STABILITY OF TRAVELING WAVES IN NON-COOPERATIVE SYSTEMS WITH NONLOCAL DISPERSAL OF EQUAL DIFFUSIVITIES

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Abstract. In this work, we first prove a stability theorem for traveling waves in a class of non-cooperative reaction-diffusion systems with nonlocal dispersal of equal diffusivities. Our stability criterion is in the sense that the initial perturbation is such that a suitable weighted relative entropy function along with its Fourier transform are integrable. Then we apply our main theorem to derive the stability of traveling waves for some specific examples of non-cooperative systems arising in ecology and epidemiology.

1. Introduction

It is well-known that, in contrast to the classical random diffusion, reaction-diffusion systems with nonlocal dispersal can better model the long range movements and nonadjacent interactions of individuals. This can be seen from many applied science models arising in physics, material science, population dynamics and so on (cf. [14, 11, 4, 22]). In fact, when the dispersal kernel is highly concentrated, it is known that the model with nonlocal dispersal tends to the classical diffusion model. Moreover, it is noted in [1] that the dynamics of models with nonlocal dispersal is quite rich. Therefore, the study of models with nonlocal dispersal has attracted a lot of attention in recent years. However, there are certain difficulties arisen in the study of nonlocal dispersal models. One of them is that there is no regularizing effect for the nonlocal dispersal model in contrast to the classical diffusion case [2, 15].

In this work, the following general reaction-diffusion system with nonlocal dispersal is to be investigated:

$$(u_i)_t(x,t) = d_i\mathcal{N}_i[u_i](x,t) + u_i(x,t)f_i(u(x,t)),\ (x,t)\in\mathbb{R}\times(0,\infty),\ 1\le i\le m, \tag{1.1}$$

where $u(x,t) := (u_1(x,t),\dots,u_m(x,t))$, $d_i > 0$, $f_i \in C^1([0,\infty)^m)$, $1 \le i \le m$, m is a positive integer and

$$\mathcal{N}_i[u_i](x,t) := \int_R J_i(y)u_i(x-y,t)dy - u_i(x,t) = (J_i * u_i - u_i)(x,t),$$

in which the kernel function J_i satisfies the following properties:

(J1) The kernel J_i is nonnegative symmetric (w.r.t. $x = 0$) and smooth in $\mathbb{R}$;

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(J2) it holds that
$$\int_{\mathbb{R}} J_i(y)dy = 1;$$
(J3) it holds that $\int_{\mathbb{R}} J_i(y)e^{\lambda y}dy < \infty$ for all $\lambda \in (0, \hat{\lambda}_i)$ and
$$\int_{\mathbb{R}} J_i(y)e^{\lambda y}dy \to \infty \text{ as } \lambda \uparrow \hat{\lambda}_i$$
for some $\hat{\lambda}_i \in (0, \infty]$.

System (1.1) arises in many applications, such as population dynamics in ecology and epidemiology. In population dynamics, we are particularly interested in the propagation phenomena of species in the biological model. More precisely, it describes the invasion of certain species to other species in ecological systems (cf. [9, 24]), or the spreading of certain diseases in epidemic models (cf. [7, 23]). Among many different approaches towards the propagation phenomena, the existence vs non-existence of traveling waves and the spreading dynamics of solutions with localized initial data are two most important subjects to be explored. Although there are some well-known difficulties in the study of nonlocal dispersal models, some abstract theory from dynamical systems can be applied to derive propagation properties for nonlocal models when it is of cooperative type. For the theory and application of monotone semiflow to derive the minimal wave speed of traveling waves and the spreading speed, we refer the reader to, e.g., [3, 26, 21, 27, 20, 18, 16, 10] and the references cited therein.

We are mainly concerned with traveling wave solutions of (1.1) connecting two different constant equilibria $\{E^{\pm}\}$. More specifically, the form of a traveling wave solution of (1.1) is
$$u_i(x,t) = \phi_i(z),\ z := x - ct,\ 1 \le i \le m,$$
with an unknown function $\Phi := (\phi_1, \dots, \phi_m)$ (the wave profile) and an unknown positive constant c (the wave speed) such that
$$\Phi(-\infty) = E^-,\ \Phi(\infty) = E^+. \tag{1.2}$$
The existence vs non-existence of traveling waves for systems with nonlocal dispersal has been studied quite extensively in past years. We refer the reader to, e.g., [5, 6] for scalar equations, [36, 35, 8, 30, 31, 37, 25] for predator-prey systems and [33, 17, 19, 32, 29] for epidemic models.

The main goal of this work is to derive the stability of traveling wave solutions for system (1.1). Therefore, we shall always assume that (1.1) admits positive traveling wave solutions $\{c, \Phi\}$ for all $c \ge c^*$ for some positive constant c^*. Hereafter Φ is positive means $\phi_i > 0$ in $\mathbb{R}$ for all i. The stability of traveling wave solutions in cooperative systems with nonlocal dispersal can be derived by sandwich method using the order-preserving property of the cooperative systems. See, e.g., [34]. For non-cooperative systems with nonlocal dispersal, due to the lack of comparison principle, little is done for the stability of traveling waves. In this work, motivated by [13], we provide a simple approach to tackle the stability of traveling waves for non-cooperative systems with nonlocal dispersal. Due to some technical difficulty, we shall only consider the equal diffusivities case in this work. Hereafter, we shall assume that $d_i = 1$ and $J_i = J$, $1 \le i \le m$, for some kernel J satisfying (J1)-(J3) with $\hat{\lambda} \in (0, \infty]$. Also, we set $\mathcal{N}[u_i] := J * u_i - u_i$.

To study the stability of traveling waves, it is more convenient to use the so-called moving coordinate $z = x - ct$. Hence, for a positive solution u of (1.1), the corresponding function $\{u_i = u_i(z,t)\}$ in terms of z-coordinate satisfies

$$(u_i)_t = \mathcal{N}[u_i] + c(u_i)_z + u_i f_i(u). \tag{1.3}$$

Note that a traveling wave $\{c, \Phi\}$ satisfies

$$\mathcal{N}[\phi_i] + c\phi_i'(z) + \phi_i f_i(\Phi(z)) = 0, \tag{1.4}$$

hence Φ is a stationary solution of (1.3) for the given wave speed c.

For a given set of positive constant $\{\sigma_i \mid 1 \le i \le m\}$ and a positive function $\Psi(z) = (\psi_1(z), \dots, \psi_m(z))$, $z \in \mathbb{R}$, we define as in [13] the following relative entropy function

$$\mathcal{E}[\Psi](z) := \sum_{i=1}^{m} \sigma_i \mathcal{E}_i[\psi_i](z), \quad \mathcal{E}_i[\psi_i](z) := \psi_i(z) - \phi_i(z) - \phi_i(z) \ln \frac{\psi_i(z)}{\phi_i(z)}. \tag{1.5}$$

It is easy to see that $\mathcal{E}[\Psi] \ge 0$ in $\mathbb{R}$ and $\mathcal{E}[\Psi](z_0) = 0$ if and only if $\Psi(z_0) = \Phi(z_0)$. For a function $u(z,t)$, the relative entropy function can be expressed as follows:

$$W(z,t) := \mathcal{E}[u(\cdot,t)](z), \ (z,t) \in \mathbb{R} \times [0,\infty). \tag{1.6}$$

Also, for a given positive constant R, we define the quantity

$$c_R := \inf_{0<\lambda<\hat{\lambda}} G(\lambda), \quad G(\lambda) := \frac{\left[\int_{\mathbb{R}} J(y) e^{\lambda y} dy - 1\right] + R}{\lambda}. \tag{1.7}$$

Note that the quantity c_R is well-defined such that $c_R > 0$. This can be easily seen by the fact that $G(0^+) = G(\hat{\lambda}^-) = \infty$ and $G(\lambda) > 0$ for $\lambda \in (0, \hat{\lambda})$.

With the above notation, we now state our main theorem on the stability of traveling waves for system (1.1) as follows.

Theorem 1.1. *Assume $c_R \ge c^*$ for some positive constant R. Let u be the solution to* (1.3) *for a given positive continuous initial data u_0. Denote a positive traveling wave solution of* (1.1) *for some $c \ge c_R$ by $\{c, \Phi\}$. Suppose that there exists a set of positive constant $\{\sigma_i \mid 1 \le i \le m\}$ such that*

$$W_t \le \mathcal{N}[W] + cW_z + RW, \ (z,t) \in \mathbb{R} \times (0,\infty) \tag{1.8}$$

for the relative entropy function W of u defined by (1.6)*. Let $\lambda_c < \hat{\lambda}$ be the smallest positive root of $G(\lambda) = c$. Then, under the condition $e^{\lambda_c z}\mathcal{E}[u_0] \in L^1(\mathbb{R})$ and $\mathcal{F}[e^{\lambda_c z}\mathcal{E}[u_0]] \in L^1(\mathbb{R})$, $u(z,t) \to \Phi(z)$ as $t \to \infty$ for all $z \in \mathbb{R}$, where $\mathcal{F}$ is the Fourier transform defined on $L^1(\mathbb{R})$.*

The stability provided in Theorem 1.1 for traveling wave of (1.1) is in the sense that the initial perturbation is such that $e^{\lambda_c z}\mathcal{E}[u_0] \in L^1(\mathbb{R})$ and $\mathcal{F}[e^{\lambda_c z}\mathcal{E}[u_0]] \in L^1(\mathbb{R})$ at $t = 0$. Although the proof of Theorem 1.1 follows from the same idea as that in [13] for the case of standard local diffusion, there are some difficulties to be overcome due to the nonlocal dispersal. With a suitable weight, thanks to the convergence result derived in [2] (see Proposition 2.1 below), we are able to derive the convergence of the weighted relative entropy function to zero as $t \to \infty$. Another key of the proof of this stability theorem is to derive the inequality (1.8) with an appropriate chosen set of positive constants $\{\sigma_i\}$. Unfortunately, as in the case of classical diffusion, our method can only be applied for systems of nonlocal dispersal with equal diffusivities. Systems with non-equal diffusivities are still left for open.

The rest of this paper is organized as follows. First, we provide a very simple proof of Theorem 1.1 along with a general calculation towards (1.8) in §2. Then, in §3, we provide an application of Theorem 1.1 to various non-cooperative systems studied in [8, 31, 37, 25] for predator-prey systems and in [33, 29] for an epidemic model. Finally, a brief discussion is given in §4.

2. Proof of Theorem 1.1

First, we consider the linear problem

$$\begin{cases} v_t(x,t) = (K * v)(x,t) - v(x,t),\ x \in \mathbb{R},\, t > 0, \\ v(x,0) = v_0(x),\ x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where K is non-negative smooth function satisfying $\int_{\mathbb{R}} K(x)\,dx = 1$ and v_0 is non-negative. We have the following result.

Proposition 2.1. *Suppose that $v_0 \in L^1(\mathbb{R})$ and $\mathcal{F}[v_0] \in L^1(\mathbb{R})$. Then the solution v of (2.1) converges to zero for all $x \in \mathbb{R}$ as $t \to \infty$.*

Proof. Let us choose an even function $\tilde{v}_0$ such that $\tilde{v}_0(x) := v_0(x) + v_0(-x) \geq 0$. Note that $\tilde{v}_0 \geq v_0$, and $\tilde{v}_0 \in L^1(\mathbb{R}), \mathcal{F}[\tilde{v}_0] \in L^1(\mathbb{R})$ and $\mathcal{F}[\tilde{v}_0]$ is real-valued. We denote the solution of (2.1) with initial data $\tilde{v}_0$ by $\tilde{v}(x,t)$ Recall from the proof of [2, Theorem 1.3] that

$$v(x,t) = \mathcal{F}^{-1}\big[\exp\{t(\mathcal{F}[K](\xi) - 1)\}\mathcal{F}[v_0](\xi)\big](x).$$

The comparison principle gives us

$$v(x,t) \leq \tilde{v}(x,t) \leq \frac{1}{2\pi}\int_{\mathbb{R}} \exp\{t\,\mathrm{Re}\,(\mathcal{F}[K](\xi) - 1)\}|\mathcal{F}[\tilde{v}_0](\xi)|\,d\xi,\ \forall\, x \in \mathbb{R}.$$

Now, since $\mathrm{Re}\,\mathcal{F}[K](\xi) = \int_{\mathbb{R}} K(x)\cos x\xi\,dx \leq 1$ for all $\xi \in \mathbb{R}$ and its equality holds only when $x\xi = 2\pi n$ for some $n \in \mathbb{Z}$. Thus, $\mathrm{Re}\,\mathcal{F}[K] < 1$ almost all $\xi \in \mathbb{R}$. The condition $\mathcal{F}[\tilde{v}_0] \in L^1(\mathbb{R})$, and the Lebesgue's dominated convergence theorem yield the result. □

Now we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Note that $w(z) = e^{-\lambda_c z}$ satisfies

$$\mathcal{N}[w] + cw_z + Rw = 0,\ z \in \mathbb{R}$$

Define a function $V(z,t) = e^{\lambda_c z}W(z,t)$. Then, by (1.8), we obtain

$$\begin{aligned} V_t(z,t) &\leq \int_{\mathbb{R}} J(y)e^{\lambda_c y}V(z-y,t)\,dy + (R - 1 - c\lambda_c)V(z,t) + cV_z(z,t) \\ &= \int_{\mathbb{R}} J(y)e^{\lambda_c y}\{V(z-y,t) - V(z,t)\}\,dy + cV_z(z,t). \end{aligned}$$

If we define a function $U(z+ct,t)=V(z,t)$, then

$$U_t(x,t) \le \int_{\mathbb{R}} J(y)e^{\lambda_c y}\{U(x-y,t)-U(x,t)\}\,dy, \qquad x=z+ct.$$

Now we introduce a new time variable τ by the relation

$$\frac{\tau}{t} := \int_{\mathbb{R}} J(y)e^{\lambda_c y}\,dy \in (0,\infty)$$

to obtain

$$U_\tau \le \frac{J(y)e^{\lambda_c y}}{\|J(y)e^{\lambda_c y}\|_{L^1}} * U - U. \tag{2.2}$$

Hence the comparison principle for scalar equations and Proposition 2.1 imply that $U(x,\tau)\to 0$ as $\tau\to\infty$ for all $x\in\mathbb{R}$. Returning to the original variables (z,t), Theorem 1.1 is thereby proved. □

In order to apply Theorem 1.1 to some specific systems, we first assume that system (1.1) has an invariant set $\mathcal{I}\subset[0,\infty)^m$. Note that any nonnegative nontrivial solution of (1.1) is positive, by applying the strong maximum principle for scalar equation.

Secondly, we assume that

$$I := \sum_{i=1}^m \sigma_i\,(u_i-\phi_i)\,\{f_i(u)-f_i(\Phi)\} \le 0 \quad \text{ for } \; u,\Phi\in\mathcal{I} \tag{2.3}$$

for a given set of positive constants $\{\sigma_i\}$. Following [13], we now perform a derivation of (1.8) as follows. By the definition of W and using (1.3), we have

$$\begin{aligned}
W_z &= \sum_{i=1}^m \sigma_i\Big\{(u_i)_z\Big(1-\frac{\phi_i}{u_i}\Big)-\phi_i'\ln\frac{u_i}{\phi_i}\Big\}\\
W_t &= \sum_{i=1}^m \sigma_i\Big(1-\frac{\phi_i}{u_i}\Big)(u_i)_t\\
&= \sum_{i=1}^m \sigma_i\Big\{(u_i-\phi_i)f_i(u)+\frac{u_i-\phi_i}{u_i}\mathcal{N}[u_i]+\Big(1-\frac{\phi_i}{u_i}\Big)c(u_i)_z\Big\}\\
&= \sum_{i=1}^m \sigma_i\Big\{(u_i-\phi_i)(f_i(u)-f_i(\Phi))+\frac{u_i-\phi_i}{u_i}\mathcal{N}[u_i]+(u_i-\phi_i)f_i(\Phi)\Big\}\\
&\qquad +c\sum_{i=1}^m \sigma_i\phi_i'\ln\frac{u_i}{\phi_i}+cW_z.
\end{aligned}$$

Set $W_i=\mathcal{E}_i[u_i]$. Then $\mathcal{N}[W_i]=\mathcal{N}[u_i]-\mathcal{N}[\phi_i]-\mathcal{N}[\phi_i\ln(u/\phi_i)]$. Hence we obtain

$$\begin{aligned}
&W_t-\mathcal{N}[W]-cW_z\\
=\ &\sum_{i=1}^m\sigma_i\Big\{(u_i-\phi_i)(f_i(u)-f_i(\Phi))-\frac{\phi_i}{u_i}\mathcal{N}[u_i]+f_i(\Phi)(u_i-\phi_i)\\
&\qquad +\mathcal{N}[\phi_i]+\mathcal{N}[\phi_i\ln(u_i/\phi_i)]+c\phi_i'\ln\frac{u_i}{\phi_i}\Big\}\\
=\ &\sum_{i=1}^m\sigma_i\Big\{(u_i-\phi_i)(f_i(u)-f_i(\Phi))-\frac{\phi_i}{u_i}\mathcal{N}[u_i]+f_i(\Phi)\Big(W_i+\phi_i\ln\frac{u_i}{\phi_i}\Big)\\
&\qquad +\mathcal{N}[\phi_i]+\mathcal{N}[\phi_i\ln(u_i/\phi_i)]+c\phi_i'\ln\frac{u_i}{\phi_i}\Big\}.
\end{aligned}$$

By substituting

$$\begin{aligned}
-\frac{\phi_i}{u_i}\mathcal{N}[u_i]&=-\frac{\phi_i}{u_i}J*u_i+\phi_i,\\
\mathcal{N}\phi_i&=J*\phi_i-\phi_i,\\
f_i(\Phi)\phi_i\ln\frac{u_i}{\phi_i}&=-(J*\phi_i-\phi_i+c\phi_i')\ln\frac{u_i}{\phi_i},\\
\mathcal{N}[\phi_i\ln(u_i/\phi_i)]&=J*(\phi_i\ln(u_i/\phi_i))-\phi_i\ln\frac{u_i}{\phi_i},
\end{aligned}$$

and using (2.3), we deduce that

$$\begin{aligned}
&W_t-\mathcal{N}[W]-cW_z-\sum_{i=1}^m\sigma_i f_i(\Phi)W_i\\
\le\ &\sum_{i=1}^m\sigma_i\Big\{-\frac{\phi_i}{u_i}J*u_i+J*\phi_i-(J*\phi_i)\ln\frac{u_i}{\phi_i}+J*\Big(\phi_i\ln\frac{u_i}{\phi_i}\Big)\Big\}.
\end{aligned}$$

We further compute that

$$\begin{aligned}
&-\frac{\phi_i}{u_i}J*u_i+J*\phi_i-(J*\phi_i)\ln\frac{u_i}{\phi_i}+J*\Big(\phi_i\ln\frac{u_i}{\phi_i}\Big)\\
=\ &\int_{\mathbb{R}}J(z-y)\Big\{\phi_i(y)-\frac{u_i(y,t)}{u_i(z,t)}\phi_i(z)-\phi_i(y)\ln\frac{u_i(z,t)}{\phi_i(z)}+\phi_i(y)\ln\frac{u_i(y,t)}{\phi_i(y)}\Big\}\,dy\\
=\ &\int_{\mathbb{R}}J(z-y)\Big\{\phi_i(y)-\frac{u_i(y,t)}{u_i(z,t)}\phi_i(z)+\phi_i(y)\ln\frac{u_i(y,t)\phi_i(z)}{u_i(z,t)\phi_i(y)}\Big\}\,dy\\
\le\ &\int_{\mathbb{R}}J(z-y)\Big\{\phi_i(y)-\frac{u_i(y,t)}{u_i(z,t)}\phi_i(z)+\phi_i(y)\Big(\frac{u_i(y,t)\phi_i(z)}{u_i(z,t)\phi_i(y)}-1\Big)\Big\}\,dy=0.
\end{aligned}$$

Here we used the inequality $\ln X\le X-1$ for $X>0$, by setting

$$X=\frac{u(y,t)\phi_i(z)}{u(z,t)\phi_i(y)}.$$

Finally, if we also have

$$\max_{1\le i\le m}\{\|f_i(\Phi)\|_{L^\infty(\mathbb{R})}\}\le R,$$

then we can conclude from (2.3) that (1.8) holds with this R and so Theorem 1.1 is applicalbe.

3. Application of Theorem 1.1

For some non-cooperative systems arsing in ecology and epidemiology, Theorem 1.1 can be applied to characterize the stability of traveling waves. Here we provide some examples in the followings.

3.1. **Predator-prey models.**

First, in [8], we consider

$$\begin{cases}(u_1)_t(x,t) = \mathcal{N}[u_1](x,t) + r_1u_1(x,t)[1 - u_1(x,t) - au_2(x,t)], \ x \in \mathbb{R}, \ t > 0, \\ (u_2)_t(x,t) = \mathcal{N}[u_2](x,t) + r_2u_2(x,t)[-1 + bu_1(x,t) - u_2(x,t)], \ x \in \mathbb{R}, \ t > 0,\end{cases} \tag{3.1}$$

where r_1, r_2, a, b are positive constants. We assume

$$b > 1, \ \ ab < 1. \tag{3.2}$$

Then the quantity

$$c^* := \inf_{0<\lambda<\hat{\lambda}} \frac{\left[\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1\right] + r_2(b-1)}{\lambda} \tag{3.3}$$

is well-defined and $c^* > 0$.

For the existence of traveling waves, we recall from [8, 28] that system (3.1) admits a traveling wave solution $\{c, (\phi_1, \phi_2)\}$ satisfying (1.2) with

$$E^+ = (1, 0), \quad E^- = \left(\frac{1+a}{1+ab}, \frac{b-1}{1+ab}\right)$$

for any $c > c^*$; while such a traveling wave exists for $c = c^*$ if we further assume that J is compactly supported.

Note that, by the comparison for the scalar equation, $\mathcal{I} := [0,1] \times [0, b-1]$ is an invariant set of system (3.1). We choose $\sigma_1 = 1/r_1$ and $\sigma_2 = a/(r_2b)$. Then the quantity I in (2.3) is computed as

$$I = -(u_1 - \phi_1)^2 - \frac{a}{b}(u_2 - \phi_2)^2 \leq 0, \ \forall\, (u_1, u_2), (\phi_1, \phi_2) \in \mathcal{I}.$$

Hence (1.8) holds with $R := \max\{r_1, r_2(b-1)\}$.

Then Theorem 1.1 leads us to the conclusion that, for any $c \geq c_R$, traveling waves for (3.1) are stable in the sense indicated in Theorem 1.1. If an extra condition $r_1 \leq r_2(b-1)$ is enforced, then this stability result holds for all $c \geq c^*$, .

Secondly, for the predator-prey system with two weak competing predators u_1, u_2 and one prey u_3 with the nonlinearities in (1.1) defined by

$$\begin{cases} f_1(u_1, u_2, u_3) = r_1(-1 - u_1 - hu_2 + bu_3), \\ f_2(u_1, u_2, u_3) = r_2(-1 - ku_1 - u_2 + bu_3), \\ f_3(u_1, u_2, u_3) = r_3(1 - au_1 - au_2 - u_3), \end{cases}$$

where $r_1, r_2, r_3 > 0$, $b > 1$, $0 < a < 1/[2(b-1)]$ and $0 < h, k < 1$, the existence of traveling waves connecting the predator-free state $(0, 0, 1)$ and the unique positive coexistence state were obtained in [12] for the case of classical diffusion; and in [31] for the case of nonlocal dispersal. Note that the existence of waves is not restricted to the case of equal diffusivities. Applying Theorem 1.1 with the same choice of $\{\sigma_i\}$ as in [13], the stability with initial perturbation in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for traveling waves of the nonlocal dispersal case with equal diffusivities can be derived. Here we have $R = \max\{r_1(b-1), r_2(b-1), r_3\}$, since $0 \le \phi_3 \le 1$ and $0 \le \phi_1, \phi_2 \le b-1$. Moreover, we have

$$c^* := \inf_{0<\lambda<\hat{\lambda}} \frac{\left[\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1\right] + \max\{r_1, r_2\}(b-1)}{\lambda},$$

and the stability holds for any wave with speed $c \ge c_R$. Note that $c_R = c^*$, if we assume that $r_3 \le \max\{r_1, r_2\}(b-1)$. Since it is quite similar to [13, Theorem 3.7], we safely omit the further details here.

Similarly, for the predator-prey system with two weak competing preys u_1, u_2 and one predator u_3 with the nonlinearities in (1.1) defined by

$$\begin{cases} f_1(u_1, u_2, u_3) = r_1(1 - u_1 - hu_2 - au_3), \\ f_2(u_1, u_2, u_3) = r_2(1 - ku_1 - u_2 - au_3), \\ f_3(u_1, u_2, u_3) = r_3(-1 + bu_1 + bu_2 - u_3), \end{cases}$$

where $r_1, r_2, r_3, a, b > 0$ (with some further restrictions on a, b) and $0 < h, k < 1$, we can obtain a stability result described in Theorem 1.1 for the traveling waves connecting the predator-free state to the unique positive coexistence state derived in [25] for the nonlocal dispersal system with equal diffusivities. Here the predator-free state is $(u_p, v_p, 0)$, where

$$u_p := \frac{1-h}{1-hk} \in (0, 1), \quad v_p := \frac{1-k}{1-hk} \in (0, 1),$$

the critical wave speed c^* is defined by

$$c^* := \inf_{0<\lambda<\hat{\lambda}} \frac{\left[\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1\right] + r_3[b(u_p + v_p) - 1]}{\lambda},$$

and the constant $R = \max\{r_1, r_2, r_3(2b-1)\}$, using $0 \le \phi_1, \phi_2 \le 1$ and $0 \le \phi_3 \le 2b-1$. Similar to [13], we can only obtain the stability result described in Theorem 1.1 for the nonlocal dispersal system with equal diffusivities for any wave with speed $c \ge c_R$, where $c_R > c^*$, since $R \ge r_3(2b-1) > r_3[b(u_p + v_p) - 1]$.

Thirdly, in [37], they considered a predator-prey system with two preys (without inter-specific competition between preys) and one predator with/without intra-specific competition, namely, the nonlinearities in (1.1) are defined by

$$\begin{cases} f_1(u_1, u_2, u_3) = r_1(1 - u_1 - a_1u_3), \\ f_2(u_1, u_2, u_3) = r_2(1 - u_2 - a_2u_3), \\ f_3(u_1, u_2, u_3) = r_3(-1 + b_1u_1 + b_2u_2 - \gamma u_3), \end{cases}$$

where $r_1, r_2, r_3, a_1, a_2, b_1, b_2$ are positive constants with $b_1 + b_2 > 1$ and γ is a nonnegative constant to denote whether there is the intra-specific competition in the predator. It is clear that $\mathcal{I} := [0,1] \times [0,1] \times [0, b_1 + b_2 - 1]$ is an invariant set of this predator-prey system. By choosing

$$\sigma_1 = \frac{1}{r_1},\ \sigma_2 = \frac{a_1 b_2}{r_2 a_2 b_1},\ \sigma_3 = \frac{a_1}{r_3 b_1},$$

the quantity I in (2.3) can be computed as

$$I = -(u_1 - \phi_1)^2 - \frac{a_1 b_2}{a_2 b_1}(u_2 - \phi_2)^2 - \gamma \frac{a_1}{b_1}(u_3 - \phi_3)^2 \le 0,\ \forall\, (u_1, u_2, u_3), (\phi_1, \phi_2, \phi_3) \in \mathcal{I}.$$

Hence (1.8) holds with $R := \max\{r_1, r_2, r_3(b_1 + b_2 - 1)\}$.

Now Theorem 1.1 implies the stability of traveling waves connecting the predator-free state $(1,1,0)$ and the unique positive coexistence state with speed $c \ge c_R$ for this predator-prey system with nonlocal dispersal of equal diffusivities. For the existence of traveling waves, we refer the reader to [37]. The stability holds for any $c \ge c^*$, where

$$c^* := \inf_{0<\lambda<\hat{\lambda}} \frac{\left[\int_{\mathbb{R}} J(y) e^{\lambda y} dy - 1\right] + r_3(b_1 + b_2 - 1)}{\lambda},$$

if we further assume that $r_i \le r_3(b_1 + b_2 - 1)$ for $i = 1, 2$.

3.2. **An epidemic model.**

Finally, we present in this section a non-cooperative system arising in epidemiology as follows. In [33, 29], they considered a nonlocal dispersal Kermack-McKendrick epidemic model described by (1.1) with

$$f_1(u_1, u_2) = -\beta u_2,\ f_2(u_1, u_2) = \beta u_1 - \gamma,$$

where u_1 is the susceptible population, u_2 is the infective population, and β, γ are positive constants which stand for the infection rate and the removal rate, respectively. Note that this system has a family of constant stationary solutions $\{(s, 0) \mid s > 0\}$.

Let $s^* > 0$ be a fixed constant such that $\beta s^* > \gamma$. Then, under the assumption that J is compactly supported, the existence of traveling wave connecting $(s^*, 0)$ and $(s_0, 0)$ for some $s_0 \le s^*$ for system (1.1) with equal diffusivities was obtained in [33] for $c > c^*$ and in [29] for $c = c^*$, where the quantity

$$c^* := \inf_{0<\lambda<\infty} \frac{\left[\int_{\mathbb{R}} J(y) e^{\lambda y} dy - 1\right] + \beta s^* - \gamma}{\lambda}$$

is well-defined and $c^* > 0$.

It is easy to see that $[0, s^*] \times [0, \infty)$ is an invariant set of this epidemic model. By choosing $\sigma_1 = \sigma_2 = 1$, the quantity I in (2.3) is identically equal to zero. Hence (1.8) holds with $R = \beta s^* - \gamma$. We conclude from Theorem 1.1 that the stability of these traveling waves in the sense described in Theorem 1.1 holds for all $c \ge c^*$.

Discussion

The main result of our study is the stability of traveling waves for a class of non-cooperative reaction-diffusion systems with nonlocal dispersal of equal diffusivities. Our stability theorem has been designed to analyze predator-prey type models and some models in epidemiology.

To our knowledge, to derive the stability of traveling waves of non-cooperative reaction-diffusion systems with nonlocal dispersal is widely open. The significant outcome of our work is to give the first blow to this problem, although we need to assume that the diffusivities of all component are the same. The strategy is to transform the system to a scalar reaction-diffusion differential inequality for the relative entropy function. This idea is based on the idea of [13] for the standard local diffusion of the Laplacian type. Improving the results from the local diffusion problem to the nonlocal diffusion problem requires more computational effort. For example, we can not use integration by parts as in the local diffusion problem, and we need to handle the integral kernel to get the suitable differential inequality.

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