\frac{1}{4}d_1 q_1 - r_1(-z)^{3/2}[h_1(-z) - q_1(-z)^{1/2}]^2 e^{\lambda_1 z} \\
& -r_1 k\beta_2 h_1(-z)^{5/2} e^{\lambda_2 z} - r_1 a h_1 (-z)^{5/2}(e^{\lambda_3 z} + p e^{\alpha\lambda_3 z})\Big\} \\
\geq\;& (-z)^{-3/2} e^{\lambda_1 z}\Big\{\frac{1}{4}d_1 q_1 - r_1 h_1^2 (-z)^{7/2} e^{\lambda_1 z} \\
& -r_1 k\beta_2 h_1(-z)^{5/2} e^{\lambda_2 z} - r_1 a h_1 (-z)^{5/2}(e^{\lambda_3 z} + p e^{\alpha\lambda_3 z})\Big\} \\
\geq\;& (-z)^{-3/2} e^{\lambda_1 z}\Big\{\frac{1}{4}d_1 q_1 - r_1 h_1^2\left(\frac{7}{2\lambda_1 e}\right)^{7/2} - r_1 k\beta_2 h_1\left(\frac{5}{2\lambda_2 e}\right)^{5/2} \\
& -r_1 a h_1\left(\frac{5}{2\lambda_3 e}\right)^{5/2} - r_1 a h_1 p\left(\frac{5}{2\alpha\lambda_3 e}\right)^{5/2}\Big\} \geq 0,
\end{aligned}$$

if we choose $q_1 > 1$ such that

$$q_1 > \frac{4r_1h_1}{d_1}\Big[h_1\left(\frac{7}{2\lambda_1 e}\right)^{7/2} + k\beta_2\left(\frac{5}{2\lambda_2 e}\right)^{5/2} + a\left(\frac{5}{2\lambda_3 e}\right)^{5/2} + ap\left(\frac{5}{2\alpha\lambda_3 e}\right)^{5/2}\Big]. \tag{3.16}$$

We conclude that the functions defined by (3.11), (3.3) and (3.4) are a pair of upper-lower-solutions of (1.5)-(1.7), provided that (3.12)-(3.16) hold.

Case 2. $r_1 d_1 = r_2 d_2$**.** In this case, we have $s = 2\sqrt{r_1 d_1(a-1)} = s\sqrt{r_2 d_2(a-1)}$ when $s = s^*$. Hence $g_i(\lambda) = 0$ has a positive double root λ_i, $i = 1, 2$. Note that $s = 2d_1\lambda_1 = 2d_2\lambda_2$. Then we replace the functions in (3.3) by

$$\overline{\phi}_2(z) := \begin{cases} h_2(-z)e^{\lambda_2 z}, & z \leq z_{20}, \\ \beta_2, & z \geq z_{20}, \end{cases} \quad \underline{\phi}_2(z) := \begin{cases} [h_2(-z) - q_2(-z)^{1/2}]e^{\lambda_2 z}, & z \leq z_{02}, \\ 0, & z \geq z_{02}, \end{cases} \tag{3.17}$$

where $h_2 := \beta_2\lambda_2 e$, $z_{20} := -1/\lambda_2$, $z_{02} := -(q_2/h_2)^2$ and $q_2 > 1$ is a constant to be determined.

As in Case 1, we can verify that the functions $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ defined by (3.11), (3.17) and (3.4) are a pair of upper-lower-solutions of (1.5)-(1.7), provided that α, p, $q_2 > 1$ and $q_1 > 1$ satisfy conditions

$$\begin{aligned}
&0 < \alpha < \min\{\lambda_1/(2\lambda_3), \lambda_2/(2\lambda_3), \alpha_0\},\\
&p > \max\left\{a\beta\sqrt{e}, \frac{r_3[a\beta + bh_1/(\alpha\lambda_3 e) + bh_2/(\alpha\lambda_3 e)]}{-[d_3(\alpha\lambda_3)^2 - s(\alpha\lambda_3) - (1-a\beta)r_3]}\right\},\\
&q_2 > \frac{4r_2h_2}{d_2}\Big[hh_1\left(\frac{7}{2\lambda_1 e}\right)^{7/2} + h_2\left(\frac{7}{2\lambda_2 e}\right)^{7/2} + a\left(\frac{5}{2\lambda_3 e}\right)^{5/2} + ap\left(\frac{5}{2\alpha\lambda_3 e}\right)^{5/2}\Big],\\
&q_1 > \frac{4r_1h_1}{d_1}\Big[h_1\left(\frac{7}{2\lambda_1 e}\right)^{7/2} + kh_2\left(\frac{7}{2\lambda_2 e}\right)^{7/2} + a\left(\frac{5}{2\lambda_3 e}\right)^{5/2} + ap\left(\frac{5}{2\alpha\lambda_3 e}\right)^{5/2}\Big].
\end{aligned}$$

Combining this with Lemma 3.1 and Proposition 2.2, we conclude this section with

Theorem 3.2. *Under condition* (1.4)*, for each* $s \geq s^*$*, there is a bounded positive solution* (ϕ_1, ϕ_2, ϕ_3) *to system* (1.5)-(1.7) *such that* $(\phi_1, \phi_2, \phi_3)(-\infty) = (0, 0, 1)$.

Indeed, the existence of a solution (ϕ_1, ϕ_2, ϕ_3) to system (1.5)-(1.7) follows from Proposition 2.2 and clearly this solution is nonnegative and bounded in $\mathbb{R}$, due to the properties of our constructed upper-lower-solutions. Moreover, it follows from the strong maximum principle for scalar equation that each component of the solution is positive in $\mathbb{R}$. Hence Theorem 3.2 follows.

4. Existence of traveling wave solutions

To derive the existence of traveling wave connecting $(0, 0, 1)$ and (u^*, v^*, w^*), it suffices to derive the right-hand tail limit of (ϕ_1, ϕ_2, ϕ_3) such that

$$(\phi_1, \phi_2, \phi_3)(\infty) = (u^*, v^*, w^*) \tag{4.1}$$

for solution (ϕ_1, ϕ_2, ϕ_3) obtained in Theorem 3.2.

To derive the right-hand tail limit of wave profile, we need to make the further restriction (1.9) on b. Moreover, we change the range of β from $[2b(a-1)/a, 1/a)$ to $(2b(a-1)/a, \min\{1-h, 1-k\}(a-1)/a^2)$. Note that, under condition (1.9), it is evident that

$$\emptyset \neq (2b(a-1)/a, \min\{1-h, 1-k\}(a-1)/a^2) \subset [2b(a-1)/a, 1/a).$$

With this choice, the functions, (3.2)-(3.4), (3.11), (3.17), constructed before are also upper-lower-solutions. Also, we have $w^* > 1 - a\beta$.

To proceed further, we set

$$\phi_i^- := \liminf_{z\to\infty} \phi_i(z),\ \phi_i^+ := \limsup_{z\to\infty} \phi_i(z),\ i = 1, 2, 3.$$

First, we prepare a lemma as follows.

Lemma 4.1. *Let*

$$b_1 \in (0, (a-1)(1-k) - a^2\beta], \quad b_2 \in (0, (a-1)(1-h) - a^2\beta]. \tag{4.2}$$

Then $\phi_i^- \geq b_i > 0$, $i = 1, 2$.

Proof. We first consider (1.1) with $(u, v, w)(x, t) = (\phi_1, \phi_2, \phi_3)(x + st)$. Using $w = \phi_3 \geq 1 - a\beta$ and $v = \phi_2 \leq \beta_2$, it follows from (1.1) that $u = \phi_1$ satisfies

$$u_t \geq d_1 u_{xx} + r_1 u\{-1 - k\beta_2 + a(1 - a\beta) - u\},\ x \in \mathbb{R},\ t > 0,$$

along with $u(x, 0) = \phi_1(x)$. We compute that

$$-1 - k\beta_2 + a(1 - a\beta) = (a-1)(1-k) - a^2\beta \geq b_1.$$

Recall that $u(x, t) = \phi_1(x + st)$ for $s \geq s^* > 0$. Therefore, it follows from [1, 2] and the comparison principle that

$$\phi_1^- = \liminf_{z\to\infty} \phi_1(z) = \liminf_{z\to\infty} u(0, z/s) \geq b_1 > 0.$$

Similarly, we can derive that $\phi_2^- \geq b_2 > 0$ and the lemma follows. □

Next, we follow a method used in [5] (see also [9]) by constructing a sequence of shrinking rectangles as follows. For $\theta \in [0, 1]$, we define

$$m_1(\theta) := \theta u^* + (1-\theta)(b_1 - \varepsilon),\ M_1(\theta) := (1-\theta)(\beta_1 + \varepsilon) + \theta u^*,$$
$$m_2(\theta) := \theta v^* + (1-\theta)(b_2 - \varepsilon),\ M_2(\theta) := (1-\theta)(\beta_2 + \varepsilon) + \theta v^*,$$
$$m_3(\theta) := \theta w^* + (1-\theta)(b_3 - \varepsilon^2),\ M_3(\theta) := (1-\theta)(\beta_3 + \varepsilon^2) + \theta w^*,$$

where $\beta_1 = \beta_2 = a - 1$, $\beta_3 := 1$, b_1 and b_2 are defined by (4.2), $b_3 := 1 - a\beta$ and ε is a small positive constant such that

$$\varepsilon < \min\left\{\frac{1-h}{a}, \frac{1-k}{a}, \frac{a\beta - 2b(a-1)}{2b}, \frac{b_1 + b_2}{2}\right\}. \tag{4.3}$$

One should note that the above choices of left and right endpoints of rectangles are different from the ones in [9, 5].

It is clear that $\beta_1 > u^*$, $\beta_2 > v^*$ and $\beta_3 > w^*$. By choosing b_1 and b_2 smaller, if it is necessary, we can ensure that Lemma 4.1 holds such that $0 < b_1 < u^*$ and

$0 < b_2 < v^*$. Then, since $w^* > b_3$, we see that $m_i(\theta)$ (resp. $-M_i(\theta)$) is a monotone increasing function of $\theta \in [0,1]$, $i = 1,2,3$, such that

$$(m_1, m_2, m_3)(1) = (M_1, M_2, M_3)(1) = (u^*, v^*, w^*).$$

Then it suffices to show that the set

$$B := \{\theta \in [0,1) \mid m_i(\theta) < \phi_i^- \leq \phi_i^+ < M_i(\theta),\ i = 1,2,3.\}$$

is nonempty and $\sup B = 1$.

Clearly, B is nonempty, since $0 \in B$. Indeed, it follows from Lemma 4.1 and the definitions of upper-lower-solutions that

$$\begin{aligned}
m_1(0) &= b_1 - \varepsilon < b_1 \leq \phi_1^- \leq \phi_1^+ \leq \beta_1 < \beta_1 + \varepsilon = M_1(0),\\
m_2(0) &= b_2 - \varepsilon < b_2 \leq \phi_2^- \leq \phi_2^+ \leq \beta_2 < \beta_2 + \varepsilon = M_2(0),\\
m_3(0) &= b_3 - \varepsilon^2 < b_3 \leq \phi_3^- \leq \phi_3^+ \leq \beta_3 < \beta_3 + \varepsilon^2 = M_3(0).
\end{aligned}$$

Hence $0 \in B$ and $B \neq \emptyset$.

To show $\sup B = 1$, we argue by a contradiction and assume that $\sup B = \theta_0 \in (0,1)$. For notational simplicity, we omit the index and let $\theta = \sup B \in (0,1)$. We also set $l_i = l_i(\theta)$ and $r_i = r_i(\theta)$, $i = 1,2,3$, by

$$\begin{aligned}
l_1 &:= -1 - m_1(\theta) - kM_2(\theta) + am_3(\theta), \quad r_1 := -1 - M_1(\theta) - km_2(\theta) + aM_3(\theta),\\
l_2 &:= -1 - hM_1(\theta) - m_2(\theta) + am_3(\theta), \quad r_2 := -1 - hm_1(\theta) - M_2(\theta) + aM_3(\theta),\\
l_3 &:= 1 - bM_1(\theta) - bM_2(\theta) - m_3(\theta), \quad r_3 := 1 - bm_1(\theta) - bm_2(\theta) - M_3(\theta).
\end{aligned}$$

Then it is easy to verify, using condition (4.3), that

$$\begin{aligned}
l_1 \geq &(1-\theta)[\varepsilon(1-k-a\varepsilon)] > 0,\ r_1 = -(1-\theta)[kb_2 + \varepsilon(1-k-a\varepsilon)] < 0,\\
l_2 \geq &(1-\theta)[\varepsilon(1-h-a\varepsilon)] > 0,\ r_2 = -(1-\theta)[hb_1 + \varepsilon(1-h-a\varepsilon)] < 0,\\
l_3 = &\ (1-\theta)\{[a\beta - 2b(a-1)] - \varepsilon(2b-\varepsilon)\} > 0,\\
r_3 = &-(1-\theta)[b(b_1 + b_2 - 2\varepsilon) + \varepsilon^2] < 0.
\end{aligned}$$

Now, by passing to the limit, we have

$$m_i(\theta) \leq \phi_i^- \leq \phi_i^+ \leq M_i(\theta), \quad i = 1,2,3,$$

for $\theta = \sup B$. Then one of the following equalities must hold

$$\phi_i^- = m_i(\theta),\ \phi_i^+ = M_i(\theta), \quad i = 1,2,3,$$

since $\theta \notin B$.

Finally, we derive a contradiction as follows. We only treat the case that $\phi_1^- = m_1(\theta)$, since the other five cases are similar. Suppose that $\phi_1^- = m_1(\theta)$. If ϕ_1 is ultimately monotone, then $\phi_1(\infty) = m_1(\theta)$. By integrating (1.5) from 0 to n for any $n \in \mathbb{N}$, we obtain

$$-d_1\phi_1'(n)+d_1\phi_1'(0)+s\phi_1(n)-s\phi_1(0) = r_1\int_0^n \phi_1(z)(-1-\phi_1-k\phi_2+a\phi_3)(z)dz. \tag{4.4}$$

Since

$$\liminf_{z\to\infty}\{\phi_1(z)(-1-\phi_1-k\phi_2+a\phi_3)(z)\} \geq m_1(\theta)l_1(\theta) > 0,$$

the right-hand side of (4.4) tends to infinity as $n \to \infty$. But, the left-hand side of (4.4) is bounded uniformly for all $n \in \mathbb{N}$, a contradiction.

On the other hand, suppose that $\phi_1(z)$ is oscillatory as $z \to \infty$. We then choose a sequence of local minimal points $\{z_n\}$ of ϕ_1 such that $z_n \to \infty$ and $\phi_1(z_n) \to m_1(\theta)$ as $n \to \infty$. Note that $d_1\phi_1''(z_n) - s\phi_1'(z_n) \geq 0$ for all n. But,

$$\liminf_{n\to\infty}\{\phi_1(z_n)(-1-\phi_1-k\phi_2+a\phi_3)(z_n)\} \geq m_1(\theta)l_1(\theta) > 0,$$

again a contradiction. We conclude that $\sup B = 1$ and so (4.1) is proved.

5. Determination of the minimal wave speed

In this section, we show that there are no traveling wave solutions with speed $s < s^*$ connecting $(0,0,1)$ to (u^*,v^*,w^*) for system (1.1)-(1.3). Hence we conclude that s^* is the minimal speed for traveling waves connecting $(0,0,1)$ and (u^*,v^*,w^*).

Proposition 5.1. *For $s < s^*$, system* (1.5)-(1.7) *has no positive solution such that* (1.8) *holds.*

Proof. For contradiction, suppose that there is a traveling wave solution of (1.1)-(1.3) with speed $s < s^* = \max\{2\sqrt{r_1d_1(a-1)}, 2\sqrt{r_2d_2(a-1)}\}$. Without loss of generality we may assume that $r_1d_1 \geq r_2d_2$. Hence $s < 2\sqrt{r_1d_1(a-1)}$.

First, we claim that $s > 0$. Indeed, since $(\phi_1,\phi_2,\phi_3)(-\infty) = (0,0,1)$ and $a-1 > 0$, there is a sufficiently large constant $R > 0$ such that

$$-1-\phi_1(y)-k\phi_2(y)+a\phi_3(y) \geq (a-1)/2, \ \forall\, y \leq -R.$$

By integrating (1.5) from $-\infty$ to any $z \le -R$, we obtain

$$\begin{aligned}\frac{r_1(a-1)}{2}\int_{-\infty}^{z}\phi_1(y)dy &\le r_1\int_{-\infty}^{z}\{\phi_1(-1-\phi_1-k\phi_2+a\phi_3)\}(y)dy\\ &= -d_1\phi_1'(z)+s\phi_1(z)\le -d_1\phi_1'(z),\end{aligned}$$

if $s \le 0$. Integrating over z from $-\infty$ to any $-R$, we deduce that

$$\frac{r_1(a-1)}{2}\int_{-\infty}^{-R}\int_{-\infty}^{z}\phi_1(y)dydz \le -d_1\phi_1(-R) < 0,$$

a contradiction. This shows that $s > 0$.

Then the proposition can be proved as that of [5, Theorem 2.6] with the help of the spreading property for logistic parabolic scalar equation derived by [1, 2].

For reader's convenience, we provide some details as follows. First, we choose a positive constant ϵ such that $\delta := a(1-\epsilon) - k\epsilon - 1 > 0$ and $s < 2\sqrt{r_1 d_1 \delta}$. Note that the function $(u, v, w)(x, t) = (\phi_1, \phi_2, \phi_3)(x + st)$ satisfies

$$u_t = d_1 u_{xx} + r_1 u(-1 - u - kv + aw),\ x \in \mathbb{R},\ t > 0, \tag{5.1}$$

such that $u(x, 0) = \phi_1(x)$.

Next, due to (1.8), we can find two positive constants K_1 and K_2 such that

$$(K_1 u + w)(z) \ge 1 - \epsilon, \quad v(z) \le K_2 u(z) + \epsilon,\ \forall z \in \mathbb{R}. \tag{5.2}$$

Indeed, (5.2) can be easily seen by using (1.8), a positive lower bound for u in compact interval, the positivity of w and the boundedness of v. It follows (5.1) and (5.2) that u satisfies the inequality

$$u_t \ge d_1 u_{xx} + r_1 u\{\delta - (1 + aK_1 + kK_2)u\},\ x \in \mathbb{R}, t > 0.$$

Finally, setting $y(t) := -(2\sqrt{r_1 d_1 \delta} + s)t/2$, we obtain from [1] and the comparison principle that

$$\liminf_{t\to\infty} u(y(t), t) \ge \frac{a(1-\epsilon) - k\epsilon - 1}{1 + aK_1 + kK_2} > 0,$$

since $|y(t)| < 2\sqrt{r_1 d_1 \delta}t$ for all $t > 0$ (using $s > 0$ and $s < 2\sqrt{r_1 d_1 \delta}$). However, since

$$y(t) + st = \frac{s - 2\sqrt{r_1 d_1 \delta}}{2}t \to -\infty \quad \text{as } t \to \infty,$$

we have $u(y(t), t) = \phi_1(y(t) + st) \to 0$ as $t \to \infty$, a contradiction. This proves the proposition. $\square$

Remark 5.2. We remark that the non-existence result also holds under a weaker assumption on the right asymptotics than in (1.8), for example,

$$\liminf_{z\to\infty}\phi_1(z)>0,\ 0<\liminf_{z\to\infty}\phi_2(z)\le\limsup_{z\to\infty}\phi_2(z)<\infty,\ \liminf_{z\to\infty}\phi_3(z)>0.$$

Indeed, (5.2) holds under the above weaker condition and so Proposition 5.1 follows.

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