RECENT DEVELOPMENTS ON WAVE PROPAGATION IN 2-SPECIES COMPETITION SYSTEMS

JONG-SHENQ GUO AND CHANG-HONG WU

ABSTRACT. In this paper, we shall survey some recent results on the wave propagation in 2-species competition systems with Lotka-Volterra type nonlinearity. This includes systems with continuous and discrete diffusion (or migration). We are interested in both monostable case and bistable with strong competition case. Questions on minimal speed for the monostable case, uniqueness of wave speed and propagation failure in the bistable case, monotonicity and uniqueness of wave profile for both cases are addressed. Finally, we give some open problems on wave propagation in 2-species competition systems.

1. INTRODUCTION

The classical Lotka-Volterra competition-diffusion system for two species in a 1-d habitat (e.g., along a river) can be written by the following system:

$$u_t = d_1 u_{xx} + u(1-u-kv), \quad x,t \in \mathbb{R}, \tag{1.1}$$

$$v_t = d_2 v_{xx} + rv(1-v-hu), \quad x,t \in \mathbb{R}, \tag{1.2}$$

where $u(x,t), v(x,t)$ are populations of two competing species, d_1, d_2 are diffusion coefficients of species u, v and h, k are (inter-specific) competition coefficients of species u, v, respectively. Here we assume that, by taking suitable scales of population, the carrying capacity is 1 for each species. Also, the intrinsic growth rate of species u is normalized to be 1 and the intrinsic growth rate of species v is given by r. All parameters d_1, d_2, h, k, r are assumed to be positive.

The global dynamics for the related kinetic system:

$$u_t = u(1-u-kv), \quad t \in \mathbb{R},$$
$$v_t = rv(1-v-hu), \quad t \in \mathbb{R},$$

can be easily described. In fact, there are always three equilibria $(0,0), (1,0), (0,1)$. In the case when both $h, k < 1$ or $h, k > 1$, we have the fourth equilibrium (co-existence state)

$$(u^*, v^*) = \left(\frac{1-k}{1-hk}, \frac{1-h}{1-hk} \right).$$

This work is partially supported by the National Science Council of Taiwan under the grant NSC 99-2115-M-032-006-MY3. We would like to thank the anonymous referee for the valuable suggestions which improve the presentation of this paper.

By a phase plane analysis in the first quadrant $\{u, v > 0\}$, we have

(A) $(u,v)(t) \to (1,0)$ as $t \to \infty$, if $0 < k < 1 < h$;
(B) $(u,v)(t) \to (0,1)$ as $t \to \infty$, if $0 < h < 1 < k$;
(C) $(u,v)(t) \to$ one of $\{(1,0),(0,1),(u^*,v^*)\}$ as $t \to \infty$, if $h,k > 1$, depending on the initial value;
(D) $(u,v)(t) \to (u^*,v^*)$ as $t \to \infty$, if $0 < h,k < 1$.

We remark that, for case (C), there exists an invariant separatrix such that solutions of the kinetic system with its initial data on this seperatrix converge to (u^*,v^*) as $t \to \infty$ (cf. [31]).

We say that the species u is a strong (weak, resp.) competitor if $h > 1$ ($h < 1$, resp.). When $h,k > 1$ it is the case of strong competition. In this case, it is also called bistable case, since both $(1,0)$ and $(0,1)$ are stable. We call the case when $h,k < 1$ the weak competition (co-existence) case. For the case $0 < k < 1 < h$ (or, $0 < h < 1 < k$), one species is superior than the other. In this case, there is only one stable equilibrium and we call it the monostable case. In this survey, we shall concentrate on the monostable case (A) and the bistable case (C). The case (B) is similar to the case (A).

One interesting research topic is the dynamics with diffusion effect. In this aspect, we are interested in the *traveling wavefront* solution of (1.1)-(1.2) connecting $(0,1)$ and $(1,0)$ in the form

$$(u,v)(x,t) = (\varphi,\psi)(\xi), \quad \xi := x + ct,$$

where c is the wave speed and (φ,ψ) is the wave profile. This reduces the problem (1.1)-(1.2) into the following 4-d dynamical system:

$$c\varphi' = d_1\varphi'' + \varphi(1 - \varphi - k\psi),\ \xi \in \mathbb{R},$$
$$c\psi' = d_2\psi'' + r\psi(1 - \psi - h\varphi),\ \xi \in \mathbb{R},$$

with boundary conditions

$$(\varphi,\psi)(-\infty) = (0,1), \quad (\varphi,\psi)(+\infty) = (1,0).$$

There are many interesting questions to be asked in this topic. For example, what is the wave speed in the strong competition bistable case? Can one determine the sign of this wave speed? Are there stationary solutions and does propagation failure occur? Can we characterize the minimal speed of invading by superior species in monostable case? Are wavefront profiles for each admissible speed unique (up to translations)? Can we say something about the stability of traveling wavefront?

For PDE system (1.1)-(1.2), there have been many studies in past years. For the positive stationary solutions in bistable case, we refer the reader to Kan-on [35] for the instability of stationary solutions and Kan-on [33] for the standing waves. The existence of wavefront in bistable case can be found in the works of Gardner [18] (using degree theory) and Conley & Gardner [15] (using Morse index). The uniqueness of wave speed is given by Kan-on [32] and some studies of stability of wavefront in bistable case can be found in the papers of Gardner [18] and Kan-on & Fang [36].

For the wavefront in monostable case, the existence of wavefront is carried out by Hosono [28] using a singular perturbation analysis. Then the existence of the minimal speed is proved by Kan-on [34]. We also refer to Okubu, Maini, Williamson & Murray [47], Hosono [29, 30] and Lewis, Li & Weinberger [39] for the question of the minimal speed.

Other than traveling wavefront solutions, there are solutions with two fronts approaching each other from both ends of the real line. These are the so-called two-front entire solutions and they are constructed by Morita & Tachibana [45]. Here an entire solution means a solution defined for all $x, t \in \mathbb{R}$.

When we divide the habitat into discrete regions or niches, the continuous model (1.1)-(1.2) is reduced to the following 2-component lattice dynamical system (LDS):

$$u_j'(t) = d_1\mathcal{D}_2[u_j](t) + u_j(t)[1 - u_j(t) - kv_j(t)], \tag{1.3}$$

$$v_j'(t) = d_2\mathcal{D}_2[v_j](t) + rv_j(t)[1 - v_j(t) - hu_j(t)], \tag{1.4}$$

where $\mathcal{D}_2[u_j] := (u_{j+1} - u_j) + (u_{j-1} - u_j)$, $j \in \mathbb{Z}$, $t \in \mathbb{R}$. In fact, the system (1.3)-(1.4) can be thought as a spatial discrete version of the continuous model (1.1)-(1.2). Notice that the diffusion coefficients d_1, d_2 in (1.3)-(1.4) are different from the ones in (1.1)-(1.2). Continuous model is actually the limiting model of LDS. It is believed that LDS is more realistic than the continuous model, since we can only measure the populations at discrete points.

As in the continuous case, the traveling wavefront for LDS can be put in the form

$$(u_j(t), v_j(t)) = (U(\xi), V(\xi)), \quad \xi = j + ct.$$

It connects from $(0, 1)$ to $(1, 0)$. Here again c is the wave speed and U, V are the wave profiles. This renders the problem (P): to find $(c, U, V) \in \mathbb{R} \times C^1(\mathbb{R}) \times C^1(\mathbb{R})$ such that

$$\begin{cases} cU' = d_1D_2[U] + U(1 - U - kV), \\ cV' = d_2D_2[V] + rV(1 - V - hU), \\ (U, V)(-\infty) = (0, 1), \quad (U, V)(+\infty) = (1, 0), \\ 0 \le U, V \le 1, \end{cases} \tag{1.5}$$

where $D_2[\varphi](\xi) := \varphi(\xi + 1) + \varphi(\xi - 1) - 2\varphi(\xi)$ for $\varphi = U, V$. Note that problem (P) is an infinitely dimensional dynamical system due to the nonlocal operator D_2.

Before the discussion of 2-species competition system, we first give a very brief survey on LDS for single species as follows.

The investigation on wave propagation of lattice dynamical systems for one-species has a long history, tracking back to the work of Bell [3]. In [3], the author proposed the discrete bistable (Nagumo-type) equation as a model for conduction in myelinated nerve axons:

$$\frac{du_j}{dt} = d(u_{j+1} + u_{j-1} - 2u_j) + f(u_j),\ j \in \mathbb{Z}, \tag{1.6}$$

where $d > 0$ stands for diffusion(migration) rate and f is Lipschitz continuous satisfying $f(0) = f(a) = f(1) = 0$, $f(u) < 0$ for $0 < u < a$ and $f(u) > 0$ for $a < u < 1$. In contrast to its continuum case, there is no wave propagation of (1.6) when the diffusion rate is small enough. This phenomenon is called propagation failure, established by Keener [37]. Subsequently the existence, uniqueness (up to translations) and stability of traveling wavefronts with nonzero wave speed was proved by Zinner [53, 54]. Note that the (nonzero) wave speed is unique. For related works on (1.6) for the bistable case, see, e.g., [27, 16, 14, 4, 6, 44, 8, 7, 5] and references therein.

For the monostable case, i.e., when f is a Lipschitz continuous function with $f(0) = 0 = f(1)$ and $f(u) > 0$ for $0 < u < 1$, under the nondegeneracy and the so-called Fisher-KPP assumption that $f(u) \le f'(0)u$ for all $u \in [0, 1]$, Zinner, Harris, and Hudson [55] established the existence of traveling wavefronts. More recently, Chen and Guo [11, 12] generalized the work of [55] and studied the stability, monotonicity, and uniqueness (up to translations) of traveling wavefronts. One should note that due to the monostable nonlinearity the admissible wave speed is a semi-infinite interval. Also, the uniqueness (up to translations) of wave profile is for each admissible wave speed. For related works on monostable case, see, e.g., [17, 10, 42, 21] and references therein. In particular, [10] allows the nonlinearity having degeneracy at both equilibria and [21] is for higher dimensional lattice.

The aforementioned works are for the homogeneous media. For the periodic media, we refer the reader to [19] for the monostable nonlinearity and [13] for the bistable nonlinearity. See also [26, 23].

From now on, we shall concentrate on the 2-species competition system. In extending the results on single species to 2-species, there arises a certain degree of difficulties. We shall point out these difficulties case by case in the following sections.

We organize this paper as follows. In section 2, we survey the results of wavefront of problem (1.5) in the bistable case. The results on the monostable case are summarized in section 3. Some comments and remarks shall also be given. Then a brief discussion for the existence of two-front entire solutions for both monostable and bistable cases is given in section 4. In section 5, we discuss the minimal speed of PDE model (1.1)-(1.2). Finally, we give some open problems in this research area in section 6.

2. Wavefront in bistable case

In this section, we assume the nonlinearity is bistable, i.e., $h, k > 1$. First, we have the following theorems on the propagation failure and the existence of stationary solutions.

Theorem 1 ([24]). *Given $r > 0, h, k > 1$. When d_1 and d_2 are small enough, there is no traveling wavefront solution of (1.3)-(1.4) with nonzero speed connecting $(0, 1)$ and $(1, 0)$.*

Theorem 2 ([24]). *Given $r > 0, h, k > 1$. Then there are infinitely many stationary solutions of (1.3)-(1.4), provided d_1 and d_2 are small enough.*

More precisely, let us set two rectangles

$$I_1 := [0, x_1] \times [y_1, 1] \text{ and } I_2 := [x_2, 1] \times [0, y_2],$$

where $y_1 \in (\max\{1/2, 1/k\}, 1)$, $x_2 \in (\max\{1/2, 1/h\}, 1)$. Choose x_1 and y_2 such that $0 < x_1, y_2 \ll 1$ so that the following conditions

$$(2u - 1)(2v - 1) + hu(2u - 1) + kv(2v - 1) > 0, \ (u, v) \in I_1 \cup I_2,$$
$$x_1 < \frac{1 - y_1}{h}, \ y_2 < \frac{1 - x_2}{k}$$

hold. For such I_1 and I_2, if d_1 and d_2 are small enough, then (1.3)-(1.4) has a unique stationary solution $\{(u_j, v_j)\}_{j\in\mathbb{Z}}$ such that $(u_j, v_j) \in I_{s_j}$ for all $j \in \mathbb{Z}$ for any given infinite sequence $\{s_j\}_{j\in\mathbb{Z}}$ with $s_j \in \{1, 2\}$ for all $j \in \mathbb{Z}$.

For 1-component LDS with bistable nonlinearity, that weak coupling (or small migration coefficient) implies the existence of stationary solutions and propagation failure can be found in [37, 44, 8]. When $d_1, d_2 \ll 1$, the species almost do not have migration tendencies. Intuitively, propagation failure occurs. Theorem 1 is proved by constructing two invariant sets as I_1, I_2, and using the comparison principle. From the biological point of view, the stationary solutions we constructed in Theorem 2 has the property that these two species do not like to live together due to the strong competition. A similar result to Theorem 2 can be found in the work of [43] by using the implicit function theorem. Our approach here (based on the Smale horseshoe theory, cf. [37, 48]) gives more information on the behavior of stationary solutions.

Next, we have the following theorems on the monotonicity of wave profiles and the uniqueness of nonzero wave speed.

Theorem 3 ([24]). *Given $h, k > 1$ and $r, d_1, d_2 > 0$. The wave profiles of any solution (c, U, V) of (P) with nonzero speed are strictly monotone, i.e., $U' > 0$ and $V' < 0$ in $\mathbb{R}$.*

Theorem 4 ([24]). *Given $h, k > 1$ and $r, d_1, d_2 > 0$. Let (c_i, U_i, V_i), $i = 1, 2$, be two arbitrary solutions of (P) with nonzero speeds. Then $c_1 = c_2$.*

The proofs of Theorems 3 and 4 rely on a detailed study of asymptotic behaviors of wave tails. The main idea of deriving the asymptotic behaviors of wave tails is to construct some auxiliary functions to compare with wave profiles. The idea is from an original idea of the work [10] for monostable one component LDS. But, for the 2-species case we need to understand the exact behavior of $U/(1-V)$ near $(U,V)=(0,1)$ and the method for the single species case cannot be applied directly. The main difficulty is that $U(\xi)/(1-V(\xi))$ can tend to zero as $\xi\to-\infty$ which means that the decay rate of wave tail of $1-V$ can be different from that of U. Therefore, the issue of the uniqueness becomes more challenging. For the uniqueness of wave speed, unfortunately we cannot exclude the case that both nonzero and zero speeds co-exist. We expect that the speed is unique and it is either nonzero or zero, but not both. This is left for an open problem.

For the existence of wavefront, we refer the reader to [51] for more details. The stability and uniqueness (up to translations) of wave profiles is under investigation.

3. Wavefront in monostable case

In this section, we consider the wavefronts in the monostable case. For convenience, we assume the following:

$$(\mathbf{A1}):\ 0<k<1<h,\ d_1=1, d_2=d>0, \text{ and } r>0.$$

As usual, in the monostable case we expect to have the minimal speed which is defined by

$$c_{min}=c_{min}(r,d,h,k):=\inf\{c\in\mathbf{R}\mid (\mathrm{P}) \text{ has a solution}\}.$$

Theorem 5 ([25]). *Assume* $(\mathbf{A1})$. *Then there exists* $c_{min}>0$ *such that the problem* (P) *admits a solution* (c,U,V) *satisfying* $U'(\cdot)>0$ *and* $V'(\cdot)<0$ *on* $\mathbb{R}$ *if and only if* $c\ge c_{min}$.

It seems that the existence of solutions of problem (P) is more complicated than that of continuous case. Fortunately, by transforming the problem into a system of integral equations with monotone property, it becomes easier to derive the existence of solutions of problem (P) than the continuous case. To derive the existence of traveling wavefront, we may use the monotone iteration method with the help of a pair of super-sub-solutions (cf. e.g., [2, 52, 11]); or, consider a sequence of truncated problems with the help of a super-solution (a method developed in [12]). By applying the aforementioned methods, we can easily obtain a limit function satisfying our 2-component LDS. The main difficulty here for the existence, in comparing with the single species case, is to check that this limit function satisfies the desired boundary conditions.

The next question is to characterize the minimal speed by parameters in the problem. Let $0 < k < 1$, we define the positive number:

$$c_* = c_*(k) := \min_{\lambda>0} \left\{ \frac{(e^\lambda + e^{-\lambda} - 2) + (1-k)}{\lambda} \right\}.$$

We explain why we have such a definition as follows. Linearizing the U-equation

$$cU' = D_2[U] + U(1 - U - kV)$$

around the unstable state $(0, 1)$, we get the characteristic equation

$$\Phi(c, \lambda) := c\lambda - [(e^\lambda + e^{-\lambda} - 2) + (1-k)] = 0. \quad (3.1)$$

Note that c_* is independent of d and r, h. Moreover, (3.1) has at least one positive real root if and only if $c \geq c_*$. Then we have the following characterization of the minimal speed for certain ranges of h, k, r, d.

Theorem 6 ([25, 20])**.** *Assume* (**A1**). *Then $c_{min} \geq c_*(k)$. Moreover, there exists a constant $d_* = d_*(k) > 2$ such that $c_{min} = c_*(k)$, provided that $d \leq d_*$ and $(h, k, r, d) \in A_1 \cup A_2 \cup A_3$, where*

$$A_1 := \{d \in (0, d_*], hk \leq 1, r > 0\},$$
$$A_2 := \left\{ d \in (0, 1], hk > 1, 0 < r \leq \frac{1-k}{hk-1} \right\},$$
$$A_3 := \left\{ d \in (1, d_*), hk > 1, 0 < r \leq \frac{d_* - d}{d_* - 1} \frac{1-k}{hk-1} \right\}.$$

Indeed, the lower bound estimate follows from

Lemma 3.1 ([25])**.** *Let (c, U, V) be a solution of (P). Then $c > 0$ and the limit*

$$\lim_{\xi \to -\infty} \frac{U'(\xi)}{U(\xi)} = \Lambda(c)$$

exists and $\Lambda(c)$ is a positive root of the equation (3.1).

The second part of Theorem 6 was proved by constructing a pair of super-sub-solutions. The restrictions on the parameters in Theorem 6 are due to the lack of suitable super-solutions. We shall come back to the discussion of the minimal speed later in this paper.

As in the bistable case, we also have the monotonicity of wave profiles as follows.

Theorem 7 ([25])**.** *Assume* (**A1**). *Then all wave profiles are strictly monotone.*

The proof is also based on a detailed analysis of wave tails and employing the sliding method. As in the bistable case, the main difficulty here, in comparing with the one component lattice dynamical system, is the lack of exact information about the limit of $U/(1-V)$ as $\xi \to -\infty$.

Finally, the uniqueness of wave profile for a given admissible speed can be proved, at least for $d \in (0, 1]$.

Theorem 8 ([25]). *Assume* (**A1**) *and* $d \leq 1$. *Then the wave profile is unique up to translations for a given wave speed* $c \geq c_{min}$.

Here we first use the bilateral Laplace transform and a modified version of Ikehara's Theorem (an idea from Carr-Chmaj [9]) to derive the more precise exponential tails of wave profiles. Then a sliding method [12] can be applied with the help of the strong comparison principle to finish the proof of Theorem 8. Unfortunately, we were unable to prove the uniqueness for $d > 1$ due to some technical reasons. We left this case as an open problem.

4. Two-front entire solutions

In this section, we briefly describe some results on the two-front entire solutions in the 2-component lattice dynamical system. When $0 < k < 1 < h$, the species u is stronger than v and the species u invades v so that eventually v will be extinct. To know how the stronger species invades the weaker one, the study of entire solutions is an important issue. In particular, the 2-front entire solutions behave as two traveling wavefronts moving towards each other from both sides of space axis. This provides another invasion way of the stronger species to the weaker one.

To study the 2-front entire solution, it is convenient to consider the following related continuum system:

$$u_t(x,t) = D_2[u(\cdot,t)](x) + [u(1-u-kv)](x,t), \tag{4.1}$$

$$v_t(x,t) = dD_2[v(\cdot,t)](x) + [rv(1-v-hu)](x,t) \tag{4.2}$$

for $x, t \in \mathbb{R}$, where $d, r, h, k > 0$ and

$$D_2[w](x) := w(x+1) + w(x-1) - 2w(x).$$

Theorem 9 ([22]). *Assume* (**A1**). *Let* (c_i, U_i, V_i) *be a solution of* (P) *satisfying*

$$\frac{U_i(\xi)}{1 - V_i(\xi)} \geq \eta_0 \quad \text{for all} \quad \xi \leq 0. \tag{4.3}$$

and let θ_i *be a given constant,* $i = 1, 2$. *Then there exists an entire solution* $(u(x,t), v(x,t))$ *of* (4.1)-(4.2) *such that*

$$\lim_{t\to-\infty} \sup_{x\geq(c_2-c_1)t/2} \{|u(x,t) - U_1(x+c_1t+\theta_1)| + |v(x,t) - V_1(x+c_1t+\theta_1)|\} = 0,$$

$$\lim_{t\to-\infty} \sup_{x\leq(c_2-c_1)t/2} \{|u(x,t) - U_2(-x+c_2t+\theta_2)| + |v(x,t) - V_2(-x+c_2t+\theta_2)|\} = 0,$$

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \{|1-u(x,t)| + |v(x,t)|\} = 0.$$

In [45], the assumption (4.3) is crucial in constructing two-front entire solutions. Also, they provide some conditions via the eigenvalues of the linearized system around equilibria $(0,1)$ and $(1,0)$ to assure (4.3) holds. Indeed, for our discrete problem, the condition (4.3) holds under the extra condition $0 < d \le 1$ (see [25, Remark 3.1]).

By a similar argument as in [22], we can also prove a similar theorem on two-front entire solutions for the bistable case. We omit the details here.

5. Minimal speed for PDE model

In this section, we consider the continuous model (1.1)-(1.2) with $d_1 = 1, d_2 = d$. We also define the minimal speed of traveling fronts of (1.1)-(1.2) by

$$c_{min} = \inf\{c > 0 \mid \text{ traveling front with speed } c \text{ exists }\}.$$

In [34], Kan-on proved that $c_{min} \ge 2\sqrt{1-k}$. Then Hosono [29] conjectured that:

$$0 < k < 1 < h \Rightarrow c_{min} = 2\sqrt{1-k},$$

under the condition

$$(h,k,r) \in \{hk \le 1, r > 0\} \cup \{hk > 1, 0 < r \le r_*\}$$

for some $r_* = r_*(h,k,d) > 0$ for certain d.

The constant $2\sqrt{1-k}$ is actually from evaluating $2\sqrt{f_u(0,1)}$ at the unstable state $(0,1)$, where $f(u,v) := u(1-u-kv)$ is the nonlinearity defined in the u-equation. This is precisely the same as the case of single KPP-equation [38], namely, the minimal speed is given by $2\sqrt{f'(0)}$ for the equation $u_t = u_{xx} + f(u)$ under certain conditions on f. Recently, Lewis, Li, Weinberger [39] proved that $c_{min} = 2\sqrt{1-k}$, if $0 < d \le 2$ and $rhk \le r + (2-d)(1-k)$, i.e., $(h,k,r,d) \in B_1 \cup B_2$, where

$$B_1 := \{d \in (0,2], hk \le 1, r > 0\},$$
$$B_2 := \left\{d \in (0,2], hk > 1, 0 < r \le \frac{(2-d)(1-k)}{hk-1}\right\}.$$

A natural question is whether this is an optimal result for the so-called *linear determinacy.* Here the linear determinacy means that the minimal speed ($2\sqrt{f_u(0,1)}$ or $2\sqrt{f'(0)}$) is exactly given by the quantity ($f_u(0,1)$ or $f'(0)$) from the linearization of the associated equation around the unstable equilibrium ($(0,1)$ or 0). In fact, the notion of minimal speed is closely related to the so-called *spreading speed.* We shall not address this issue here and only refer the reader to, e.g., [1, 49, 50, 39, 40, 41].

It will be of interest to investigate whether the linear determinacy hold. Murray predicted that the minimal speed is $2\sqrt{1-k}$ by a heuristic argument [46]. Later, Hosono [29] gave a numerical example showing that minimal speed can be larger than $2\sqrt{1-k}$. This gave a counter-example to Murray's conjecture. To the best of our knowledge, there is no analytical results for Hosono's conjecture. However, Hosono's results suggest that the linear determinacy may not hold.

Through the study on LDS, we are able to extend the result of [39]. For this, using the finite difference scheme, we can approximate the system (1.1)-(1.2) by the following discretized system (P_τ):

$$\begin{cases} \hat{u}_j'(t) = \tau^{-2}[\hat{u}_{j+1}(t) + \hat{u}_{j-1}(t) - 2\hat{u}_j(t)] + \hat{u}_j(t)(1 - \hat{u}_j(t) - k\hat{v}_j(t)), \\ \hat{v}_j'(t) = d\tau^{-2}[\hat{v}_{j+1}(t) + \hat{v}_{j-1}(t) - 2\hat{v}_j(t)] + r\hat{v}_j(t)(1 - \hat{v}_j(t) - h\hat{u}_j(t)), \end{cases}$$

$j \in \mathbb{Z}$, $t \in \mathbb{R}$, where $\hat{u}_j(t) := \hat{u}(j\tau, t)$, $\hat{v}_j(t) := \hat{v}(j\tau, t)$ and τ is the spatial mesh size. It follows from Theorem 6 that the minimal speed of (P_τ) is given by

$$c_*(k;\tau) = \min_{\lambda>0} \left\{ \frac{\tau^{-2}(e^\lambda + e^{-\lambda} - 2) + (1-k)}{\lambda} \right\}$$

under the assumptions that $0 < k < 1 < h$, $0 < d \le d_*$, $r > 0$ and $(h,k,r,d) \in A_1 \cup A_2 \cup A_3$. Moreover, it is shown in [25] that

$$\tau c_*(k;\tau) \to 2\sqrt{1-k} \quad \text{as} \quad \tau \to 0^+. \tag{5.1}$$

Based on the above observations, we can prove the following theorem on the linear determinacy for the system (1.1)-(1.2). Recall the constant $d_* = d_*(k) > 2$ defined in Theorem 6.

Theorem 10 ([20]). *Suppose that $0 < k < 1 < h$, $d > 0$, and $r > 0$. If $d \le d_*$ and $(h,k,r,d) \in A_1 \cup A_2 \cup A_3$, then the minimal speed c_{min} of the traveling front of (1.1)-(1.2) is equal to $2\sqrt{1-k}$. More precisely, (1.1)-(1.2) admits a traveling front connecting $(0,1)$ and $(1,0)$ with speed c if and only if $c \ge 2\sqrt{1-k}$.*

This theorem gives some extension of the result of [39]. Note that $d_*(k) > 2$. Clearly, $B_1 \subset A_1$. Writing $B_2 = B_{20} \cup B_{21}$ with

$$B_{20} := \left\{ d \in (0,1], hk > 1, 0 < r \le \frac{(2-d)(1-k)}{hk-1} \right\},$$

$$B_{21} := \left\{ d \in (1,2], hk > 1, 0 < r \le \frac{(2-d)(1-k)}{hk-1} \right\},$$

we have $B_{21} \subset A_3 \cap \{d \in (1,2]\}$ and $A_2 \subset B_{20}$.

For reader's convenience, we give a rough idea of the proof of Theorem 10 as follows. For the details, we refer the reader to [20].

Proof of Theorem 10. It suffices to prove the existence of front for any speed $c > 2\sqrt{1-k}$. Given $c > 2\sqrt{1-k}$. First, by (5.1), there exists $\tau_0 > 0$ such that

$$\tau c_*(k;\tau) < c, \ \forall \tau \in (0, \tau_0].$$

For any speed c/τ, by Theorem 6, (P_τ) has a traveling front $(c/\tau, \tilde{U}^{\tau,c}, \tilde{V}^{\tau,c})$, if $\tau \in (0, \tau_0]$. Take $\tau_n = 1/n$ and $c_n \downarrow c$ as $n \to \infty$. Let

$$(\tilde{U}_n, \tilde{V}_n) = (\tilde{U}^{\tau_n,c_n}, \tilde{V}^{\tau_n,c_n}), \quad (U_n, V_n)(\xi) = (\tilde{U}_n, \tilde{V}_n)(\xi/\tau_n).$$

Then (U_n, V_n) satisfies

$$c_n(U_n)'(\xi) = n^2[U_n(\xi + 1/n) + U_n(\xi - 1/n) - 2U_n(\xi)] + U_n(\xi)[1 - U_n(\xi) - kV_n(\xi)],$$
$$c_n(V_n)'(\xi) = n^2 d[V_n(\xi + 1/n) + V_n(\xi - 1/n) - 2V_n(\xi)] + rV_n(\xi)[1 - V_n(\xi) - hU_n(\xi)].$$

Next, we use the discrete Fourier transform and the variation-of-constant formula to derive the equi-continuity of (U_n, V_n). Then, by taking a suitable subsequence of (U_n, V_n) and passing to the limit, the limit function (φ, ψ) satisfies (1.1)-(1.2).

Finally, we show that (φ, ψ) connecting $(0, 1)$ to $(1, 0)$ from $-\infty$ to ∞. This completes the proof. □

6. Some open problems

In this section, we list some interesting open problems in the study of discrete 2-species competition system as follows.

(1) The uniqueness (up to translations) of traveling wavefront for the monostable case with $d > 1$ and the bistable case.
(2) The stability of traveling wave solutions for both monostable and bistable cases.
(3) In the bistable case, it is important to determine the sign of the (unique) wave speed. This would tell us which species wins the competition. The question is to determine how the parameters in the system influence the sign of the speed.
(4) In the bistable case, it is expected that there exists a critical value d^* so that the propagation failure occurs when $d < d^*$ and nonzero speed traveling wave solutions exist when $d > d^*$. It will be very interesting to characterize the d^* by the parameters in the system.
(5) Concerning 2-front entire solutions, it would be nice if one could construct 2-front entire solutions for our LDS by relaxing the boundedness assumption on $[1 - V_i]/U_i$. Moreover, can we find some other new type entire solutions? This would give us different invading/competiton phenomena.
(6) In the bistable case, it is interesting to determine the stability of the non-monotone stationary solutions (when the migration rates are small) which we have constructed.

For the continuous PDE monostable case, a very difficult and interesting question is to determine the optimal condition for the linear determinacy by the parameters of the system. Note that all of the above results are for the homogeneous environment. However, in reality the environment is always heterogeneous. Therefore, it is very important to study the competition system when the environment is heterogeneous.

References

[1] D.G. Aronson, H.F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, in Partial Differential Equations and Related Topics, ed. J. A. Goldstein, Lecture Notes in Mathematics, **446**, pages 5-49. Springer-Verlag, Berlin, 1975.

[2] C. Atkinson, G.E.H. Reuter, *Deterministic epidemic waves*, Math. Proc. Cambridge Philos. Soc. **80** (1976), 315–330.

[3] J. Bell, *Some threshold results for models of myelinated nerves*, Math. Biosci. **54** (1981), 181–190.

[4] P.W. Bates, A. Chmaj, *A discrete convolution model for phase transitions*, Arch. Rational Mech. Anal. **150** (1999), 281–305.

[5] P.W. Bates, X. Chen, A. Chmaj, *Travelling waves of bistable dynamics on a lattice*, SIAM J. Math. Anal. **35** (2003), 520–546.

[6] J.W. Cahn, J. Mallet-Paret, E. Van Vleck, *Traveling wave solutions of ODEs on a two-dimensional spacial lattice*, SIAM J. Appl. Math. **59** (1999), 455–493.

[7] A. Carpio, L.L. Bonilla, *Depinning transitions in discrete reaction-diffusion equations*, SIAM J. Appl. Math. **63** (2003), 1056–1082.

[8] A. Carpio, S.J. Chapman, S.P. Hastings, J.B. McLeod, *Wave solutions for a discrete reaction-diffusion equation*, Eur. J. Appl. Math. **11** (2000), 399–412.

[9] J. Carr, A. Chmaj, *Uniqueness of travelling waves for nonlocal monostable equations*, Proc. Amer. Math. Soc. **132** (2004), 2433–2439.

[10] X. Chen, S.-C. Fu, J.-S. Guo, *Uniqueness and asymptotics of traveling waves of monostable dynamics on lattices*, SIAM J. Math. Anal. **38** (2006), 233–258.

[11] X. Chen, J.-S. Guo, *Existence and asymptotic stability of travelling waves of discrete quasilinear monostable equations*, J. Differential Equations **184** (2002), 549–569.

[12] X. Chen, J.-S. Guo, *Uniqueness and existence of travelling waves for discrete quasilinear monostable dynamics*, Math. Ann. **326** (2003), 123–146.

[13] X, Chen, J.-S. Guo, C.-C. Wu, *Traveling waves in discrete periodic media for bistable dynamics*, Arch. Ration. Mech. Anal. **189** (2008), 189–236.

[14] S.-N. Chow, J. Mallet-Paret, W. Shen, *Traveling waves in lattice dynamical systems*, J. Differential Equations **149** (1998), 248–291.

[15] C. Conley, R. Gardner, *An application of the generalized Morse index to traveling wave solutions of a competitive reaction diffusion model*, Indiana Univ. Math. J. **33** (1984), 319–343.

[16] G. Fáth, *Propagation failure of traveling waves in a discrete bistable medium*, Physica D **116** (1998), 176–180.

[17] S.-C. Fu, J.-S. Guo, S.-Y. Shieh,*Traveling wave solutions for some discrete quasilinear parabolic equations*, Nonl. Anal. **48** (2002), 1137–1149.

[18] R.A. Gardner, *Existence and Stability of Travelling wave solutions of competition models: a degree theoretic approach*, J. Differential Equations **44** (1982), 343–364.

[19] J.-S. Guo, F. Hamel, *Front propagation for discrete periodic monostable equations*, Math. Ann. **335** (2006), 489–525.

[20] J.-S. Guo, X. Liang, *The minimal speed of traveling fronts for the Lotka-Volterra competition system*, J. Dynam. Differential Equations **23** (2011), 353–363.

[21] J.-S. Guo, C.-H. Wu, *Existence and uniqueness of traveling waves for a monostable 2-D lattice dynamical system*, Osaka J. Math. **45** (2008), 327-346.

[22] J.-S. Guo, C.-H. Wu, *Entire solutions for a two-component competition system in a lattice*, Tohoku Math. J. **62** (2010), 17-28.

[23] J.-S. Guo, C.-H. Wu, *Front propagation for a two-dimensional periodic monostable lattice dynamical system*, Discrete Conti. Dyn. Syst. **26** (2010), 197–223.

[24] J.-S. Guo, C.-H. Wu, *Wave propagation for a two-component lattice dynamical system arising in strong competition models*, J. Differential Equations **250** (2011), 3504–3533.

[25] J.-S. Guo, C.-H. Wu, *Traveling wave front for a two-component lattice dynamical system arising in competition models*, preprint.

[26] J.-S. Guo, C.-C. Wu, *Uniqueness and stability of traveling waves for periodic monostable lattice dynamical system*, J. Differential Equations **246** (2009), 3818–3833.

[27] D. Hankerson, B. Zinner, *Wavefronts for a cooperative tridiagonal system of differential equations*, J. Dyn. Differential Equations **5** (1993), 359–373.

[28] Y. Hosono, *Singular perturbation analysis of travelling waves for diffusive Lotka-Volterra competitive models*, Numerical and Applied Mathematics, Part II (Paris, 1988), Baltzer, Basel, (1989) 687–692.

[29] Y. Hosono, *The minimal speed of traveling fronts for a diffusive Lotka-Volterra competition model*, Bulletin of Math. Biology **60** (1998), 435–448.

[30] Y. Hosono, *Traveling waves for a diffusive Lotka-Volterra competition model I: singular perturbations*, Discete Contin. Dyn. Syst. Ser. B **3** (2003), 79–95.

[31] M. Iida, T. Muramatsu, H. Ninomiya, E. Yanagida, *Diffusion-induced extinction of a superior species in a competition system*, Japan J. Indust. Appl. Math. **15** (1998), 223–252.

[32] Y. Kan-on, *Parameter dependence of propagation speed of travelling waves for competition-diffusion equations*, SIAM J. Math. Anal. **26** (1995), 340–363.

[33] Y. Kan-on, *Existence of standing waves for competition-diffusion equations*, Japan J. Indust. Appl. Math. **13** (1996), 117–133.

[34] Y. Kan-on, *Fisher wave fronts for the Lotka-Volterra competition model with diffusion*, Nonlinear Anal. **28** (1997), 145–164.

[35] Y. Kan-on, *Instability of stationary solutions for a Lotka-Volterra competition model with diffusion*, J. Math. Anal. Appl. **208** (1997), 158–170.

[36] Y. Kan-on, Q. Fang, *Stability of monotone travelling waves for competition-diffusion equations*, Japan J. Indust. Appl. Math. **13** (1996), 343–349.

[37] J.P. Keener, *Propagation and its failure in coupled systems of discrete excitable cells*, SIAM J. Appl. Math. **47** (1987), 556–572.

[38] A.N. Kolmogorov, I.G. Petrovsky, N.S. Piskunov, *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un probléme biologique*, Bull. Univ. Moskov. Ser. Internat., Sect. **A 1** (1937), 1-25.

[39] M.A. Lewis, B. Li, H.F. Weinberger, *Spreading speed and linear determinacy for two-species competition models*, J. Math. Biol. **45** (2002), 219–233.

[40] B. Li, H.F. Weinberger, M.A. Lewis, *Spreading speeds as slowest wave speeds for cooperative systems*, Math. Biosci. **196** (2005), 82–98.

[41] X. Liang, X.-Q. Zhao, *Asymptotic speeds of spread and traveling waves for monotone semiflows with application*. Comm. Pure Appl. Math. **60** (2007), 1–40.

[42] S. Ma, X.-Q. Zhao, *Global asymptotic stability of minimal fronts in monostable lattice equations*, Discrete Contin. Dyn. Syst. (Ser. A) **21** (2008), 259–275.

[43] R.S. MacKay, J.-A. Sepulchre, *Multistability in networks of weakly coupled bistable units*. Physica D **82** (1995), 243-254.

[44] J. Mallet-Paret, *The global structure of traveling waves in spatially discrete dynamical systems*, J. Dyn. Diff. Eq. **11** (1999), 49-127.

[45] Y. Morita, K. Tachibana, *An entire solution to the Lotka-Volterra competition-diffusion equations*, SIAM J. Math. Anal. **40** (2009), 2217–2240.

[46] J. D. Murray, *Mathematical Biology*, New York, Springer, 1989.

[47] A. Okubo, P.K. Maini, M.H. Williamson, J.D. Murray, *On the spatial spread of the grey squirrel in Britain*, Proc. R. Soc. Lond. B, **238** (1989), 113–125.

[48] S.-F. Shieh, *Horseshoes for coupled discrete nonlinear Schrodinger equations*, J. Math. Phys. **50** 022701 (2009); doi:10.1063/1.3077226.

[49] H.F. Weinberger, *On spreading speeds and traveling waves for growth and migration models in a periodic habitat*, J. Math. Biol. **45** (2002), 511–548.

[50] H.F. Weinberger, M.A. Lewis, B. Li, *Analysis of the linear conjecture for spread in cooperative models*, J. Math. Biol. **45** (2002), 183–218.

[51] C.-C. Wu, *Existence of traveling wavefront for discrete bistable competition model*, Discrete Contin. Dyn. Syst. Ser. B. (to appear).

[52] J. Wu, X. Zou, *Asymtotic and periodic boundary value problems of mixed FDEs and wave solutions of lattice differential equations*, J. Differential Equations **135** (1997), 315–357.

[53] B. Zinner, *Stability of traveling wavefronts for the discrete Nagumo equations*, SIAM J. Math. Anal. **22** (1991), 1016–1020.

[54] B. Zinner, *Existence of traveling wavefronts for the discrete Nagumo equations*, J. Differential Equations **96** (1992), 1–27.

[55] B. Zinner, G. Harris, W. Hudson, *Traveling wavefronts for the discrete Fishers equation*, J. Differential Equations **105** (1993), 46–62.

Department of Mathematics, Tamkang University, Tamsui, New Taipei City 25137, Taiwan; and MIMS, Meiji University, 1-1-1 Higashimita, Tamaku, Kawasaki 214-8571, Japan.
E-mail address: jsguo@mail.tku.edu.tw

Department of Mathematics, Tamkang University, Tamsui, New Taipei City 25137, Taiwan
E-mail address: 896400012@ntnu.edu.tw