CONVERGENCE TO STATIONARY SOLUTIONS IN A ONE-SPECIES SYSTEM OF FISHER-KPP TYPE OVER A RIVER NETWORK

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Abstract. In this paper, we derive the convergence to stationary solutions for a one-species system of Fisher-KPP type in a simple two-branches river network. Our method is based on the spectrum analysis of the associated linear operator to a system satisfied by the weighted entropy function of original solution on a suitable weighted Hilbert space.

1. Introduction

Recently, the study of solution behaviors for certain biological systems defined in river networks has attracted more and more attention. For this, we refer the reader to, e.g., [6, 7, 5, 2, 1, 4, 8]. Among others, the simplest river network is the two-branches system. In particular, the following system is considered by Du, Lou, Peng and Zhou ([2]):

$$\begin{cases}\partial_t u_L = \partial_x^2 u_L - \beta_L \partial_x u_L + u_L(1-u_L), & x \in (0,+\infty) := \mathbb{R}_L,\ t>0,\\ \partial_t u_U = \partial_x^2 u_U - \beta_U \partial_x u_U + u_U(1-u_U), & x \in (-\infty,0) := \mathbb{R}_U,\ t>0\end{cases} \tag{1.1}$$

along with the boundary conditions

$$u_L(0,t) = u_U(0,t), \quad a_L\partial_x u_L(0,t) = a_U \partial_x u_U(0,t), \quad t>0, \tag{1.2}$$

where $\beta_L, \beta_U, a_L, a_U$ are positive constants in which β_L, β_U are the water flow speeds and a_L, a_U account for the cross-section area of the lower and upper river branches, respectively. Note that the condition

$$a_L\beta_L = a_U\beta_U \tag{1.3}$$

holds due to the conservation of flow at the junction point $x=0$.

It is shown in [2] that, under the assumption

$$\beta_U \geq 2 > \beta_L, \tag{1.4}$$

there exists a one parameter family of stationary solutions of (1.1)-(1.2). More precisely, there exists $\alpha_0 \in (0,1)$ such that for each $\alpha \in [\alpha_0, 1)$ there exist unique positive monotone increasing

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functions $\phi_L(x;\alpha)$ and $\phi_U(x;\alpha)$ satisfying

$$\begin{cases} -\phi_L'' + \beta_L \phi_L' = \phi_L - \phi_L^2, & x \in (0,+\infty), \\ -\phi_U'' + \beta_U \phi_U' = \phi_U - \phi_U^2, & x \in (-\infty,0), \\ \phi_L(0) = \phi_U(0) = \alpha \in (0,1), \\ a_L \phi_L'(0) = a_U \phi_U'(0), \end{cases} \tag{1.5}$$

and

$$\phi_U(-\infty;\alpha) = 0, \quad \phi_L(+\infty;\alpha) = 1. \tag{1.6}$$

On the other hand, if $\alpha \in (0,\alpha_0)$, there is no stationary solution satisfying (1.5) and (1.6).

Concerning the asymptotic behavior, according to [2], if $u(x,0)$ is a compactly supported non-negative non-zero function, then the solution of (1.1)-(1.2) with this initial function at $t = 0$ converges to the minimal solution $(\phi_L(\cdot,\alpha_0), \phi_U(\cdot,\alpha_0))$ as $t \to +\infty$ locally uniformly in $\mathbb{R}$. However, the convergence to non-minimal stationary solutions is left open in [2]. The main purpose of this short note is to resolve this problem.

To this aim, for a given $\alpha \in [\alpha_0, 1)$, we define the entropy function

$$S_j(x,t) = S_j(x,t;\alpha) := u_j(x,t) - \phi_j(x;\alpha) - \phi_j(x;\alpha) \log \frac{u_j(x,t)}{\phi_j(x;\alpha)} \geq 0, \quad j = L, U, \tag{1.7}$$

where (u_L, u_U) is a positive solution of (1.1)-(1.2). Here $S_j(x,t) \geq 0$ can be shown by using $X - 1 - \log X \geq 0$ for all $X > 0$, where $X := u_j(x,t)/\phi_j(x;\alpha)$. Moreover, $S_j(x,t) = 0$ if and only if $u_j(x,t) = \phi_j(x;\alpha)$.

We now state the main theorem of this paper as follows.

Theorem 1.1. *Suppose that $\beta_U > 2$ and $\beta_L \in (0,2)$. Let (u_L, u_U) be the solution of* (1.1)-(1.2) *with initial data $(u_L, u_U)(x,0)$ satisfying*

$$e^{-\frac{\beta_j}{2}x} S_j(x,0;\alpha) \in L^2(\mathbb{R}_j), \ j = L, U, \tag{1.8}$$

for some $\alpha \in (\alpha_0, 1)$. Then $u_j(x,t) \to \phi_j(x;\alpha)$ locally uniformly in $\mathbb{R}_j$ as $t \to \infty$ for $j = L, U$.

The proof of Theorem 1.1 is given in the next section. We remark that for a given initial data $(u_L, u_U)(x,0)$ there is at most one α satisfying condition (1.8). Indeed, assume that there are $\alpha_1, \alpha_2 \in (\alpha_0, 1)$ such that (1.8) holds for $\alpha = \alpha_i$, $i = 1, 2$. Then, by Theorem 1.1, $u_L(0,t) \to \phi_L(0;\alpha_i)$ as $t \to \infty$ for $i = 1, 2$. Hence $\alpha_1 = \phi_L(0;\alpha_1) = \phi_L(0;\alpha_2) = \alpha_2$.

2. Proof of Theorem 1.1

In the following, let $\alpha \in (\alpha_0, 1)$ be fixed. By a simple calculation, we have

$$\partial_t S_j = \partial_x^2 S_j - \beta_j \partial_x S_j + (1 - \phi_j) S_j - (u_j - \phi_j)^2 - \left(\frac{\sqrt{\phi_j}\,\partial_x u_j}{u_j} - \frac{\phi_j'}{\sqrt{\phi_j}} \right)^2, j = L, U. \tag{2.1}$$

See also [3] for the detailed calculation. It is easy to check the boundary conditions

$$S_L(0,t) = S_U(0,t), \quad t > 0,$$

and

$$a_L \partial_x S_L(0,t) = a_U \partial_x S_U(0,t), \quad t > 0,$$

hold. From (2.1), we have

$$\partial_t S_j \leqq \partial_x^2 S_j - \beta_j \partial_x S_j + (1 - \phi_j) S_j, \quad j = L, U.$$

First, we consider the entropy equation

$$\partial_t \hat{S}_j = \partial_x^2 \hat{S}_j - \beta_j \partial_x \hat{S}_j + (1-\phi_j)\hat{S}_j, \quad j = L, U, \tag{2.2}$$

posed on two half-lines $\mathbb{R}_L = (0,\infty)$ and $\mathbb{R}_U = (-\infty, 0)$, coupled at $x = 0$ by

$$\hat{S}_L(0,t) = \hat{S}_U(0,t), \quad a_L \partial_x \hat{S}_L(0,t) = a_U \partial_x \hat{S}_U(0,t). \tag{2.3}$$

We assume $\beta_U \geq 2 > \beta_L$. Define branchwise

$$\hat{S}_j(x,t) = e^{\frac{\beta_j}{2}x} \hat{Z}_j(x,t), \ j = L, U. \tag{2.4}$$

Then

$$\partial_t \hat{Z}_j = \partial_x^2 \hat{Z}_j + V_j(x)\hat{Z}_j, \ j = L, U$$

where

$$V_j(x) = 1 - \phi_j(x) - \frac{\beta_j^2}{4}, \ j = L, U.$$

Note that $V_j \in L^\infty(\mathbb{R}_j)$, $j = L, U$. For the boundary conditions, the continuity condition in (2.3) gives us $\hat{Z}_L(0,t) = \hat{Z}_U(0,t)$ for all $t \geq 0$. Moreover, the flux condition in (2.3) yields

$$a_L\Big(\partial_x \hat{Z}_L(0,t) + \frac{\beta_L}{2}\hat{Z}_L(0,t)\Big) = a_U\Big(\partial_x \hat{Z}_U(0,t) + \frac{\beta_U}{2}\hat{Z}_U(0,t)\Big)$$

for $t \geq 0$. Thus we conclude from (1.3) that

$$a_L \partial_x \hat{Z}_L(0,t) = a_U \partial_x \hat{Z}_U(0,t).$$

Now we define the operator

$$\mathcal{L}Z := (Z_L'' + V_L Z_L, \ Z_U'' + V_U Z_U)$$

with domain

$$\mathcal{D}(\mathcal{L}) = \{Z := (Z_L, Z_U) \mid \ Z_j \in H^2(\mathbb{R}_j), Z_L(0) = Z_U(0), a_L Z_L'(0) = a_U Z_U'(0)\}.$$

Let us introduce the weighted Hilbert space.

$$X := L^2(a_L dx) \oplus L^2(a_U dx)$$

with the weighted inner product

$$\langle Z, Y\rangle_X = a_L \int_0^\infty Z_L \overline{Y_L}\, dx + a_U \int_{-\infty}^0 Z_U \overline{Y_U}\, dx.$$

Hereafter $\overline{Y_j}$ denotes the complex conjugate of Y_j. We compute

$$a_L \int_0^\infty Z_L'' \overline{Y_L}\, dx = -a_L Z_L'(0)\overline{Y_L(0)} - a_L \int_0^\infty Z_L' \overline{Y_L}' dx$$

and

$$a_U \int_{-\infty}^0 Z_U'' \overline{Y_U}\, dx = a_U Z_U'(0)\overline{Y_U(0)} - a_U \int_{-\infty}^0 Z_U' \overline{Y_U}' dx.$$

Therefore, the total boundary contribution is

$$-a_L Z_L'(0)\overline{Y_L(0)} + a_U Z_U'(0)\overline{Y_U(0)} = (-a_L Z_L'(0) + a_U Z_U'(0))\overline{Y_L(0)} = 0,$$

by the vertex conditions

$$Z_L(0) = Z_U(0), \ a_L Z_L'(0) = a_U Z_U'(0).$$

Thus

$$\langle \mathcal{L}Z, Y\rangle_X = -a_L \int_0^\infty Z_L' \overline{Y_L}' dx - a_U \int_{-\infty}^0 Z_U' \overline{Y_U}' dx + a_L \int_0^\infty V_L Z_L \overline{Y_L} + a_U \int_{-\infty}^0 V_U Z_U \overline{Y_U},$$

and, by symmetry, we conclude that

$$\langle \mathcal{L}Z, Y\rangle_X = \langle Z, \mathcal{L}Y\rangle_X, \ Z, Y \in \mathcal{D}(\mathcal{L}),$$

since both V_L and V_U are real-valued. Therefore, we have proved the following proposition.

Proposition 2.1. *The operator $\mathcal{L}$ is self-adjoint on the weighted Hilbert space X.*

Since $V_j \in L^\infty(\mathbb{R}_j)$ and

$$V_L(x) \to c_L := -\frac{\beta_L^2}{4}, \quad V_U(x) \to c_U := 1 - \frac{\beta_U^2}{4},$$

as $x \to \pm\infty$, the operator $\mathcal{L}$ is a compact perturbation of the constant-coefficient operators

$$\mathcal{L}_L^\infty = \frac{d^2}{dx^2} + c_L, \quad \mathcal{L}_U^\infty = \frac{d^2}{dx^2} + c_U$$

on the corresponding half-lines. Hence, by Weyl's theorem on the stability of the essential spectrum,

$$\sigma_{\text{ess}}(\mathcal{L}) = (-\infty, \max\{c_L, c_U\}].$$

If $\beta_U > 2$, then $c_U < 0$, and therefore

$$\sigma_{\text{ess}}(\mathcal{L}) \subset (-\infty, 0).$$

We discuss the absence of nonnegative eigenvalues from Sturm–Liouville approach.

Theorem 2.2. *If $\beta_U > 2$ and $\beta_L \in (0, 2)$, then $\mathcal{L}$ has no nonnegative eigenvalues.*

Proof. Suppose $\mathcal{L}Z = \lambda Z$ with $\lambda \geq 0$. Since λ lies above the essential spectrum, it must be a discrete eigenvalue and $Z \in L^2(\mathbb{R}_L) \oplus L^2(\mathbb{R}_U)$.

For $|x|$ sufficiently large, $V_j(x)$ is arbitrarily close to c_j, hence the eigenvalue equation becomes asymptotically

$$\mathcal{L}_j^\infty W = W'' + c_j W = \lambda W.$$

Thus

$$W'' = (\lambda - c_j)W.$$

Because $c_j < 0$ and $\lambda \geq 0$,

$$\lambda - c_j > 0.$$

Hence the asymptotic solutions are proportional to

$$e^{\pm\sqrt{\lambda - c_j}\, x}.$$

On $\mathbb{R}_L = (0, \infty)$ and $\mathbb{R}_U = (-\infty, 0)$, square-integrability forces

$$W_L(x) = Const.\ e^{-\sqrt{\lambda - c_L}\, x}, \quad W_U(x) = Const.\ e^{\sqrt{\lambda - c_U}\, x}. \tag{2.5}$$

First, we shall prove that $\lambda = 0$ is not an eigenvalue. For contradiction, we assume that there exists $(Z_L, Z_U) \in \mathcal{D}(\mathcal{L}) \setminus \{(0, 0)\}$ such that $\mathcal{L}(Z_L, Z_U) = (0, 0)$. From the standard theory of the second order linear ordinary differential equation, there exist constants $A, B \in \mathbb{R}$ such that

$$e^{\frac{\beta_L}{2}x} Z_L(x) = A\phi_L(x) + B\phi_L(x)\Big(\int_0^x \frac{e^{\beta_L s}}{\phi_L^2(s)} ds\Big). \tag{2.6}$$

Indeed, one can check that $F(x) := e^{-\frac{\beta_L}{2}x}\phi_L$ satisfies

$$F'' + \left(1 - \phi_L - \frac{\beta_L^2}{4}\right) F = 0,\ x \in \mathbb{R}_L.$$

By setting $F(x) := e^{-\frac{\beta_L}{2}x}\phi_L(x)G(x)$, so that G satisfies

$$G'' + \left(2\frac{\phi_L'}{\phi_L} - \beta_L\right) G' = 0,$$

then (2.6) follows from the variation of constants method. Note that $Z_L \approx W_L$ as $x \to \infty$ for some $W_L(x)$. Therefore, $\phi_L(\infty) = 1$, (2.5) and (2.6) give us

$$Const.\ \approx e^{\frac{\beta_L}{2}x}Z_L(x) = A\phi_L(x) + B\phi_L(x)\Big(\int_0^x \frac{e^{\beta_L s}}{\phi_L^2(s)}ds\Big) \approx A + Be^{\beta_L x} + O(1)\ \text{ as } x \to \infty.$$

Thus, we conclude that $B = 0$.

Similarly, we have

$$e^{\frac{\beta_U}{2}x}Z_U(x) = C\phi_U(x) + D\phi_U(x)\Big(\int_0^x \frac{e^{\beta_U s}}{\phi_U^2(s)}ds\Big)$$

for some constants C, D. Recall from Theorem 1.1 (iv) of [2], when $\beta_U > 2$ and $\alpha \in (\alpha_0, 1)$,

$$\phi_U(x) \approx Const.\ e^{\frac{\beta_U - \sqrt{\beta_U^2 - 4}}{2}x}\ \text{ as } x \to -\infty. \tag{2.7}$$

Using $Z_U \approx W_U$ with $\lambda = 0$ as $x \to -\infty$, we deduce that

$$e^{\frac{\beta_U}{2}x}Z_U(x) \approx Const.\ e^{\frac{\beta_U + \sqrt{\beta_U^2 - 4}}{2}x}\ \text{ as } x \to -\infty,$$

and hence $C = 0$, due to (2.7). Therefore, we have $e^{\frac{\beta_L}{2}x}Z_L = A\phi_L$ on $\mathbb{R}_L$ and $e^{\frac{\beta_U}{2}x}Z_U(x) = D\phi_U(x)\Big(\int_0^x \frac{e^{\beta_U s}}{\phi_U^2(s)}ds\Big)$ on $\mathbb{R}_U$ for some $A, D \in \mathbb{R}$. Moreover, by substituting $x = 0$, we have $Z_U(0) = 0$. Then the junction condition yields $Z_L(0) = Z_U(0) = 0$, which gives us $A = 0$ and so $Z_L \equiv 0$. Then $a_L Z_L'(0) = a_U Z_U'(0)$ yields $Z_U'(0) = 0$. It follows from the standard uniqueness theory of initial value problem that $Z_U \equiv 0$. This contradiction shows that $\lambda = 0$ is not an eigenvalue.

Next, we shall prove that $\lambda > 0$ is not an eigenvalue. For contradiction, we assume that there exists $(\psi_L, \psi_U) \in \mathcal{D}(\mathcal{L}) \setminus \{(0,0)\}$ such that $\mathcal{L}(\psi_L, \psi_U) = \lambda(\psi_L, \psi_U)$ for some $\lambda > 0$. Without loss of generality, we may assume that $\psi_L(0) \geq 0$. Because ϕ_j are positive, the functions

$$P_j := \frac{\psi_j}{e^{-\frac{\beta_j}{2}x}\phi_j}, \quad j = L, U,$$

are well-defined. By a simple calculation, we obtain that

$$P_j'' + 2\frac{\frac{d}{dx}\big(e^{-\frac{\beta_j}{2}x}\phi_j\big)}{e^{-\frac{\beta_j}{2}x}\phi_j}P_j' = \lambda P_j, \qquad j = L, U \tag{2.8}$$

along with the boundary conditions

$$P_L(0) = P_U(0) \tag{2.9}$$

and

$$a_L P_L'(0) = a_U P_U'(0). \tag{2.10}$$

According to (2.5), $\phi_L(\infty)=1$ and (2.7), we have

$$\psi_L(x)\approx Const.\ e^{-\sqrt{\lambda+\beta_L^2/4}\,x},\quad e^{-\frac{\beta_L}{2}x}\phi_L(x)\approx Const.\ e^{-\frac{\beta_L}{2}x},\ \text{ as } x\to\infty,$$
$$\psi_U(x)\approx Const.\ e^{\sqrt{\lambda+\beta_U^2/4-1}\,x},\quad e^{-\frac{\beta_U}{2}x}\phi_U(x)\approx Const.\ e^{-\sqrt{\beta_U^2/4-1}\,x},\ \text{ as } x\to-\infty.$$

We see that $P_j(x)$ decays to zero exponentially fast as $x\to\pm\infty$ for $j=L,U$, respectively. Since $P_L(0)\geq 0$, $P_L(\infty)=0$ and, by (2.8), P_L cannot take a positive maximum or a negative minimum in $(0,\infty)$, we obtain that P_L is monotone decreasing in $\mathbb{R}_L$. In particular, $P_L\geq 0\geq P_L'$ in $\mathbb{R}_L$. If $\psi_L(0)=0$, then $P_L\equiv 0$ on $\mathbb{R}_L$. Hence $P_U(0)=P_L(0)=0$ and $P_U'(0)=0$, due to (2.10) and $P_L'(0)=0$. It follows from the uniqueness theory of initial value problem for (2.8) that $P_U\equiv 0$. This implies that $(\psi_L,\psi_U)=(0,0)$, a contradiction. Hence $\psi_L(0)>0$.

On the other hand, since $P_U(0)=P_L(0)>0$ and $P_U(-\infty)=0$, it follows from (2.8) again that P_U is monotone increasing in $\mathbb{R}_U$. However, $P_U'(0)\leq 0$, due to (2.10) and $P_L'(0)\leq 0$. Hence $P_U'(0)=0$ and so $P_L'(0)=0$. It then follows from (2.8) that $\lim_{x\to 0^+}P_L''(x)>0$. This implies that $P_L'(x)>0$ for sufficiently small $x\in\mathbb{R}_L$, a contradiction to $P_L'\leq 0$ in $\mathbb{R}_L$. We conclude that any positive λ is not an eigenvalue of $\mathcal{L}$. Therefore, we have proved the theorem. ☐

Now we prove the exponential stability, when $\beta_U>2$ and $\beta_L\in(0,2)$. We assume that

$$e^{-\frac{\beta_j}{2}x}S_j(x,0)\in L^2(\mathbb{R}_j),\ j=L,U.$$

Let $\hat{S}(x,t)$ be the solution of (2.2) and (2.3) with the initial function $S_j(x,0)$, $j=L,U$, at $t=0$. Recall $\hat{Z}$ from (2.4). Since the spectrum of $\mathcal{L}$ is contained in $(-\infty,-\eta]$ for some $\eta>0$, the spectral theorem implies that

$$\|\hat{Z}_j(\cdot,t)\|_{L^2(\mathbb{R}_j)}\leq Ce^{-\eta t},\ j=L,U,$$

for some positive constant C. Thus, by the comparison principle, we have

$$e^{-\frac{\beta_j}{2}x}S_j(x,t)\to 0\quad\text{in}\quad L^2(\mathbb{R}_j),\ j=L,U,\ \text{ as } t\to\infty.$$

In particular, by the parabolic regularity, $S_j(x,t)\to 0$ locally uniformly in $\mathbb{R}_j$, $j=L,U$, as $t\to\infty$. Thereby we have completed the proof of Theorem 1.1. ☐

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