STABILIZATION TO A POSITIVE EQUILIBRIUM FOR SOME REACTION-DIFFUSION SYSTEMS

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Abstract. The aim of this paper is to study the asymptotic behavior of solutions for some reaction-diffusion systems in biology. First, we establish a Liouville type theorem for entire solutions of these reaction-diffusion systems. Based on this theorem, we derive the stabilization of the solutions of the reaction-diffusion system to the unique positive constant state, under the condition that this positive constant state is globally stable in the corresponding kinetic systems. Two specific examples about spreading phenomena from ecology and epidemiology are given to illustrate the application of this theory.

1. Introduction

In this paper, we consider the following reaction-diffusion system

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + f_i(u),\ x \in \mathbb{R}^N,\ t \in \mathbb{R},\ i = 1, \cdots, m, \tag{1.1}$$

where $m, N \in \mathbb{N}$, $d_i > 0$ and $f_i : \mathbb{R}^m \to \mathbb{R}$ is a C^1 function for $i = 1, \cdots, m$. System (1.1) arises in many ecological systems (such as predator-prey systems) and epidemic models. We refer the reader to [3, 19, 18, 15, 10] and the references cited therein.

Throughout this paper we assume that the corresponding kinetic system of (1.1):

$$\frac{du_i}{dt} = f_i(u),\ i = 1, \cdots, m \tag{1.2}$$

has a unique positive equilibrium $u^* := (u_1^*, \cdots, u_m^*)$. In the study of biological and/or epidemiological models, one of the fundamental questions is the long time behavior of their solutions. In particular, when the unique equilibrium u^* is globally stable for the kinetic system and the spatial dependence is taken into account, it is interesting to see whether u^* is also stable for the reaction-diffusion system (1.1). Our first challenge of this paper is to answer this question. We remark that a similar question was addressed in [15] for reaction-diffusion systems on bounded domains with zero Neumann boundary condition.

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In the following, we give two interesting examples, one in ecology and the other in epidemiology, of reaction-diffusion system (1.1).

The first example is the control of introduced rabbits to protect native birds from introduced cat predation in an island. When we consider the case without rabbits and the control(s), the full 3-species model is reduced to

$$B_t = d_b \Delta B + r_b \left(1 - \frac{B}{K}\right) B - \mu C, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.3}$$

$$C_t = d_c \Delta C + r_c \left(1 - \mu \frac{C}{B}\right) C, \qquad x \in \mathbb{R}^N,\ t > 0, \tag{1.4}$$

where $d_b, d_c, r_b, r_c, K, \mu$ are positive constants. In (1.3)-(1.4), the function B (C, resp.) denotes the population density of birds (cats, resp.), d_b (d_c, resp.) is the diffusion coefficient of birds (cats, resp.), the parameter μ is the intake of birds per individual cat per unit time and K is the carrying capacity of the birds. For the detailed biological background of the full 3-species model, we refer the reader to [4, 5]. Since the birds population $B(x,t)$ may reach zero in a finite time for some point x so that (1.4) becomes singular, we call system (1.3)-(1.4) a singular predator-prey system. For the recent studies of this singular predator-prey system, we refer the reader to [13, 9, 11, 12, 2].

The second example is the following epidemic model

$$S_t = d_1 \Delta S + \mu - \mu S - \frac{\beta S I}{1 + \alpha I}, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.5}$$

$$I_t = d_2 \Delta I + \frac{\beta S I}{1 + \alpha I} - (\mu + \sigma) I, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.6}$$

from an SIR (susceptible-infective-removed) epidemic model. Here the parameters $\mu, \beta, \alpha, \sigma$ are all positive constants in which μ denotes the death rates of susceptible, infective and removed populations. The parameter σ is the removed/recovery rate, β is the infective transmission rate and α is the saturation level in the Holling type II incidence function. Since the equation for the removed population is decoupled from the equations for S and I, the study of the full SIR model is reduced to system (1.5)-(1.6). We refer to, e.g., [1, 17] for more description of this SIR model.

We now state the following Liouville-type theorem, one of the main theorems in this paper.

Theorem 1.1. *Let $u = (u_1, \cdots, u_m)$ be an entire solution of the reaction-diffusion system* (1.1) *such that $c_i \le u_i \le C_i$ on $\mathbb{R}^N \times \mathbb{R}$ for some positive constants c_i and C_i for $i = 1, \cdots, m$. Assume further that the kinetic system* (1.2) *admits a nonnegative bounded Lyapunov functional of the form:*

$$F(u) = \sum_{i=1}^{m} F_i(u_i),\ u \in \mathbb{R}_+^m,$$

where $F_i : [c_i, C_i] \to [0, +\infty)$ is a strictly convex C^2 function with $F_i(u_i^*) = 0$ for each i such that

$$\sum_{i=1}^{m} F_i'(u_i) f_i(u) \leq -\nu F(u),\ u \in \prod_{i=1}^{m} [c_i, C_i], \tag{1.7}$$

for some positive constant ν. Then $u \equiv u^*$.

One of the typical examples of Lyapunov functional (cf. [6, 9, 7]) is

$$F_i(u_i) := a_i g(u_i/u_i^*),\ g(z) := z - 1 - \ln(z),$$

for some positive constant a_i, $i = 1, \cdots, m$. Here an entire solution is defined to be a classical solution which exists for all $t \in \mathbb{R}$.

The main idea of the proof of Theorem 1.1 is to find a suitable Lyapunov functional for the reaction-diffusion system (1.1) with a Lyapunov functional of the kinetic system (1.2) at hand. For a reaction-diffusion system on a bounded domain with zero Neumann boundary condition, in [15], the authors take the integral of a Lyapunov functional for the kinetic system (1.2) over the spatial domain to obtain a Lyapunov functional of the original reaction-diffusion system. However, such construction is impossible for the Cauchy problem, since a Lyapunov functional for the kinetic system is not integrable on the whole space. Hence it is natural to put a suitable weight (or, cut-off) function so that the Lyapunov functional is integrable over the whole space and becomes a Lyapunov functional of the reaction-diffusion system. Such an idea was used in, e.g., [9, 8, 7]. The innovative feature of Theorem 1.1 is that it holds for any spatial dimension N and without any restrictions on the diffusion coefficients.

Remark 1.2. As a consequence, the restriction $N \leq 2$ in [9, Theorem 1.5(ii)] can be removed. Hence Theorem 1.5 in [9] holds for all $k \in (0, 1/s_0)$ in which $s_0 \in (1/5, 1/4)$ is the unique positive root of the polynomial $32s^2 + 16s^2 - s - 1$. Also, the equal diffusion condition in [8, Theorem 2.6] is not needed. Furthermore, most results of [7] for one spatial dimension can be extended to general N dimension. To see this, first we note that [7, Lemmas 4.1-4.3] hold for any spatial dimension N. Secondly, [7, Lemma 3.2] (one of the key lemmas) can be proved by the same argument along any direction $\mathbf{e}$ (as that in [8]) so that it holds for any dimension N. Thirdly, other results in section 3 of [7] can be easily extended to general N dimension. Hence Theorems 2.1-2.3 and 2.5 in [7] hold for general N dimension. Note that Theorem 2.4 of [7] holds (with the same proof) for any spatial dimension.

For the above mentioned singular predator-prey system (1.3)-(1.4) and epidemic model (1.5)-(1.6), we immediately obtain two new results by applying Theorem 1.1.

For the singular predator-prey system, we have

Theorem 1.3. *Suppose that* $r_b \geq 4$*. Let* (B, C) *be the solution of* (1.3)-(1.4) *supplemented with the initial condition*

$$B(x,0) = B_0(x) \in [\underline{B}K, K], \quad C(x,0) = C_0(x) \leq K/\mu, \quad x \in \mathbb{R}^N, \tag{1.8}$$

where $\underline{B} := 1/2 - \sqrt{1/4 - 1/r_b} > 0$ *and* $C_0(x)$ *is a nonnegative continuous function with nonempty support. Then, for any* $c \in (0, c^*)$,

$$\lim_{t\to\infty} \sup_{|x|\leq ct} \{|B(x,t) - B^*| + |C(x,t) - C^*|\} = 0, \tag{1.9}$$

where $c^* := 2\sqrt{d_c r_c}$ *and* $(B^*, C^*) := (K(1 - 1/r_b), K(1 - 1/r_b)/\mu)$*.*

Next, it is easy to see that the disease-free state $(1, 0)$ is a constant state of (1.5)-(1.6). If we further assume that

$$\beta > \mu + \sigma =: \theta, \tag{1.10}$$

then there is the unique positive equilibrium state (S^*, I^*), where

$$S^* := \frac{\theta + \alpha\mu}{\beta + \alpha\mu}, \; I^* := \frac{\mu}{\theta}(1 - S^*).$$

We have the following theorem on the asymptotic behavior of system (1.5)-(1.6).

Theorem 1.4. *Let the condition* (1.10) *be enforced and let* $c_* := 2\sqrt{d_2(\beta - \theta)}$*. Then for any solution* (S, I) *of* (1.5)-(1.6) *with the initial condition*

$$S(x,0) = S_0(x) \in [\underline{S}, 1], \quad I(x,0) = I_0(x) \leq \kappa := \frac{\beta - \theta}{\alpha\theta}, \quad x \in \mathbb{R}^N, \tag{1.11}$$

for some positive constant $\underline{S} \in (0, 1)$*, where* $I_0(x)$ *is a non-negative continuous function with nonempty support, we have*

$$\lim_{t\to\infty} \sup_{|x|\leq ct} \{|S(x,t) - S^*| + |I(x,t) - I^*|\} = 0 \tag{1.12}$$

for any $c \in (0, c_*)$*.*

The rest of this paper is organized as follows. We first give a proof of Theorem 1.1 in section 2. Then we study the singular predator-prey system (1.3)-(1.4) in section 3 to prove Theorem 1.3, and the epidemic model (1.5)-(1.6) in section 4 to prove Theorem 1.4, respectively. Finally, in section 5, we provide a detailed proof of (4.2) (below in §4).

2. Proof of Theorem 1.1

Proof. For a given $R > 0$, motivated by the proof of [7, Lemma 4.1], we introduce the functional

$$\mathcal{F}_R(t) = \mathcal{F}_R[u](t) := \int_{\mathbb{R}^N} \rho(x) F(u(x,t)) dx, \tag{2.1}$$

where

$$\rho(x) := \exp\Big(-\frac{(1+|x|^2)^{1/2}}{R}\Big),\ x \in \mathbb{R}^N.$$

It is easy to check that

$$\max_{\mathbb{R}^N} \rho = e^{-1/R} < 1, \quad \Delta\rho \le \frac{\rho}{R^2}. \tag{2.2}$$

Since F is nonnegative and bounded on $\prod_{i=1}^m [c_i, C_i]$, the functional $\mathcal{F}_R(t)$ is well-defined, nonnegative and uniformly bounded.

We compute the time derivative of (2.1) and substitute (1.1) to obtain

$$\frac{d}{dt}\mathcal{F}_R(t) = \sum_{i=1}^m \int_{\mathbb{R}^N} \rho(x)[d_i F_i'(u_i)\Delta u_i]\, dx + \sum_{i=1}^m \int_{\mathbb{R}^N} \rho(x) F_i'(u_i) f_i(u)\, dx.$$

Note that, since $c_i \le u_i \le C_i$ for $1 \le i \le m$, by the parabolic regularity theory we obtain that $\{\nabla u_i \mid 1 \le i \le m\}$ are uniformly bounded in $\mathbb{R}^N$ for all $t \in \mathbb{R}$. Then, using the integration by parts and the exponential decay of ρ and $\nabla\rho$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N} \rho(x) F(u(x,t))\, dx + \sum_{i=1}^m \int_{\mathbb{R}^N} \rho(x) d_i F_i''(u_i)|\nabla u_i|^2\, dx \\
= & -\sum_{i=1}^m \int_{\mathbb{R}^N} d_i \nabla\rho(x)\cdot\nabla F_i(u_i)\, dx + \sum_{i=1}^m \int_{\mathbb{R}^N} \rho(x) F_i'(u_i) f_i(u)\, dx \\
= & \sum_{i=1}^m \int_{\mathbb{R}^N} [d_i \Delta\rho(x)] F_i(u_i)\, dx + \sum_{i=1}^m \int_{\mathbb{R}^N} \rho(x) F_i'(u_i) f_i(u)\, dx \\
\le & -\Big(\nu - \frac{\max\{d_i\}}{R^2}\Big)\int_{\mathbb{R}^N} \rho(x) F(u(x,t))\, dx,
\end{aligned}$$

by (1.7) and (2.2). It follows from $F_i'' \ge 0$ for all i that

$$\frac{d}{dt}\mathcal{F}_R(t) \le -\frac{\nu}{2}\mathcal{F}_R(t),\ \forall\, t \in \mathbb{R}, \tag{2.3}$$

if we choose R sufficiently large. It follows from (2.3) that $\mathcal{F}_R(t)e^{\nu t/2}$ is non-increasing in $t \in \mathbb{R}$. Since $\mathcal{F}_R(t) \ge 0$ for all t and $\mathcal{F}_R(t)e^{\nu t/2} \to 0$ as $t \to -\infty$ due to the uniform boundedness of $\mathcal{F}_R$, we obtain

$$\mathcal{F}_R(t) = 0,\ \forall\, t \in \mathbb{R},\ \forall\, R \gg 1. \tag{2.4}$$

Consequently, $F(u(x,t)) = 0$ for all $x \in \mathbb{R}^N$, $t \in \mathbb{R}$. Hence the theorem is proved. ∎

3. The singular predator-prey system

In this section, we study the singular predator-prey system and prove Theorem 1.3. In the sequel, for notational simplicity (with a suitable scaling), we set

$$d_b = d > 0, \quad d_c = 1, \quad K = 1, \quad \mu = 1$$

in (1.3)-(1.4). Then system (1.3)-(1.4) is reduced to

$$\begin{cases} B_t = d\Delta B + r_b\,(1-B)\,B - C, & x \in \mathbb{R}^N,\ t>0, \\ C_t = \Delta C + r_c\,(1 - C/B)\,C, & x \in \mathbb{R}^N,\ t>0. \end{cases} \tag{3.1}$$

Suppose that $r_b \geq 4$. Let (B, C) be a solution of system (3.1) with the initial condition

$$B(x,0) = B_0(x) \in [\underline{B}, 1], \quad C(x,0) = C_0(x) \leq 1, \quad x \in \mathbb{R}^N,$$

where the function $C_0(x)$ is a nonnegative continuous function with non-empty compact support. Then the solution (B, C) exists globally in time and satisfies (cf. [2])

$$\underline{B} \leq B(x,t) \leq 1, \quad 0 \leq C(x,t) \leq 1,\ x \in \mathbb{R}^N,\ t > 0. \tag{3.2}$$

Moreover, we have the following spreading property for the predator from [2, Theorem 2.1]:

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} C(x,t) \geq \underline{B}, \quad \forall\, c \in (0, c^*), \tag{3.3}$$

where $c^* = 2\sqrt{r_c}$. One may notice that it only requires the condition that $B_0(x) \in [\underline{B}, 1]$ in the proof of [2, Theorem 2.1].

In order to apply Theorem 1.1, we first verify condition (1.7) in the following lemma.

Lemma 3.1. *The associated kinetic system of* (3.1) *admits a Lyapunov functional F such that condition* (1.7) *holds for* $(B, C) \in [\underline{B}, 1] \times [m, 1]$ *for some positive constant ν, where m is a positive constant in* $(0, 1)$.

Proof. Consider the function $F = F(B, C)$ (cf. [6, 10]) defined by

$$F(B,C) := F_1(B) + F_2(C),\ B \in [\underline{B}, 1],\ C \in [m, 1],$$

where

$$F_1(B) := r_c\Big[(B - B^*) - B^* \ln\Big(\frac{B}{B^*}\Big)\Big], \quad F_2(C) := (C - C^*) - C^* \ln\Big(\frac{C}{C^*}\Big).$$

Then the directional derivative of F along the vector $(r_b B(1-B) - C, r_c C(1 - C/B))$, denoted by $\mathscr{X}F$, is calculated as

$$(\mathscr{X}F)(B,C) = r_c(B - B^*)\Big\{r_b(1-B) - \frac{C}{B}\Big\} + r_c(C - C^*)\Big(1 - \frac{C}{B}\Big).$$

Note that

$$r_b(1-B)-\frac{C}{B}=-\Big(r_b-\frac{1}{B}\Big)(B-B^*)-\frac{C-C^*}{B},$$
$$(C-C^*)\Big(1-\frac{C}{B}\Big)=\frac{(C-C^*)(B-B^*)}{B}-\frac{(C-C^*)^2}{B},$$

using $B^*=C^*=1-1/r_b$. Thus

$$(\mathscr{X}F)(B,C)=-r_c\Big(r_b-\frac{1}{B}\Big)(B-B^*)^2-\frac{r_c}{B}(C-C^*)^2. \tag{3.4}$$

Since $B\ge \underline{B}$, we have

$$r_b-\frac{1}{B}\ge r_b-\frac{1}{\underline{B}}=\frac{1}{\underline{B}}(r_b\underline{B}-1)>0.$$

It follows from (3.4) that

$$(\mathscr{X}F)(B,C)\le -\frac{r_c}{\underline{B}}(r_b\underline{B}-1)(B-B^*)^2-r_c(C-C^*)^2,\ \forall\,(B,C)\in[\underline{B},1]\times[m,1]. \tag{3.5}$$

Using (3.5), we claim that there exists a sufficiently small positive constant ν such that

$$\mathscr{X}F(B,C)\le -\nu F(B,C),\ \forall\,(B,C)\in[\underline{B},1]\times[m,1]. \tag{3.6}$$

Indeed, given γ and Γ such that $0<\gamma<1<\Gamma<\infty$. Then for any positive constant a there is a small enough positive constant $b=b(a,\gamma)$ such that $h(x):=-a(x-1)^2+b[(x-1)-\ln x]\le 0$ for all $x\in[\gamma,\Gamma]$. This follows from $h(0^+)=\infty$, $h(1)=h'(1)=h'(b/(2a))=0$ and $h''(1)<0$, if b is chosen so that $b<2a\gamma$. Hence, for

$$\gamma:=\underline{B}/B^*,\Gamma:=1/B^*,a:=\frac{r_c(B^*)^2}{\underline{B}}(r_b\underline{B}-1),b\in(0,2a\gamma),$$

we obtain

$$-a\left(\frac{B}{B^*}-1\right)^2\le -b\left[\left(\frac{B}{B^*}-1\right)-\ln\left(\frac{B}{B^*}\right)\right],\ \forall\,B\in[\underline{B},1].$$

Similarly, for

$$\gamma_1:=\min\{1/2,m/C^*\},\Gamma_1:=1/C^*,a_1:=r_c(C^*)^2,b_1\in(0,2a_1\gamma_1),$$

we have

$$-a_1\left(\frac{C}{C^*}-1\right)^2\le -b_1\left[\left(\frac{C}{C^*}-1\right)-\ln\left(\frac{C}{C^*}\right)\right],\ \forall\,C\in[m,1].$$

Hence (3.6) holds if the constant ν is chosen so that

$$0<\nu<\min\left\{2(r_b\underline{B}-1),2\min\{C^*/2,m\}r_c\right\}.$$

Hence (1.7) is deduced and the lemma is proved. □

Now we are ready to apply Theorem 1.1 to prove Theorem 1.3.

Proof of Theorem 1.3. We use a contradiction argument. Suppose that there exist $c_0 \in (0, c^*)$, $\delta > 0$ and a sequence $\{(x_k, t_k)\} \subset \mathbb{R}^N \times (0, \infty)$ with $t_k \to \infty$ and $|x_k| \leq c_0 t_k$ such that

$$|B(x_k, t_k) - B^*| + |C(x_k, t_k) - C^*| \geq \delta, \ \forall\, k \in \mathbb{N}. \tag{3.7}$$

Consider the sequence of functions

$$B_k(x,t) := B(x + x_k, t + t_k), \quad C_k(x,t) := C(x + x_k, t + t_k).$$

By (3.3) with $c \in (c_0, c^*)$, there exists a positive constant τ large enough such that

$$\underline{B}/2 \leq C_k(x,t) \leq 1, \ \text{ if } |x + x_k| \leq c(t + t_k) \text{ and } t + t_k \geq \tau. \tag{3.8}$$

Then, using (3.2), (3.8) and the standard parabolic estimates, there is a subsequence of $\{(B_k, C_k)\}$, still denoted by $\{(B_k, C_k)\}$, such that

$$(B_k, C_k) \to (B_\infty, C_\infty) \ \text{ as } k \to \infty$$

locally uniformly on $\mathbb{R}^N \times \mathbb{R}$, where (B_∞, C_∞) is an entire solution of (3.1) such that $\underline{B} \leq B_\infty \leq 1$ and $\underline{B}/2 \leq C_\infty \leq 1$. Hence $(B_\infty, C_\infty) \equiv (B^*, C^*)$, by Lemma 3.1 and Theorem 1.1. This contradicts (3.7) and so we finish the proof. □

4. The epidemic model (1.5)-(1.6)

In this section, we study the epidemic model and prove Theorem 1.4. Let (S, I) be a solution of (1.5)-(1.6) with the initial condition (1.11). Then we have

$$\kappa_1 \leq S(x,t) \leq 1, \ 0 \leq I(x,t) \leq \kappa, \ x \in \mathbb{R}^N, \ t \geq 0, \tag{4.1}$$

for some positive constant $\kappa_1 := \min\{\underline{S}, \alpha\mu/(\beta + \alpha\mu)\}$. Indeed, the lower bound of S follows from the fact that the constant κ_1 is a subsolution (for any $I \geq 0$) of (1.5). The others can be seen by the comparison principle (cf. [14]).

When $S_0 \equiv 1$, as noted in [14, section 4], we have

$$\liminf_{t \to \infty} \inf_{|x| \leq ct} I(x,t) > 0, \ \forall\, c \in (0, c_*), \tag{4.2}$$

where $c_* = 2\sqrt{d_2(\beta - \theta)}$. In fact, (4.2) holds for any initial data satisfying the condition in (1.11) by an argument used in [7, 8]. For the reader's convenience, we provide the detailed proof of (4.2) in the next section.

Now, we check that the associated kinetic system of (1.5)-(1.6) has a Lyapunov functional

$$F(S, I) := (1 + \alpha I^*)[(S - S^*) - S^* \ln(S/S^*)] + [(I - I^*) - I^* \ln(I/I^*)],$$

such that the property (1.7) holds for $(S,I)\in[\varepsilon,1]\times[\varepsilon,\kappa]$ for any small $\varepsilon>0$ (see also [16]). To see this, we first compute the directional derivative of F along the vector

$$(\mu-\mu S-\beta SI/(1+\alpha I),\beta SI/(1+\alpha I)-(\mu+\sigma)I),$$

denoted by $\mathscr{X}F$, as follows

$$\begin{aligned}
&\mathscr{X}F(S,I)\\
=\;&(1+\alpha I^*)\frac{1}{S}(S-S^*)\left\{\mu(1-S)-\frac{\beta SI}{1+\alpha I}\right\}+\frac{1}{I}(I-I^*)\left\{\frac{\beta SI}{1+\alpha I}-(\mu+\sigma)I\right\}\\
=\;&(1+\alpha I^*)\frac{1}{S}(S-S^*)\left\{\mu(S^*-S)+\frac{\beta S^*I^*}{1+\alpha I^*}-\frac{\beta SI}{1+\alpha I}\right\}\\
&+\frac{1}{I}(I-I^*)\left\{\frac{\beta SI}{1+\alpha I}-\frac{\beta S^*I^*}{1+\alpha I^*}-(\mu+\sigma)(I-I^*)\right\}\\
=\;&-(1+\alpha I^*)\left[\frac{\mu}{S}+\frac{\beta I^*}{(1+\alpha I^*)S}\right](S-S^*)^2-\left[\frac{\mu+\sigma}{I}-\frac{\beta S^*}{(1+\alpha I^*)I}\right](I-I^*)^2\\
&+(1+\alpha I^*)\left(\frac{\beta I^*}{1+\alpha I^*}-\frac{\beta I}{1+\alpha I}\right)(S-S^*)+\left(\frac{\beta S}{1+\alpha I}-\frac{\beta S^*}{1+\alpha I^*}\right)(I-I^*).
\end{aligned}$$

Moreover, since $\mu+\sigma=\beta S^*/(1+\alpha I^*)$ and

$$\begin{aligned}
&(1+\alpha I^*)\left(\frac{\beta I^*}{1+\alpha I^*}-\frac{\beta I}{1+\alpha I}\right)(S-S^*)+\left(\frac{\beta S}{1+\alpha I}-\frac{\beta S^*}{1+\alpha I^*}\right)(I-I^*)\\
=\;&\left\{-\frac{\beta}{1+\alpha I}+\frac{\beta}{1+\alpha I^*}\right\}(S-S^*)(I-I^*)-\frac{\alpha\beta S}{(1+\alpha I)(1+\alpha I^*)}(I-I^*)^2\\
=\;&\frac{\alpha\beta(S-S^*)}{(1+\alpha I)(1+\alpha I^*)}(I-I^*)^2-\frac{\alpha\beta S}{(1+\alpha I)(1+\alpha I^*)}(I-I^*)^2\\
=\;&-\frac{\alpha\beta S^*}{(1+\alpha I)(1+\alpha I^*)}(I-I^*)^2,
\end{aligned}$$

we obtain

$$\mathscr{X}F(S,I)=-\left[\frac{\mu(1+\alpha I^*)}{S}+\frac{\beta I^*}{S}\right](S-S^*)^2-\frac{\alpha\beta S^*}{(1+\alpha I)(1+\alpha I^*)}(I-I^*)^2. \tag{4.3}$$

Using (4.3) and following the same proof as that in Lemma 3.1, we can find a positive constant ν (depending on ε) such that condition (1.7) holds.

We are ready to give a proof of Theorem 1.4 as follows.

Proof of Theorem 1.4. We repeat the proof of Theorem 1.3. Suppose, on the contrary, that there exist $c_0\in(0,c_*)$, $\delta>0$ and a sequence $\{(x_k,t_k)\}\subset\mathbb{R}^N\times(0,\infty)$ with $t_k\to\infty$ and $|x_k|\le c_0t_k$ such that

$$|S(x_k,t_k)-S^*|+|I(x_k,t_k)-I^*|\ge\delta,\ \forall\,k\in\mathbb{N}.$$

Then the sequence $\{(S_k, I_k)\}$, defined by

$$S_k(x,t) := S(x + x_k, t + t_k), \quad I_k(x,t) := I(x + x_k, t + t_k),$$

has a subsequence, still denoted by $\{(S_k, I_k)\}$, such that

$$(S_k, I_k) \to (S_\infty, I_\infty) \text{ as } k \to \infty$$

locally uniformly on $\mathbb{R}^N \times \mathbb{R}$. Then, by (4.1), (S_∞, I_∞) is a nonnegative bounded entire solution of (1.5)-(1.6) such that $\kappa_1 \le S_\infty \le 1$ and $0 \le I_\infty \le \kappa$. Also, by (4.2) with $c \in (c_0, c_*)$, there exists a positive constant $\varepsilon_0 > 0$ such that $I_\infty(x,t) \ge \varepsilon_0$ for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Therefore, Theorem 1.4 follows by applying Theorem 1.1. $\square$

5. Proof of (4.2)

In this section, we provide a proof of the property (4.2) when the initial data (S_0, I_0) satisfying the condition in (1.11). The idea of the proof is motivated by [7, 8]. We only present here the case when $N = 1$. The general higher dimensional case can be proved by taking all possible directions $e \in S^{N-1}$ as that in [8].

We divide our discussion into the following three steps.

Step 1. claim that for any $c \in [0, c_*)$ there exists a positive constant $\delta_1(c)$, independent of (S_0, I_0), such that

$$\limsup_{t\to\infty} I(ct,t) \ge \delta_1(c). \tag{5.1}$$

For contradiction, we assume that there are sequences $\{(S_{0,n}, I_{0,n})\}$ and $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \sup_{t\ge t_n} I_n(ct,t) = 0,$$

where (S_n, I_n) is the corresponding solution of (1.5)-(1.6) with initial datum $(S_{0,n}, I_{0,n})$. For any $R > 0$, by passing to the limit as $n \to \infty$ and applying the strong maximum principle, we obtain

$$\limsup_{n\to\infty} \{ \sup_{t\ge t_n, |x-ct|\le R} I_n(x,t) \} = 0. \tag{5.2}$$

Next, for any $x_n \in [ct - R, ct + R]$, by extracting a subsequence with the help of standard parabolic estimates, the limit

$$(S_\infty, I_\infty)(x,t) := \lim_{n\to\infty} (S_n, I_n)(x + x_n, t + t_n), \ x \in \mathbb{R}, \ t \in \mathbb{R},$$

exists and is an entire solution of (1.5)-(1.6). Since $I_\infty(0,t) = 0$ for all $t > 0$ due to (5.2), $I_\infty \equiv 0$ by the strong maximum principle. Hence S_∞ satisfies

$$(S_\infty)_t = d_1 \Delta S_\infty + \mu - \mu S_\infty, \ x \in \mathbb{R}, \ t \in \mathbb{R}.$$

Note that $1 \geq S_\infty \geq \kappa_1$ in $\mathbb{R} \times \mathbb{R}$, by (4.1). Hence $S_\infty \equiv 1$. This proves that

$$\limsup_{n\to\infty}\{\sup_{|x-ct|\leq R, t\geq t_n} S_n(x,t)\} = 1. \tag{5.3}$$

Now, due to $c < c_*$, there is a small positive constant η such that

$$\frac{c^2}{4d_2} < \beta(1-3\eta) - (\mu+\sigma).$$

Also, there is R large enough such that

$$\frac{\lambda_1}{R^2} < (\theta_1 - \beta\eta) - \frac{c^2}{4d_2}, \quad \theta_1 := \beta(1-2\eta) - (\mu+\sigma),$$

where $\lambda_1 > 0$ is the principal eigenvalue of the eigenvalue problem

$$-d_2\Delta\phi(x) = \lambda_1\phi(x),\ |x| < 1, \quad \phi(x) = 0,\ |x| = 1. \tag{5.4}$$

Hereafter we fix an eigenfunction ϕ so that

$$\int_{|x|\leq 1} \phi(x)dx = 1.$$

Then for a fixed sufficiently large n it follows from (1.5), (5.2) and (5.3) that

$$(I_n)_t(x,t) \geq d_2\Delta I_n(x,t) + \beta(1-\eta)\frac{I_n(x,t)}{1+\eta} - (\mu+\sigma)I_n(x,t)$$

for all $x \in (ct-R, ct+R)$, $t \geq t_n$. Note that $(1-\eta)/(1+\eta) > 1-2\eta$ for any $\eta > 0$. Hence the function $\hat{I}_n(x,t) := I_n(x+ct,t)$ satisfies

$$(\hat{I}_n)_t(x,t) \geq d_2\Delta\hat{I}_n(x,t) + c(\hat{I}_n)_x(x,t) + \theta_1\hat{I}_n(x,t),\ |x| < R,\ t \geq t_n.$$

On the other hand, it follows from (5.4) that the function $\psi(x) := e^{-\frac{cx}{2d_2}}\phi\left(x/R\right)$ satisfies

$$-d_2\Delta\psi(x) - c\psi_x = \left(\frac{\lambda_1}{R^2} + \frac{c^2}{4d_2}\right)\psi,\ |x| < R, \quad \psi(x) = 0,\ |x| = R.$$

Then, by comparison, we obtain that

$$\hat{I}_n(x,t) \geq Ae^{\beta\eta t}e^{-\frac{cx}{2d_2}}\phi\left(\frac{x}{R}\right),\ |x| \leq R,\ t \geq t_n, \tag{5.5}$$

if the positive constant A is chosen small enough such that

$$\hat{I}_n(x,t_n) \geq Ae^{\beta\eta t_n}e^{-\frac{cx}{2d_2}}\phi\left(\frac{x}{R}\right),\ |x| \leq R.$$

Putting $x = 0$ in (5.5), we obtain

$$\lim_{t\to\infty} I_n(ct,t) = \lim_{t\to\infty} \hat{I}_n(0,t) = \infty,$$

which is a contradiction to the boundedness of I_n. Hence the claim (5.1) is proved.

Step 2. claim that for any $c \in [0, c_*)$ there exists a positive constant $\delta_2(c)$, independent of (S_0, I_0), such that

$$\liminf_{t\to\infty} I(ct,t) \geq \delta_2(c). \tag{5.6}$$

To this aim, we follow the proof of [7, Lemma 3.2]. Proceed by a contradiction and assume that there are sequences $\{(S_{0,n}, I_{0,n})\}$ and $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} I_n(ct_n, t_n) = 0, \tag{5.7}$$

where (S_n, I_n) is the solution of (1.5)-(1.6) with initial datum $(S_{0,n}, I_{0,n})$. On the other hand, by (5.1), we can choose a time sequence $\{t'_n\}$ with $t'_n < t_n$ for all n, $t'_n \to \infty$ as $n \to \infty$ and

$$I_n(ct'_n, t'_n) \geq \frac{\delta_1(c)}{2}, \ \forall\, n.$$

Then we define

$$\tau_n := \sup\{t \geq t'_n \mid I_n(ct, t) \geq \delta_1(c)/2\}.$$

It easy to see that

$$I_n(c\tau_n, \tau_n) = \delta_1(c)/2, \quad I_n(ct, t) \leq \delta_1(c)/2, \ \forall\, t \in (\tau_n, t_n). \tag{5.8}$$

Next, up to extraction of a subsequence, we have

$$(S_n, I_n)(x + c\tau_n, t + \tau_n) \to (S_\infty, I_\infty)(x, t) \text{ as } n \to \infty$$

locally uniform in $\mathbb{R} \times \mathbb{R}$, where (S_∞, I_∞) is an entire solution of (1.5)-(1.6). Note that $t_n - \tau_n \to \infty$ as $n \to \infty$. Otherwise, if (up to extraction of a subsequence) $\{t_n - \tau_n\}$ converges to a finite limit τ_0, then

$$I_\infty(c\tau_0, \tau_0) = \lim_{n\to\infty} I_n(c(t_n - \tau_n) + c\tau_n, (t_n - \tau_n) + \tau_n) = \lim_{n\to\infty} I_n(ct_n, t_n) = 0,$$

by (5.7). It follows from the strong maximum principle that $I_\infty \equiv 0$, which contradicts with $I_\infty(0, 0) = \delta_1(c)/2$ due to (5.8). Hence $t_n - \tau_n \to \infty$ as $n \to \infty$. This together with (5.8) also implies that

$$I_\infty(ct, t) \leq \delta_1(c)/2, \ \forall\, t \geq 0,$$

a contradiction to (5.1). Therefore, claim (5.6) is proved.

Step 3. we derive (4.2) by a contradiction argument. We only consider the case $x \in [0, ct]$. The case for $x \in [-ct, 0]$ can be done similarly.

For a contradiction, we assume that there are sequences $\{(S_{0,n}, I_{0,n})\}$ and $\{(c_n, t_n)\}$ with $t_n \to \infty$ as $n \to \infty$ and $c_n \in [0, c]$ for all n such that

$$\lim_{n\to\infty} I_n(c_n t_n, t_n) = 0. \tag{5.9}$$

Without loss of generality, we may assume that $c_n \to \hat{c} \in [0, c]$. Set

$$t'_n := c_n t_n / c_0, \quad c_0 := (\hat{c} + c^*)/2 \in (\hat{c}, c_*).$$

Then $\{t'_n\}$ has no bounded subsequences. Otherwise, up to extraction of a subsequence, suppose that $c_n t_n \to x_0 \in \mathbb{R}$ as $n \to \infty$ for some $x_0 \in \mathbb{R}$. Note that, up to extraction of a subsequence, the limit function

$$(S_\infty, I_\infty)(x,t) := \lim_{n\to\infty} (S_n, I_n)(x + c_n t_n, t + t_n)$$

is an entire solution of (1.5)-(1.6) such that $I_\infty(0,0) = 0$ and so, by the strong maximum principle, $I_\infty \equiv 0$. However, as $n \to \infty$ we have

$$I_n(0, t_n) = I_n(-c_n t_n + c_n t_n, t_n) \to I_\infty(-x_0, 0) = 0,$$

which contradicts (5.6) with $c = 0$. Hence we may assume without loss of generality that $t'_n \to \infty$ as $n \to \infty$.

Next, by (5.6), we have

$$\liminf_{t\to\infty} I_n(c_0 t, t) \geq \delta_2(c_0) \text{ for all } n.$$

It follows from the fact $t'_n \to \infty$ as $n \to \infty$ that

$$t'_n < t_n, \quad I_n(c_n t_n, t'_n) = I_n(c_0 t'_n, t'_n) \geq \delta_2(c_0)/2, \quad \forall\, n \gg 1.$$

Then we introduce

$$\tau_n := \sup\{t \geq t'_n \mid I_n(c_n t_n, t) \geq \delta_0\},\ \delta_0 := \min\{\delta_2(c_0)/2, \delta_1(0)/2\}.$$

Note that $I_n(c_n t_n, \tau_n) = \delta_0$ for all n large due to (5.9). As before, we have $t_n - \tau_n \to \infty$ as $n \to \infty$, by a limiting argument with the help of the strong maximum principle. It follows that, up to extraction of a subsequence,

$$(S_n, I_n)(x + c_n t_n, t + \tau_n) \to (S_\infty, I_\infty)(x,t) \text{ as } n \to \infty$$

locally uniform in $\mathbb{R} \times \mathbb{R}$, where (S_∞, I_∞) is an entire solution of (1.5)-(1.6) such that

$$I_\infty(0,0) = \delta_0, \quad I_\infty(0,t) \leq \delta_0,\ \forall\, t \geq 0.$$

This contradicts (5.1) with $c = 0$. The proof is thereby complete. □

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