Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

THE SIGN OF TRAVELING WAVE SPEED IN BISTABLE DYNAMICS

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(Communicated by the associate editor name)

Abstract. We are concerned with the sign of traveling wave speed in bistable dynamics. This question is related to which species wins the competition in multiple species competition models. It is well-known that the wave speed is unique for traveling wave connecting two stable states. In this paper, we first review some known results on the sign of wave speed in bistable two species competition models. Then we derive rigorously the sign of bistable wave speed for a special three species competition model describing the competition in two different circumstances: (1) two species are weak competitors and one species is a strong competitor; (2) three species are very strong competitors. It is interesting to observe that, under certain conditions on the parameters, two weaker competitors can wipe out the strongest competitor.

1. Introduction.

In this paper, we are concerned with the sign of wave speed of traveling wave solution to a reaction-diffusion system. Traveling wave solution has been an important object in the study of pattern formations over the last ten decades. Typically a traveling wave solution connects two constant steady states at left and right ends of the spatial domain such that it keeps the same shape for all times and moves with a constant speed. There are the so-called front type wave,

2010 *Mathematics Subject Classification.* Primary: 35K55, 35K57; Secondary: 92D25, 92D40.

Key words and phrases. Bistable, competition model, traveling wave, wave speed, uniqueness.

This work was partially supported by the Ministry of Science and Technology of Taiwan under the grants 105-2115-M-032-003-MY3 and MOST 108-2636-M-024-001 and by JSPS KAKENHI Grant Numbers JP15K04996 and JP18K03412.

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pulse type wave and the mixture of both front and pulse types. Here we focus mainly on the front type waves.

A wave is of monostable type, if one of these two steady states is stable and the other is unstable in the ODE sense (i.e. without diffusion). It is of bistable type, if both steady states are stable in the ODE sense. For the monostable waves, there exist a continuum of wave speeds. On the other hand, for the bistable case, it is well-known that the admissible wave speed is unique (in most cases).

In the competition model, the sign of wave speed gives us the information on which species wins the competition. The sign of wave speed decides which species becomes dominant and eventually occupies the whole habitat. Therefore, it is an important task to determine the sign of this unique wave speed in bistable dynamics. However, less attention were paid on the sign of wave speed (cf. [1, 9, 10, 19]) in past years. The main purpose of this work is to provide a new result on a special 3-species bistable competition system.

Consider the following system of reaction-diffusion equations

$$u_t^j = d_j u_{xx}^j + f_j(u^1, \cdots, u^N),\ x \in \mathbb{R},\ t \in \mathbb{R},\ j = 1, \cdots, N, \tag{1}$$

where $N \in \mathbb{N}$. Assume that there are two constant states $u_\pm := (u_\pm^1, \cdots, u_\pm^N)$ of (1) such that they are stable in the ODE sense. To be a front, we meant that $u_-^j \neq u_+^j$ for all j.

We say that $(u^1, \cdots, u^N)(x,t)$ is a traveling front solution of (1) with wave speed s connecting u_- and u_+, if

$$u^j(x,t) = U^j(\xi),\ j = 1, \cdots, N,\ \xi := x - st,$$

for some functions $\{U^j \mid j = 1, \cdots, N\}$ (*the wave profile*) such that $U^j(\pm\infty) = u_\pm^j$ for each j. Hence $(s, \{U^j\})$ is a traveling front of (1) connecting u_- and u_+, if it satisfies

$$-s(U^j)'(\xi) = d_j (U^j)''(\xi) + f_j(U^1, \cdots, U^N)(\xi),\ \xi \in \mathbb{R},\ j = 1, \cdots, N, \tag{2}$$

and the boundary conditions

$$U^j(-\infty) = u_-^j,\ U^j(\infty) = u_+^j,\ j = 1, \cdots, N.$$

Notice that the wave moves to the right if and only if $s > 0$. In this case, u tends the state u_- as $t \to \infty$ and we say that the state u_- wins the competition.

Recently, the following special 3-species competition system was studied ([11, 13, 14]):

$$u_t = d_1 u_{xx} + a_1 u(1 - u - b_2 v),\ x \in \mathbb{R}, t > 0, \tag{3}$$

$$v_t = d_2 v_{xx} + a_2 v(1 - v - b_1 u - b_3 w),\ x \in \mathbb{R}, t > 0, \tag{4}$$

$$w_t = d_3 w_{xx} + a_3 w(1 - w - b_2 v),\ x \in \mathbb{R}, t > 0, \tag{5}$$

where $d_i, a_i, b_i,\ i = 1, 2, 3$, are positive constants. The special nonlinearity models that there is no competition between species u and w. One should note that, in general, a 3-species competition system is not a monotone system. However, under this special circumstance, system (3)-(5) is a monotone system which enjoys a comparison principle. There are two special states $(1, 0, 1)$ and $(0, 1, 0)$, the former is the case when v loses and the latter one is when v wins. To determine whether species v wins the competition, we consider the traveling waves connecting these two states.

Under the assumption

$$b_2 > 1, \quad b_1 + b_3 < 1,$$

the traveling wave of (3)-(5) connecting $(1, 0, 1)$ and $(0, 1, 0)$ is of monostable type and this case was studied in [11] including the spatial discrete case for a lattice dynamical system. Indeed, the discrete three species competition system associated with system (3)-(5) is:

$$u_j'(t) = \hat{d}_1 \mathcal{D}_2[u_j](t) + a_1[u_j(1 - u_j - b_2 v_j)](t), \ t \in \mathbb{R}, \ j \in \mathbb{Z}, \tag{6}$$

$$v_j'(t) = \hat{d}_2 \mathcal{D}_2[v_j](t) + a_2[v_j(1 - b_1 u_j - v_j - b_3 w_j)](t), \ t \in \mathbb{R}, \ j \in \mathbb{Z}, \tag{7}$$

$$w_j'(t) = \hat{d}_3 \mathcal{D}_2[w_j](t) + a_3[w_j(1 - b_2 v_j - w_j)](t), \ t \in \mathbb{R}, \ j \in \mathbb{Z}, \tag{8}$$

where $\hat{d}_i$, a_i, b_i, $i = 1, 2, 3$, are positive constants and $\mathcal{D}_2[z_j](t) := z_{j+1}(t) + z_{j-1}(t) - 2z_j(t)$. In this case, the species v always wins the competition, since there exists the positive minimal speed to system (3)-(5).

On the other hand, under the assumption

$$b_2 > 1 > b_1, \ b_2 > 1 > b_3, \ b_1 + b_3 > 1, \tag{9}$$

both states $(1, 0, 1)$ and $(0, 1, 0)$ are stable. Intuitively, species v should win the competition, since v is a strong competitor and u (w, resp.) is a weak competitor in the absence of w (u, resp.). However, putting u and w together (under the condition $b_1 + b_3 > 1$), it is possible that v loses the competition. It is one of the questions to be addressed in this paper. For system (6)-(8), the existence of traveling front connecting $(1, 0, 1)$ and $(0, 1, 0)$ is derived in [13], while the stability and uniqueness of traveling fronts were addressed in [14]. Since our main concern here is the sign of wave speed for traveling fronts of bistable type, we shall not address the existence of traveling fronts to (3)-(5) connecting $(1, 0, 1)$ and $(0, 1, 0)$ with (9) here. We only refer the reader to [22, 7] for some general theory to the existence of traveling waves.

By some numerical simulations on system (3)-(5) with (9), it is found that v wins the competition, if $b_2 > b_1 + b_3$, and v loses the competition when $b_2 < b_1 + b_3$. One of the purposes of this paper is to give a rigorous proof of this numerical observation. In addition, we also consider the case that $b_i \gg 1$ for each i. In this case, we will investigate how the diffusion rates affect the sign of the wave speed by a singular limit analysis. This is motivated by a recent work on two species case by Girardin and Nadin [9] in which they provide some results on the wave speed sign when both species are very strong competitors.

The rest of this paper is organized as follows. First, in §2 we shall review some existing results for the 1 and 2 species competition models. Then the 3-species case is treated in §3. In particular, we derive the strict monotonicity of wave profile and the uniqueness of wave speed and wave profiles (up to translations) for system (3)-(5). Finally, we give some criteria to determine the sign of wave speed under certain conditions on the parameters.

As one can see, the understanding of wave speed sign in bistable dynamics is far from complete even for 2-species competition case. For example, should the diffusion coefficients and growth rates be taken into account in the determination of wave speed sign? Next, nothing is known about the sign of wave speed in the discrete lattice dynamical systems. In fact, there is a possibility of propagation failure for small diffusion which makes the question more subtle (see, e.g., [12]). Finally, the sign of wave speed for the 3-species case, both discrete and continuous cases are still largely left open.

2. Review of some existing results.

2.1. The scalar case: $N = 1$. Let (s, U) be a traveling front of (1) with $N = 1$. Then (2) becomes

$$-sU'(\xi) = dU''(\xi) + f(U(\xi)),\ \xi \in \mathbb{R}. \tag{10}$$

Multiplying (10) by U' and integrating it over $(-\infty, \infty)$, we deduce

$$-s\int_{-\infty}^{\infty}(U')^2 = \int_{-\infty}^{\infty} f(U)U'd\xi = \int_0^1 f(u)du.$$

Therefore, the sign of wave speed s is determined by the sign of the integral of f over $[0, 1]$.

2.2. Two species case. Consider the following Lotka-Volterra competition diffusion system

$$\begin{cases} u_t = u_{xx} + u(1-u-kv),\ x \in \mathbb{R},\ t \in \mathbb{R}, \\ v_t = dv_{xx} + av(1-v-hu),\ x \in \mathbb{R},\ t \in \mathbb{R}, \end{cases} \tag{11}$$

where $u = u(x,t)$ and $v = v(x,t)$ represent population densities of two competing species, and a, h, k, d are positive constants in which $1, a$ are the intrinsic growth rates, $1, d$ are the diffusion coefficients, h, k are the inter-specific competition coefficients. Here the carrying capacity is normalized to be 1 (by taking a suitable unit) for each species.

The u-equation in system (11) can be deduced by taking suitable scales of time and space variables. The parameters h and k influence the asymptotic behaviors of (u, v) and it is the bistable case when $h, k > 1$. Indeed, both constant states $(0, 1)$ and $(1, 0)$ are stable in the ODE sense. For the existence and stability of traveling waves to (11), we refer to [20, 8, 3, 15, 16, 17, 18, 19], etc. In particular, in the bistable case, Kan-on [15] derived the existence of traveling fronts such that the speed is unique and the wave profile is monotone and unique up to translations.

We are concerned with the monotone traveling fronts (s, U, V) connecting $(0, 1)$ and $(1, 0)$, i.e., $(s, U, V) = (s, U, V)(\xi)$ satisfies

$$U'' + sU' + U(1-U-kV) = 0 < U',\ \xi \in \mathbb{R}, \tag{12}$$

$$dV'' + sV' + aV(1-V-hU) = 0 > V',\ \xi \in \mathbb{R}, \tag{13}$$

with the boundary conditions

$$(U, V)(-\infty) = (0, 1),\ (U, V)(+\infty) = (1, 0), \tag{14}$$

where

$$(a, h, k, d) \in \mathcal{P} := \{(a, h, k, d) \mid a > 0,\ h > 1,\ k > 1,\ d > 0\}$$

and $s = s(a, h, k, d)$.

First, motivated by the scalar case, we try some integrations and obtain

Theorem 2.1 ([10]). (1) *Suppose that $a = d$, $1 < h \le 2$ and $k \ge 2$. Then $s(a, h, k, d) \ge 0$. Moreover, $s = 0$ only when $h = k = 2$.*
(2) *For $a = d$, we have $s < 0$ when $h \ge 2$, $1 < k \le 2$ and $(h, k) \ne (2, 2)$.*
(3) *Suppose that $a > d$. Then $s(a, h, k, d) > 0$, if $1 < h \le 1 + d/a$ and $k \ge 2$.*
(4) *Suppose that $a < d$. Then $s(a, h, k, d) < 0$, if $h \ge 2$ and $1 < k \le 1 + a/d$.*

It seems from Theorem 2.1 that the sign of wave speed only depends on the sign of $(k-h)$. This fits perfectly with the intuition that the stronger competitor wins the competition.

Next, by the change of the variables $(\tilde{U},\tilde{V})=(U,aV)$, problem (**P**) is reduced to the following problem ($\tilde{\mathbf{P}}$):

$$\tilde{U}''+\tilde{s}\tilde{U}'+\tilde{U}(1-\tilde{U}-c\tilde{V})=0<\tilde{U}',\ \xi\in\mathbb{R}, \tag{15}$$

$$d\tilde{V}''+\tilde{s}\tilde{V}'+\tilde{V}(a-b\tilde{U}-\tilde{V})=0>\tilde{V}',\ \xi\in\mathbb{R}, \tag{16}$$

$$(\tilde{U},\tilde{V})(-\infty)=(0,a),\ (\tilde{U},\tilde{V})(+\infty)=(1,0), \tag{17}$$

where $(a,b,c,d)\in\tilde{\mathcal{P}}:=\{(a,b,c,d)\mid 0<1/c<a<b,\ d>0\}$, $\tilde{s}=\tilde{s}(a,b,c,d)=s$. Here, for given a and d, we have the following relations between parameters (h,k) and (b,c):

$$(h,k)=(b/a,ac),\quad (b,c)=(ah,k/a). \tag{18}$$

We recall from [15] the following property of monotone dependence on parameters:

$$\frac{\partial}{\partial a}\tilde{s}(a,b,c,d)>0,\ \frac{\partial}{\partial b}\tilde{s}(a,b,c,d)<0,\ \frac{\partial}{\partial c}\tilde{s}(a,b,c,d)>0,\ (a,b,c,d)\in\tilde{\mathcal{P}}. \tag{19}$$

From (19) and $(h,k)=(b/a,ac)$, we also have

$$\frac{\partial}{\partial k}s(a,h,k,d)>0>\frac{\partial}{\partial h}s(a,h,k,d),\ (a,h,k,d)\in\mathcal{P}. \tag{20}$$

Also, in [15], it is proved that for any $d>0$ and for any positive numbers b,c with $b>1/c$ there exists a unique positive number $\bar{a}=\bar{a}(b,c,d)\in(1/c,b)$ such that $\tilde{s}(\bar{a},b,c,d)=0$. It then follows from (19) that (for b,c,d fixed) $\tilde{s}(a,b,c,d)>0$ when $a\in(\bar{a},b)$; while $\tilde{s}(a,b,c,d)<0$ when $a\in(1/c,\bar{a})$. However, the dependence of $\bar{a}$ on b,c,d is not clear.

Our question is that, for a given (a,b,c,d) (or, (a,h,k,d)), can we determine the sign of $\tilde{s}$ (or, s)? To this aim, we first recall some information on $s=0$ from [10] as follows.

Lemma 2.2. *There holds $s(1,h,h,1)=0$ for all $h>1$. If $s(a,h,k,d)=0$ for some $(a,h,k,d)\in\mathcal{P}$, then $s(d,k,h,a)=0$ and $s(la,h,k,ld)=0$ for all $l>0$. In particular, $s(d,h,h,d)=0$ for all $d>0,h>1$.*

The lemma can be proved by the uniqueness of wave speed and a suitable change of variables. Moreover, The following theorem follows easily from Lemma 2.2 and (20).

Theorem 2.3 ([10])**.** *Suppose that $a=d$. Then we have*

$$s(a,h,k,d)=\begin{cases}>0, & \text{if}\quad k>h>1;\\ =0, & \text{if}\quad h=k>1;\\ <0, & \text{if}\quad h>k>1.\end{cases}$$

Hence the case when $a=d$ is completely understood. However, for $a\neq d$, we only have

Theorem 2.4 ([10])**.** *Suppose that $a>d$. Then $s(a,h,k,d)>0$, if $h>1$ and $k\geq(a/d)^2h$. For $a<d$, we have $s(a,h,k,d)<0$, if $k>1$ and $h\geq(d/a)^2k$.*

Theorem 2.4 is also proved by applying Lemma 2.2. Note that $\mathrm{sign}(s)=\mathrm{sign}(k-h)$ in Theorem 2.4 (and Theorems 2.1, 2.3).

It is interesting to observe the following scaling property.

Theorem 2.5 ([10]). *For any $l > 0$, $s(a, h, k, d)$ and $s(la, h, k, ld)$ have the same sign.*

Theorem 2.5 indicates that the sign of wave speed depends only on the ratio of intrinsic growth rate a and diffusion coefficient d of v.

Finally, we have the following results for $a > d$:

Theorem 2.6 ([10]). *Suppose that $a > d$. Then $s(a, h, k, d) > 0$, if $h > 1$, $k \geq 5a/d$ and $(3ah - d)h \leq (4a - d)k$.*

Since, for $h = k$, we have

$$(4a - d)k - (3ah - d)h = k[(4a - d) - (3ak - d)] = ak(4 - 3k) < 0$$

for $k \geq 5a/d$. Therefore, $\text{sign}(s) = \text{sign}(k - h)$ is true in Theorem 2.6.

However, we have

Theorem 2.7 ([10]). *If $a = d/4$, then we have*

$$s(a, h, k, d) = \begin{cases} > 0, & \text{if } 1 < h \leq 4/3 \text{ and } k \geq 5/4,\ (h, k) \neq (4/3, 5/4); \\ = 0, & \text{if } \ h = 4/3,\ k = 5/4; \\ < 0, & \text{if } h \geq 4/3 \text{ and } 1 < k \leq 5/4,\ (h, k) \neq (4/3, 5/4). \end{cases}$$

One of the key ingredients in the proof of Theorems 2.6 and 2.7 is the following identities

$$\int_{-\infty}^{\infty} U^2 V' = -\frac{1}{3k}, \quad \int_{-\infty}^{\infty} U' V^2 = \frac{1}{3h},$$
$$\left(\frac{k}{3} - r\right) \int_{-\infty}^{\infty} U' V^3 + (1 - rh) \int_{-\infty}^{\infty} U U' V^2 - \int_{-\infty}^{\infty} U' (V')^2 = \frac{1 - 2r}{6h},$$
$$\frac{2r}{3} \int_{-\infty}^{\infty} U' V^3 + 2rh \int_{-\infty}^{\infty} U U' V^2 - \int_{-\infty}^{\infty} U' (V')^2 = \frac{r}{3h},$$

where $r := a/d$ and (s, U, V) is a traveling front with $s = 0$.

Note that $\text{sign}(s)$ is inconsistent with $\text{sign}(k - h)$ in Theorem 2.7.

3. **Three species case.** In this section, we first consider the system (3)-(5) for a 3-species competition system. Without loss of generality, we may assume that $d_2 = a_2 = 1$ by taking suitable scales of time and space variables. Hence system (3)-(5) becomes

$$u_t = d_1 u_{xx} + a_1 u(1 - u - b_2 v),\ x \in \mathbb{R},\ t > 0,$$
$$v_t = v_{xx} + v(1 - v - b_1 u - b_3 w),\ x \in \mathbb{R},\ t > 0,$$
$$w_t = d_3 w_{xx} + a_3 w(1 - w - b_2 v),\ x \in \mathbb{R},\ t > 0.$$

Then, for a traveling front (s, U, V, W) of system (3)-(5) connecting $(1, 0, 1)$ and $(0, 1, 0)$, we have

$$-sU'(\xi) = d_1 U''(\xi) + a_1 U(\xi)(1 - U - b_2 V)(\xi),\ \xi \in \mathbb{R}, \tag{21}$$
$$-sV'(\xi) = V''(\xi) + V(\xi)(1 - V - b_1 U - b_3 W)(\xi),\ \xi \in \mathbb{R}, \tag{22}$$
$$-sW'(\xi) = d_3 W''(\xi) + a_3 W(\xi)(1 - W - b_2 V)(\xi),\ \xi \in \mathbb{R}, \tag{23}$$

such that

$$(U, V, W)(-\infty) = (1, 0, 1), \quad (U, V, W)(\infty) = (0, 1, 0). \tag{24}$$

From now on, we shall assume that

$$U' < 0,\ V' > 0,\ W' < 0 \quad \text{in } \mathbb{R}. \tag{25}$$

Note that (25) implies that

$$0 < U, V, W < 1 \quad \text{in } \mathbb{R}. \tag{26}$$

Then, following the method used in [14], we can prove the following uniqueness theorem.

Theorem 3.1. *For a traveling wave (s, U, V, W) of system (3)-(5) connecting $(1, 0, 1)$ and $(0, 1, 0)$ such that (25) holds, the wave speed s is uniquely determined and the wave profile is also unique up to translations.*

Proof. Indeed, as in [14], it is more convenient to transform system (21)-(23) to the following cooperative system

$$-s\hat{U}' = d_1\hat{U}'' + a_1(1-\hat{U})(b_2\hat{V} - \hat{U}),\ \xi \in \mathbb{R}, \tag{27}$$

$$-s\hat{V}' = \hat{V}'' + \hat{V}(1 - b_1 - b_3 - \hat{V} + b_1\hat{U} + b_3\hat{W}),\ \xi \in \mathbb{R}, \tag{28}$$

$$-s\hat{W}' = d_3\hat{W}'' + a_3\hat{W}(b_2\hat{V} - \hat{W}),\ \xi \in \mathbb{R}, \tag{29}$$

where $\hat{U} := 1 - U$, $\hat{V} := V$ and $\hat{W} := 1 - W$. Then, using (25) and (26), the same super-sub-solutions as in [14] with $j + ct$ replaced by $x - st$ based on a given traveling wave solution (s, U, V, W) of (3)-(5) can be constructed (see (3.10)-(3.12) in [14]), where the operator D_2 is replaced by the second derivative and the monotone functions $\rho_i^{\pm}(x)$ are chosen to satisfy $0 < \rho_i^{\pm}(x) \le 1$, $|(\rho_i^{\pm})'(x)| \le 1$ and $|(\rho_i^{\pm})''(x)| \le 1$ for $x \in \mathbb{R}$ and $i = 1, 2, 3$.

Finally, since cooperative systems enjoy the (strong) comparison principle, the conclusion follows by the same proof as that of [14, Theorem 4.2]. We safely omit the details. □

Remark 1. Actually, condition (25) can be achieved if the following weaker condition

$$0 \le U, V, W \le 1 \quad \text{in } \mathbb{R} \tag{30}$$

than (26) is assumed. The proof is rather standard and we only outline it as follows.

Proof of (30) ⇒ (25). First, to derive (25), we show that

$$\hat{U}', \hat{V}', \hat{W}' > 0 \ \text{ in } \mathbb{R} \setminus [-L, L] \tag{31}$$

for some large positive number L. To see (31), by Proposition A in the appendix,

$$\lim_{\xi\to-\infty} \frac{\hat{U}'(\xi)}{\hat{U}(\xi)} = \min\{\lambda_1, \lambda_2\},\ \lim_{\xi\to-\infty} \frac{\hat{V}'(\xi)}{\hat{V}(\xi)} = \lambda_2,\ \lim_{\xi\to-\infty} \frac{\hat{W}'(\xi)}{\hat{W}(\xi)} = \min\{\lambda_2, \lambda_3\}.$$

Hence $\hat{U}'(\xi), \hat{V}'(\xi), \hat{W}'(\xi) > 0$ for $\xi \in (-\infty, -L)$ for some large constant L. The case for the right tail can be treated similarly and so (31) follows.

Next, thanks to (31) and (24), the constant

$$\eta^* := \inf\{l > 0 \mid \vec{U}(\xi + l) \ge \vec{U}(\xi) \text{ for all } \xi \in \mathbb{R}\}$$

is well-defined, where $\vec{U}(\xi) := (\hat{U}, \hat{V}, \hat{W})(\xi)$. By continuity, we have $\vec{U}(\xi + \eta^*) \ge \vec{U}(\xi)$ for all $\xi \in \mathbb{R}$. We now prove that $\eta^* = 0$. For a contradiction, we assume that $\eta^* > 0$. By the strong maximum principle, we have $\vec{U}(\xi + \eta^*) > \vec{U}(\xi)$ for all $\xi \in \mathbb{R}$. Using the continuity of $\hat{U}$, $\hat{V}$ and $\hat{W}$, there exists $\hat{\eta} \in (0, \eta^*)$ such that

$$\vec{U}(\xi + \eta) > \vec{U}(\xi) \text{ for all } \xi \in [-L - \eta^*, L] \text{ and } \eta \in [\hat{\eta}, \eta^*].$$

On the other hand, since $\hat{U}' > 0$, $\hat{V}' > 0$ and $\hat{W}' > 0$ in $\mathbb{R} \setminus [-L, L]$, we have

$$\vec{U}(\xi + \eta) > \vec{U}(\xi) \text{ for all } \xi \in \mathbb{R} \setminus [-L - \eta^*, L] \text{ and } \eta \in [\hat{\eta}, \eta^*].$$

Thus, $\vec{U}(\xi+\eta) > \vec{U}(\xi)$ for all $\xi \in \mathbb{R}$ and $\eta \in [\hat{\eta}, \eta^*]$, which contradicts the definition of η^*. This implies that $\eta^* = 0$. Hence $U' \le 0$, $V' \ge 0$ and $W' \le 0$ in $\mathbb{R}$ and so (25) follows by the strong maximum principle. $\square$

3.1. **Case of two weak competitors and one strong competitor.** In this subsection, we consider the special case:

$$a_1 = d_1,\ a_3 = d_3, \quad b_3 = b_1. \tag{32}$$

Theorem 3.2. *Let condition (32) be imposed and let (s, U, V, W) be a solution to (21)-(24) such that (25) holds. If $0 < b_1 < 1 < b_2$ and $b_2 \ge 2$, then we have $s < 0$.*

Proof. First, multiplying (21) by V'/d_1, (22) by $(U+W)'$ and (23) by V'/d_3, integrating them over $(-\infty, \infty)$ and taking the sum gives

$$-2s\int_{-\infty}^{\infty}\left[\left(\frac{1}{d_1}+1\right)(U'V') + \left(1+\frac{1}{d_3}\right)(W'V')\right] = I,$$

where the facts $U'(\pm\infty) = V'(\pm\infty) = W'(\pm\infty) = 0$ are used and

$$\begin{aligned} I \ :=\ & \int_{-\infty}^{\infty} U(1-U-b_2V)V' + \int_{-\infty}^{\infty} V[1-V-b_1(U+W)](U+W)' \\ & + \int_{-\infty}^{\infty} W(1-W-b_2V)V'. \end{aligned}$$

Next, we compute

$$\begin{aligned} \int_{-\infty}^{\infty} V(1-V)(U+W)' &= \int_{-\infty}^{\infty} V(U+W)' - \int_{-\infty}^{\infty} V^2(U+W)' \\ &= -\int_{-\infty}^{\infty}(U+W)V' + 2\int_{-\infty}^{\infty}(U+W)VV'. \end{aligned}$$

Also,

$$\begin{aligned} -b_1\int_{-\infty}^{\infty} V(U+W)(U+W)' &= -b_1\int_{-\infty}^{\infty} V[(U+W)^2/2]' \\ &= \frac{b_1}{2}\int_{-\infty}^{\infty}(U+W)^2V'. \end{aligned}$$

It follows that

$$\begin{aligned} I &= \int_{-\infty}^{\infty}\{(U+W)V' - (U^2+W^2)V'\} - b_2\int_{-\infty}^{\infty}(U+W)VV' \\ &\quad -\int_{-\infty}^{\infty}(U+W)V' + 2\int_{-\infty}^{\infty}(U+W)VV' + \frac{b_1}{2}\int_{-\infty}^{\infty}(U+W)^2V' \\ &= (2-b_2)\int_{-\infty}^{\infty}(U+W)VV' + \int_{-\infty}^{\infty}\left\{\frac{b_1}{2}(U+W)^2 - (U^2+W^2)\right\}V'. \end{aligned}$$

Hence $I < 0$ and so $s < 0$, if $b_2 \ge 2$ and $b_1 < 1$. The theorem follows. $\square$

This theorem gives us the case when species v wins the competition.
Next, we prepare a lemma as follows.

Lemma 3.3. *Let condition (32) be imposed such that $d_1 = d_3$ and let (s, U, V, W) be a solution to (21)-(24) such that (25) holds. Then we have $U = W$.*

Proof. In fact, it is easy to check that (s, W, V, U) is also a traveling front of system (3)-(5). Then the lemma follows from the uniqueness of wave profile. $\square$

With Lemma 3.3, we have the following interesting theorem.

Theorem 3.4. *Let (s, U, V, W) be a traveling front of system (3)-(5) connecting $(1, 0, 1)$ and $(0, 1, 0)$ such that (25) holds. Assume that $a_1 = a_3 = d_3 = d_1 = d > 0$, $b_3 = b_1$ and $2 > b_2 > 1 > b_1 > 0$. Then $s > 0$ if $b_2 < 2b_1$, $s = 0$ if $b_2 = 2b_1$, and $s < 0$ if $b_2 > 2b_1$.*

Proof. First, recall from Lemma 3.3 that $U = W$. Next, it is easy to check that (s, V, W) satisfies

$$\begin{aligned} &V'' + sV' + V(1 - V - 2b_1W) = 0,\ \xi \in \mathbb{R}, \\ &dW'' + sW' + dW(1 - W - b_2V) = 0,\ \xi \in \mathbb{R}, \end{aligned}$$

and the boundary conditions

$$(V, W)(-\infty) = (0, 1), \quad (V, W)(\infty) = (1, 0).$$

Hence the conclusion follows from Theorem 2.3. □

Theorem 3.4 tells us that two weak competitors can wipe out a strong competitor. This phenomenon can be observed for other ranges of parameters, by using theorems mentioned in section 2. We leave the details here to the interested reader.

3.2. **Case of three very strong competitors.** In this subsection, we consider the sign of the wave speed s determined by the following system:

$$\begin{cases} -sU'(\xi) = d_1U''(\xi) + a_1U(\xi)(1 - U - \beta_{12}kV)(\xi),\ \xi \in \mathbb{R}, \\ -sV'(\xi) = V''(\xi) + V(\xi)(1 - V - kU - \beta_{23}kW)(\xi),\ \xi \in \mathbb{R}, \\ -sW'(\xi) = d_3W''(\xi) + a_3W(\xi)(1 - W - \beta_{32}kV)(\xi),\ \xi \in \mathbb{R}, \\ (U, V, W)(-\infty) = (1, 0, 1), \quad (U, V, W)(\infty) = (0, 1, 0), \\ U' > 0, \quad V' < 0, \quad W' > 0 \quad \text{in } \mathbb{R}, \end{cases} \tag{33}$$

where $d_1, d_3, a_1, a_3, \beta_{12}, \beta_{23}, \beta_{32}$ are given positive constants and k is an arbitrarily large constant.

When $W \equiv 0$, system (33) is reduced to the one studied by Girardin and Nadin [9]. They consider the infinite competition limit (as $k \to \infty$) and the corresponding limiting problem has the segregation property, which has been discussed by Dancer et al. [4, 5] for competition-diffusion systems in bounded domains. It turns out the solution of the limiting problem corresponds to semi-waves studied by Du and Lin [6] (see also [2, 9] for more complete description). Then the sign of the wave speed can be determined in terms of the property of semi-waves. Following this idea, for system (33), if taking $k \to \infty$, species u (resp., w) and species v cannot coexist in the same interval. Therefore, we may expect that in the limiting problem, species u and w will coexist in $(-\infty, \xi_0)$ (since there is no interaction between u and w); while species v will occupy (ξ_0, ∞) for some $\xi_0 \in \mathbb{R}$.

Our main result of this subsection is as follows.

Theorem 3.5. *Given positive constants $d_1, d_3, a_1, a_3, \beta_{12}, \beta_{23}, \beta_{32}$, there exists a sufficiently large N_0 (depending on $d_1, d_3, a_1, a_3, \beta_{12}, \beta_{23}, \beta_{32}$) such that*

$$s > 0, \quad \textit{when } \frac{\sqrt{d_1}}{\beta_{12}\sqrt{a_1}} + \frac{\beta_{23}\sqrt{d_3}}{\beta_{32}\sqrt{a_3}} > 1; \quad s < 0, \quad \textit{when } \frac{\sqrt{d_1}}{\beta_{12}\sqrt{a_1}} + \frac{\beta_{23}\sqrt{d_3}}{\beta_{32}\sqrt{a_3}} < 1,$$

if $k \geq N_0$.

Theorem 3.5 reveals the role of the diffusion rates for the special 3-species competition-diffusion system. It shows that under a very strong competition, if species u or species w diffuses faster enough, then species v will lose in the competition; while species v will win the competition if both species u and w diffuse slowly enough.

With the help of Proposition A, we can estimate the wave speed as follows. This result is important to determine the sign of the wave speed.

Proposition 1. *Suppose that* (s, U, V, W) *is a solution of* (33) *for a given* $k > 0$. *Then*

$$-2 < s < \min\{2\sqrt{a_1 d_1}, 2\sqrt{a_3 d_3}\}.$$

Proof. Recall from the definition of $P_i^{\pm}$ $(i = 1, 2, 3)$ in the appendix that

$$P_i^+(-s/d_i) < 0$$

for $i = 1, 3$ and $P_2^-(-s) < 0$. By the definitions of λ_2, μ_1 and μ_3, we see that

$$\frac{-s}{d_i} > \mu_i, \quad i = 1, 3; \text{ and } -s < \lambda_2. \tag{34}$$

We now assume that $s > 0$. Inspired by [15, Lemma 3.6], we consider

$$\widehat{U}(\xi) := U(\xi) e^{\frac{s}{2d_1}\xi}, \quad \xi \in \mathbb{R}.$$

Clearly, $\widehat{U}(\cdot) > 0$ and $\widehat{U}(-\infty) = 0$, since $U(-\infty) = 1$ and $s > 0$. By Proposition A and (34), we have $\widehat{U}(+\infty) = 0$. Thus, there exists $z \in \mathbb{R}$ such that $\widehat{U}''(z) \le 0$. By some simple calculations, we have

$$0 \ge \widehat{U}''(z) = e^{\frac{s}{2d_1}\xi} \left(U''(z) + \frac{s}{d_1} U'(z) + \frac{s^2}{4d_1^2} U(z) \right).$$

Using U-equation in (33),

$$\begin{aligned} 0 \quad &\ge \quad \widehat{U}''(z) > e^{\frac{s}{2d_1}\xi} \left(-\frac{a_1}{d_1} U(1-U)(z) + \frac{s^2}{4d_1^2} U(z) \right) \\ &> \quad e^{\frac{s}{2d_1}\xi} U(z) \left(-\frac{a_1}{d_1} + \frac{s^2}{4d_1^2} \right). \end{aligned}$$

This implies that

$$-\frac{a_1}{d_1} + \frac{s^2}{4d_1^2} < 0, \text{ or equivalently } -2\sqrt{d_1 a_1} < s < 2\sqrt{d_1 a_1}.$$

Note that $s > 0$. We thus obtain $s < 2\sqrt{d_1 a_1}$. Similarly, if we define $\widehat{W}(\xi) := W(\xi) e^{\frac{s}{2d_3}\xi}$, the above process can be applied to derive $s < 2\sqrt{d_3 a_3}$.

We next assume that $s < 0$ and define

$$\widehat{V}(\xi) := V(\xi) e^{\frac{s}{2}\xi}, \quad \xi \in \mathbb{R}.$$

Clearly, $\widehat{V}(\cdot) > 0$ and $\widehat{V}(+\infty) = 0$. It follows from Proposition A and (34) that $\widehat{V}(-\infty) = 0$. Then there exists $\zeta \in \mathbb{R}$ such that $\widehat{V}''(\zeta) \le 0$. Using the above argument, we have

$$0 \ge \widehat{V}''(\zeta) > e^{\frac{s}{2}\xi} V(\zeta) \left(-1 + \frac{s^2}{4} \right).$$

This implies that $-2 < s < 2$. Since $s < 0$, we thus obtain $s > -2$. This completes the proof. $\square$

We now consider the limiting problem by a singular limit process. For each $k \in \mathbb{N}$, let $\{(s_k, U_k, V_k, W_k)\}$ be the solution to (33) with $s = s_k$. Thanks to Proposition 1, we may assume, up to extract a subsequence, that $s_k \to s^*$ as $k \to \infty$ for some $s^* \in [-2, \min\{2\sqrt{a_1 d_1}, 2\sqrt{a_3 d_3}\}]$. Moreover, without loss of generality we may assume that

$$\begin{cases} \min\{U_k(0), W_k(0)\} = 1/2 \quad \text{for each } k > 1 \text{ if } s^* \le 0; \\ V_k(0) = 1/2 \quad \text{for each } k > 1 \text{ if } s^* > 0. \end{cases} \tag{35}$$

This condition makes sure that the limit functions are not null.

Since we have $\|\omega_k\|_{L^\infty(\mathbb{R})} = 1$ for $\omega = U, V, W$, it is not hard to see that $\{U_k\}$, $\{V_k\}$ and $\{W_k\}$ are equicontinuous in $[-n, n]$ for any $n \in \mathbb{N}$ (see, e.g., [9, Proposition 3.1]). By Arzelá-Ascoli theorem, up to extract a subsequence, there exists $(U_*, V_*, W_*) \in [C(\mathbb{R})]^3$ such that

$$(U_k, V_k, W_k) \to (U_*, V_*, W_*) \quad \text{in } \mathbb{R}, \text{ as } k \to \infty,$$

uniformly on any compact subset of $\mathbb{R}$. Moreover, since $U_k' < 0$, $V_k' > 0$ and $W_k' < 0$ in $\mathbb{R}$,

$$U_*' \le 0, \quad V_*' \ge 0, \quad W_*' \le 0 \quad \text{in } \mathbb{R}. \tag{36}$$

The following result shows that the limit function (U_*, V_*, W_*) has the so-called segregation property.

Proposition 2. *There exists $\xi_0 \in \mathbb{R}$ such that (s^*, U_*, V_*, W_*) satisfies*

$$\begin{cases} s^* U_*' + d_1 U_*'' + a_1 U_*(1 - U_*) = 0, \quad \xi < \xi_0; \quad U_*(\xi) = 0, \quad \xi \ge \xi_0, \\ s^* V_*' + V_*'' + V_*(1 - V_*) = 0, \quad \xi > \xi_0; \quad V_*(\xi) = 0, \quad \xi \le \xi_0, \\ s^* W_*' + d_3 W_*'' + a_3 W_*(1 - W_*) = 0, \quad \xi < \xi_0; \quad W_*(\xi) = 0, \quad \xi \ge \xi_0, \\ (U_*, W_*)(-\infty) = (1, 1), \quad V_*(+\infty) = 1. \\ U_*'(\xi) \le 0, \quad W_*'(\xi) \le 0, \quad \xi < \xi_0; \quad V_*'(\xi) \ge 0, \quad \xi > \xi_0 \end{cases} \tag{37}$$

with

$$U_*'(\xi_0) < 0, \quad W_*'(\xi_0) < 0, \quad V_*'(\xi_0) = -\frac{d_1}{a_1 \beta_{12}} U_*'(\xi_0) - \frac{d_3 \beta_{23}}{a_3 \beta_{32}} W_*'(\xi_0) > 0. \tag{38}$$

Note that $V_*'(\xi_0)$ (resp., $U_*'(\xi_0)$, $W_*'(\xi_0)$) is considered as the right (resp., left) derivative. To prove Proposition 2, we prepare the following two lemmas.

Lemma 3.6. *$U_* V_* = W_* V_* = 0$ in $\mathbb{R}$.*

Proof. We can follow the same line as in [9, Lemma 3.2] to show this result. For reader's convenience, we give the details here. Multiplying U_k-equation by a test function $\varphi \in C_0^\infty(\mathbb{R})$ and integrating over $(-\infty, \infty)$, we have

$$\begin{aligned} k a_1 \beta_{12} \Big| \int_{-\infty}^{\infty} U_k V_k \varphi \Big| \quad &\le \quad |s_k| \int_{-\infty}^{\infty} U_k |\varphi'| + d_1 \int_{-\infty}^{\infty} U_k |\varphi''| \\ &\quad + a_1 \int_{-\infty}^{\infty} U_k (1 - U_k) |\varphi|, \end{aligned}$$

where we have used the integration by parts. By Proposition 1 and the fact that $0 < U_k < 1$ for all k, we have $|\int_{-\infty}^{\infty} U_k V_k \varphi| \le C \|\varphi\|_{C^2}/k$ for some constant $C > 0$ independent of k. By taking $k \to \infty$, $U_* V_* = 0$. Similarly, we have $W_* V_* = 0$ and thus the lemma follows. $\square$

Next, multiplying U_k-equation by $-1/(a_1\beta_{12})$ and W_k-equation by $-\beta_{23}/(a_3\beta_{32})$, and then summing the two equations with V_k-equation, we obtain

$$\frac{s_k}{a_1\beta_{12}}U_k' + \frac{s_k\beta_{23}}{a_3\beta_{32}}W_k' - s_kV_k' = -\frac{d_1}{a_1\beta_{12}}U_k'' - \frac{d_3\beta_{23}}{a_3\beta_{32}}W_k'' + V_k''$$
$$-\frac{1}{\beta_{12}}U_k(1-U_k) - \frac{\beta_{23}}{\beta_{32}}W_k(1-W_k) + V_k(1-V_k).$$

It follows that

$$\frac{s^*}{a_1\beta_{12}}U_*' + \frac{s^*\beta_{23}}{a_3\beta_{32}}W_*' - s^*V_*' = -\frac{d_1}{a_1\beta_{12}}U_*'' - \frac{d_3\beta_{23}}{a_3\beta_{32}}W_*'' + V_*''$$
$$-\frac{1}{\beta_{12}}U_*(1-U_*) - \frac{\beta_{23}}{\beta_{32}}W_*(1-W_*) + V_*(1-V_*). \tag{39}$$

in the weak sense. Define

$$G(\xi) := \frac{d_1}{a_1\beta_{12}}U_*(\xi) + \frac{d_3\beta_{23}}{a_3\beta_{32}}W_*(\xi) - V_*(\xi).$$

Lemma 3.7. $G \in C^1(\mathbb{R})$.

Proof. Since U_*, V_* and W_* are continuous in $\mathbb{R}$, with (39) one can follow the process in [9, Lemma 3.5] to finish the proof. We omit the details here. □

We are ready to show Proposition 2.

Proof of Proposition 2. We only consider the case $s^* \le 0$, since the proof for the case $s^* > 0$ is parallel. Define

$$S(U_*) := \{\xi \in \mathbb{R} |\, U_*(\xi) > 0\}.$$

From (36) and (35) we have $U_*' \le 0$ and $U_*(0) \ge 1/2$. Then

$$S(U_*) := (-\infty, A) \quad \text{for some } A \in (0,\infty) \cup \{+\infty\}.$$

If $A = +\infty$, from Lemma 3.6 we see that $V_* \equiv 0$. Using the standard elliptic regularity theory, we see that $U_* \in C^2(\mathbb{R})$ satisfies

$$-s^*U_*' = d_1U_*'' + a_1U_*(1-U_*), \quad \xi \in \mathbb{R},$$

in the classical sense. Using (35) and the monotonicity of U_*, (s^*, U_*) is exactly a traveling front solution connecting 1 and 0, which gives $s^* \ge 2\sqrt{a_1d_1}$. This contradicts with $s^* \le 0$. Hence, $A \in (0,\infty)$. Similarly, we can show that there exists $B \in (0,\infty)$ such that $S(W_*) = (-\infty, B)$.

Next, we shall show that $A = B$. If $A \ne B$, without loss of generality we may assume that $A > B$. In this case, Lemma 3.6 yields that $W_* = V_* = 0$ in $[B, A]$. By Lemma 3.7, $U_*'(A) = 0$. Also, recall that $U_*(A) = 0$. By the uniqueness of solutions of ODEs, we obtain $U_* \equiv 0$, which is impossible. Therefore, we obtain $A = B$. Moreover, we have $S(V_*) = (A, \infty)$ in view of Lemma 3.6.

Define $\xi_0 = A$. Then we see that (U_*, V_*, W_*) satisfies (37) except the boundary conditions. Thanks to $0 \le U_*, V_*, W_* \le 1$ and (36), the simple phase plane analysis insures the boundary conditions. To complete the proof, it suffices to show (38). Indeed, we must have $U_*'(\xi_0^-) < 0$. Otherwise, by the uniqueness of solutions of

ODEs, $U_* \equiv 0$, which is impossible. Similarly, $W_*'(\xi_0^-) < 0$ holds true. Next, by Lemma 3.7,

$$\begin{aligned} & \frac{d_1}{a_1\beta_{12}}U_*'(\xi_0^-) + \frac{d_3\beta_{23}}{a_3\beta_{32}}W_*'(\xi_0^-) - V_*'(\xi_0^-) \\ = \quad & \frac{d_1}{a_1\beta_{12}}U_*'(\xi_0^+) + \frac{d_3\beta_{23}}{a_3\beta_{32}}W_*'(\xi_0^+) - V_*'(\xi_0^+). \end{aligned}$$

Since $V_*'(\xi_0^-) = U_*'(\xi_0^+) = W_*'(\xi_0^+) = 0$, we have

$$V_*'(\xi_0^+) = -\frac{d_1}{a_1\beta_{12}}U_*'(\xi_0^-) - \frac{d_3\beta_{23}}{a_3\beta_{32}}W_*'(\xi_0^-) > 0,$$

which implies (38). Thus, we complete the proof. □

Recall that $s^* \in [-2, \min\{2\sqrt{a_1d_1}, 2\sqrt{a_3d_3}\}]$. In fact, s^* cannot be an endpoint.

Lemma 3.8. *There holds that*

$$-2 < s^* < \min\{2\sqrt{a_1d_1}, 2\sqrt{a_3d_3}\}.$$

Proof. The proof can be done by using the argument of [9, Lemma 3.14]. For reader's convenience, we give a sketch of proof here.

Suppose that $s^* = -2$. Fix $\eta > \xi_0$ such that $V_*(\eta) = 1/2$, where ξ_0 is defined in Proposition 2. Let $\widetilde{V}$ be a traveling wave for Fisher-KPP equation with speed -2 (exactly the maximal wave speed) such that

$$\widetilde{V}'' - 2\widetilde{V} + \widetilde{V}(1-\widetilde{V}) = 0,\ \xi \in \mathbb{R}, \quad \widetilde{V}(-\infty) = 0, \quad \widetilde{V}(+\infty) = 1, \quad \widetilde{V}(\eta) = 1/2.$$

Now consider $Q(\xi) = \widetilde{V}(\xi) - V_*(\xi)$. By applying the maximum principle one has $Q \equiv 0$ (cf. [9, Lemma 3.14]). However, by Proposition 2, we see that $V_*(\xi_0) = 0$, which implies $Q(\xi_0) = \widetilde{V}(\xi_0) > 0$. This leads a contradiction with $Q \equiv 0$. Hence, $s^* > -2$. Similar process can be applied to show both $s^* < 2\sqrt{a_1d_1}$ and $s^* < 2\sqrt{a_3d_3}$. Thus, we complete the proof. □

Proposition 2 and Lemma 3.8 imply that U_*, V_* and W_* are exactly semi-waves constructed by in [6, 2, 9]. Therefore, the wave profile (U_*, V_*, W_*) of the limiting problem is unique. Let us recall a result in [9]:

Proposition 3 (Theorem 3.16 of [9])**.** *For any $s > -2$,*

$$sQ' + Q'' + (1-Q)Q = 0 \text{ in } (0, +\infty),\ Q(0) = 0$$

has a unique positive solution $Q = Q_s$. Moreover, $Q_s'(\cdot) > 0$ in $[0,\infty)$ and $Q_s(\infty) = 1$. In addition, $Q_s'(0)$ is increasing and continuous in s.

Thanks to Proposition 3, we can now show our main result of this subsection.

Proof of Theorem 3.5. Let (s^*, U_*, V_*, W_*) satisfy (37) with (38). Without of loss generality, we may assume that $\xi_0 = 0$ (ξ_0 is defined in Proposition 2). From Proposition 2, we see that U_*, V_* and W_* can be seen as three semi-wave of the Fisher-KPP equation. It turns out that we can use a similar argument in [9, Theorem 4.1] to finish the proof, which is given as follows. By a suitable scaling, we have

$$U_*'(0) = -\sqrt{\frac{a_1}{d_1}}Q'_{(-\frac{s^*}{\sqrt{a_1d_1}})}(0),\ V_*'(0) = Q'_{s^*}(0),\ W_*'(0) = -\sqrt{\frac{a_3}{d_3}}Q'_{(-\frac{s^*}{\sqrt{a_3d_3}})}(0).$$

Thus, condition (38) becomes

$$Q'_{s^*}(0) = \frac{\sqrt{d_1}}{\beta_{12}\sqrt{a_1}} Q'_{(-\frac{s^*}{\sqrt{a_1 d_1}})}(0) + \frac{\beta_{23}\sqrt{d_3}}{\beta_{32}\sqrt{a_3}} Q'_{(-\frac{s^*}{\sqrt{a_3 d_3}})}(0). \tag{40}$$

When $\frac{\sqrt{d_1}}{\beta_{12}\sqrt{a_1}} + \frac{\beta_{23}\sqrt{d_3}}{\beta_{32}\sqrt{a_3}} > 1$, using (40) and the fact $Q'_s(0) > 0$ that

$$Q'_{s^*}(0) > \min\left\{ Q'_{(-\frac{s^*}{\sqrt{a_1 d_1}})}(0), Q'_{(-\frac{s^*}{\sqrt{a_3 d_3}})}(0) \right\}. \tag{41}$$

Note that $Q'_s(0)$ is increasing in s (Proposition 3). Then we see from (41) that $s^* > 0$. Similarly, when $\frac{\sqrt{d_1}}{\beta_{12}\sqrt{a_1}} + \frac{\beta_{23}\sqrt{d_3}}{\beta_{32}\sqrt{a_3}} < 1$, we can derive that $s^* < 0$. Since $s_k \to s^*$ as $k \to \infty$, we thus complete the proof. □

4. **Appendix.** In the appendix, we provide the asymptotic behavior of the wave profile of traveling wave solutions (U, V, W) as $\xi \to \pm\infty$. We assume that (s, U, V, W) satisfies

$$\begin{cases} -sU'(\xi) = d_1 U''(\xi) + a_1 U(\xi)(1 - U - b_2 V)(\xi), \ \xi \in \mathbb{R}, \\ -sV'(\xi) = V''(\xi) + V(\xi)(1 - V - b_1 U - b_3 W)(\xi), \ \xi \in \mathbb{R}, \\ -sW'(\xi) = d_3 W''(\xi) + a_3 W(\xi)(1 - W - b_4 V)(\xi), \ \xi \in \mathbb{R}, \\ (U, V, W)(-\infty) = (1, 0, 1), \quad (U, V, W)(\infty) = (0, 1, 0), \\ 0 \le U, V, W \le 1, \ \xi \in \mathbb{R}, \end{cases} \tag{42}$$

where

$$b_2 > 1, \quad b_4 > 1, \quad b_1 + b_3 > 1. \tag{43}$$

Define

$$\begin{aligned} P_j^-(\lambda) &:= d_j \lambda^2 + s\lambda - a_j, \quad j = 1, 3, \\ P_2^-(\lambda) &:= \lambda^2 + s\lambda + 1 - b_1 - b_3, \\ P_1^+(\lambda) &:= d_1 \lambda^2 + s\lambda + 1 - b_2, \\ P_2^+(\lambda) &:= \lambda^2 + s\lambda - 1, \\ P_3^+(\lambda) &:= d_3 \lambda^2 + s\lambda + 1 - b_4, \end{aligned}$$

and λ_j (resp., μ_j) denotes the positive (resp., negative) root of P_j^- (resp., P_j^+), $j = 1, 2, 3$.

Applying a similar approach as in [21] but with some more tedious calculations, we have the following result. Here we omit the proof.

Proposition A. *Let (s, U, V, W) be a solution of (42) under condition (43). Then the following hold:*

(1) there exist $C_i > 0$, $i = 1, 2, 3$, such that

$$\lim_{\xi \to -\infty} \frac{1 - U(\xi)}{|\xi|^p e^{\min\{\lambda_1, \lambda_2\}\xi}} = C_1, \ \lim_{\xi \to -\infty} \frac{V(\xi)}{e^{\lambda_2 \xi}} = C_2, \ \lim_{\xi \to -\infty} \frac{1 - W(\xi)}{|\xi|^q e^{\min\{\lambda_3, \lambda_2\}\xi}} = C_3,$$

where

$$p = \begin{cases} 0, & \text{if } \lambda_1 \ne \lambda_2, \\ 1, & \text{if } \lambda_1 = \lambda_2, \end{cases}, \quad q = \begin{cases} 0, & \text{if } \lambda_3 \ne \lambda_2, \\ 1, & \text{if } \lambda_3 = \lambda_2, \end{cases},$$

(2) there exist $C_i > 0$, $i = 4, 5, 6$, *such that*

$$\lim_{\xi\to\infty} \frac{U(\xi)}{e^{\mu_1\xi}} = C_4, \quad \lim_{\xi\to\infty} \frac{V(\xi)}{|\xi|^r e^{\min\{\mu_1,\mu_2,\mu_3\}\xi}} = C_5, \quad \lim_{\xi\to\infty} \frac{W(\xi)}{e^{\mu_3\xi}} = C_6,$$

where

$$r = \begin{cases} 0, & \text{if } \mu_2 \neq \mu_j \text{ for } j = 1, 3, \\ 1 \text{ or } 2 & \text{if } \mu_2 = \mu_j \text{ for some } j \in \{1, 3\}, \end{cases}$$

Acknowledgments. We would like to dedicate this work to Professor Wei-Ming Ni on the occasion of his 70th birthday.

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Received xxxx 20xx; revised xxxx 20xx.

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