THE STRUCTURE OF STATIONARY SOLUTIONS TO A MICRO-ELECTRO MECHANICAL SYSTEM WITH FRINGING FIELD

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Abstract. We study the structure of stationary solutions of a micro-electro mechanical system with fringing field. It is known that there is a positive critical value such that no stationary solutions exist for the applied voltage beyond this critical value, at least two stationary solutions exist for the applied voltage below this critical value and there is a unique stationary solution at this critical value. In this paper, we verify that there are exactly two solutions below this critical value analytically. Moreover, the stability of the smaller stationary solutions is derived.

1. Introduction

In this paper, we are concerned with the dynamic deflection of an elastic membrane inside a micro-electro mechanical system (MEMS). We consider the case when the distance between the plate and the membrane is relative small compared to the length of the device. In the case when we ignore the inertia, the device embedded in an electric circuit without a capacitor, and the system is with the fringing field, the equation describing the operation of the MEMS is reduced to the following single parabolic equation

$$u_t = \Delta u + \lambda \frac{1+\delta|\nabla u|^2}{(1-u)^2},\ x \in \Omega \subset \mathbb{R}^N,\ t > 0, \tag{1.1}$$

where λ and δ are positive constants, λ is proportional to the square of the applied voltage, $\delta|\nabla u|^2$ describes the fringing field, Ω is the domain of the plate and $u = u(x,t)$ denotes the deflation of the membrane. Here we normalized the gap between the membrane and the plate on the boundary to be 1. Since the edge of the membrane is kept fixed, we have the zero Dirichlet boundary condition for u on the boundary of Ω.

For the physical background of MEMS, we refer the reader to, e.g., [22, 23, 24, 18]. There were extensive works on the analysis of MEMS model, we refer the reader to [13, 3, 6, 8,

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4, 5, 14, 9, 16, 15, 10, 11] for model without fringing field, and [26, 1, 20, 25, 7, 19, 21] for model with fringing field. However, there are still many interesting mathematical questions open, especially for model with fringing field (see, e.g., [26]).

In this paper, we are mainly interested in the stationary solutions of (1.1) with zero Dirichlet boundary condition. For this aspect, we refer to the earlier works [17, 12, 8]. We recall from [17] that, when $\delta = 0$ and $N = 1$, there is a positive finite critical value λ_* such that there are exactly two stationary solutions to (1.1) with zero Dirichlet boundary condition when $\lambda \in (0, \lambda_*)$; exactly one solution when $\lambda = \lambda_*$ and no solutions for $\lambda > \lambda_*$.

For (1.1), we shall only focus on the case when Ω is of one-dimensional and without loss of generality we assume that $\delta = 1$ and $\Omega = (-1, 1)$. Then it is reduced to study the following boundary value problem (BVP) for $u = u(x)$:

$$-u_{xx} = \lambda \frac{1 + u_x^2}{(1-u)^2}, \quad x \in (-1, 1), \tag{1.2}$$

$$u(\pm 1) = 0. \tag{1.3}$$

Recall from [26] that problem (1.2)-(1.3) has no solution for $\lambda > \lambda^*$, a unique solution for $\lambda = \lambda^*$, and at least 2 solutions for $\lambda \in (0, \lambda^*)$ for a finite $\lambda^* > 0$. However, the exact number of solutions for $\lambda \in (0, \lambda^*)$ remains open.

On of the main purposes of this paper is to prove that there are exactly 2 solutions to (1.2)-(1.3) for each $\lambda \in (0, \lambda^*)$. At a first glance, this question seems a simple-looking standard problem. In fact, there is a well-known method in dealing with this problem which consists of two steps, namely, first we transform (BVP) to an algebraic equation describing the relation between the maximum, say η, of a solution and the parameter λ of (BVP), then we analyze the derived algebraic equation to obtain our solution structure. This method is proved to be very effective for many problems, even for nonlocal problem (cf., e.g., [2, 12, 9, 15, 10]).

Depending on problems, there are different functions in the algebraic equation. The success of the above mentioned method is because an explicit formula, such as $\lambda = \Lambda(\eta)$ for some function Λ, can be derived, although the function $\Lambda(\eta)$ might be very complicated. However, due to the gradient term in (1.2), we are unable to find an explicit formula $\Lambda(\eta)$ for λ to (BVP). This is one of the major difficulties in dealing with this exact multiplicity problem, but we are able to derive an implicit relation for λ and η (see (2.7) below). Furthermore, to deal with the singularity of the integrand in (2.7), we successfully transfer (2.7) to an equivalent (implicit) relation of λ and η in which only proper integral is involved (see (3.3) below).

On the other hand, our exact multiplicity problem is equivalent to the uniqueness of critical points of λ (as a function of η), since $\lambda(\eta) = 0$ when $\eta = 0, 1$. One of standard ways to derive this uniqueness is to prove that $\lambda''(\eta) < 0$ whenever $\lambda'(\eta) = 0$. However, both derivatives of λ are implicitly defined in terms of λ. This is actually the major difficulty of this work. To overcome this difficulty, we derive some more accurate estimation for λ (see Lemmas 2.4 and 2.5 in §2), by introducing a higher order interpolation (see (2.11) below). Moreover, $\lambda''(\eta) > 0$ when η is close to 1, by our numerical simulation, and we do not know where the target critical point is, a good a priori estimate for the location of critical points is needed (see Lemma 3.2 in §3). Combining these two points, the whole proving process of the uniqueness of $\lambda'(\eta)$ is carried out with a quite involved analysis.

The rest of this paper is organized as follows. In §2, we first derive an implicit formula for λ and η (the maximum of a solution to (1.2)-(1.3)). Then we give some upper and lower bounds estimation for $\lambda(\eta)$. Next, §3 is devoted to the proof of the uniqueness of critical points of $\lambda(\eta)$. Finally, in §4, we prove the stability of the smaller stationary solutions for $\lambda \in (0, \lambda^*)$. Moreover, for reader's convenience, we provide in Appendix the derivation of some non-standard integrals used in this paper.

2. Preliminaries

In the sequel, we let $u = u(x;\lambda)$ be a (classical) solution of (1.2)-(1.3) for a given $\lambda > 0$. Our question is to determine the number of solutions of (1.2)-(1.3) for a given $\lambda \in (0, \lambda^*)$. In this section, we first derive a relation between λ and the maximum of u, i.e., $u(0;\lambda)$. Then we shall provide some estimates needed for the later purposes.

First, using a transformation introduced by Wei and Ye ([26])

$$v(x) = v(x;\lambda) := \int_0^{u(x)} e^{\lambda/(1-s)} ds, \tag{2.1}$$

problem (1.2)-(1.3) is transformed into

$$-v_{xx} = \lambda \frac{e^{\lambda/(1-u)}}{(1-u)^2}, \quad x \in (-1,1), \tag{2.2}$$

$$v(\pm 1) = 0. \tag{2.3}$$

Note that any solution u of problem (1.2)-(1.3) is strictly concave and symmetric (with respect to $x = 0$). Hence v is also strictly concave and symmetric.

Next, using (2.1) and (2.2), we rewrite (1.2) as

$$-(e^{\frac{\lambda}{1-u}} u')' = \frac{\lambda e^{\frac{\lambda}{1-u}}}{(1-u)^2}, \quad u' := u_x. \tag{2.4}$$

Multiplying both sides of (2.4) by $e^{\frac{\lambda}{1-u}} u'$ (which is v'), we have

$$-(e^{\frac{\lambda}{1-u}} u')'(e^{\frac{\lambda}{1-u}} u') = \frac{\lambda e^{\frac{\lambda}{1-u}}}{(1-u)^2}(e^{\frac{\lambda}{1-u}} u'),$$

then integrating it from 0 to $x > 0$ gives

$$-\frac{(e^{\frac{\lambda}{1-u}}u')^2}{2} = \frac{e^{\frac{2\lambda}{1-u}} - e^{\frac{2\lambda}{1-\eta}}}{2},$$

hereafter, $\eta := u(0;\lambda)$. It follows that

$$e^{\frac{\lambda}{1-u}}u' = -\sqrt{e^{\frac{2\lambda}{1-\eta}} - e^{\frac{2\lambda}{1-u}}}$$

and so

$$e^{\frac{\lambda}{1-u}}(e^{\frac{2\lambda}{1-\eta}} - e^{\frac{2\lambda}{1-u}})^{-1/2}u' = -1. \tag{2.5}$$

By integrating (2.5) from 1 to $x \in [0,1)$, we obtain

$$\int_0^{u(x)} e^{\frac{\lambda}{1-s}} \left(e^{\frac{2\lambda}{1-\eta}} - e^{\frac{2\lambda}{1-s}}\right)^{-1/2} ds = 1 - x$$

and this gives

$$\int_0^{u(x)} \left(e^{2\lambda(\frac{1}{1-\eta}-\frac{1}{1-s})} - 1\right)^{-1/2} ds = 1 - x. \tag{2.6}$$

It follows that

$$\int_0^{\eta} \left(e^{2\lambda(\frac{1}{1-\eta}-\frac{1}{1-s})} - 1\right)^{-1/2} ds = 1, \tag{2.7}$$

which is an implicit relation between λ and η.

Lemma 2.1. *For each $\eta \in (0,1)$, $\lambda = \lambda(\eta)$ is defined uniquely such that* (2.7) *holds.*

Proof. Given $\eta \in (0,1)$. Let

$$I(\eta,a) := \int_0^{\eta} \left(e^{2a(\frac{1}{1-\eta}-\frac{1}{1-s})} - 1\right)^{-1/2} ds.$$

It is clearly that $I(\eta,a)$ is continuous and decreasing in a. Since $I(\eta,a) \to \infty$ as $a \to 0^+$, and $I(\eta,a) \to 0$ as $a \to \infty$, there is a unique positive λ such that $I(\eta,\lambda) = 1$. Hence (2.7) is satisfied uniquely by this λ for the given η. $\square$

From Lemma 2.1 it follows that the corresponding solution $u(x)$ of (1.2)-(1.3) can be determined from (2.6) for a given $\eta \in (0,1)$ with $u(0) = \eta$ and $\lambda = \lambda(\eta)$.

The singularity of the integrand in (2.7) occurs at $s = \eta$, using the Taylor expansion of the integrand about $s = \eta$, we have that

$$\int_0^{\eta} \left(\frac{1-\eta}{\sqrt{2\lambda}}(\eta - s)^{-1/2} + \frac{1-\frac{\lambda}{1-\eta}}{2\sqrt{2\lambda}}(\eta - s)^{1/2} + O((\eta - s)^{3/2})\right) ds = 1. \tag{2.8}$$

Therefore,

$$\frac{2(1-\eta)}{\sqrt{2\lambda}}(\eta)^{1/2} + \frac{1-\frac{\lambda}{1-\eta}}{3\sqrt{2\lambda}}(\eta)^{3/2} + O(\eta^{5/2}) = 1.$$

The leading term implies that $\lambda = 2\eta + o(\eta)$ as $\eta \to 0^+$. Hence $\lambda(\eta)$ has a simple zero at $\eta = 0$. Furthermore, $\lambda = 2\eta - \frac{10}{3}\eta^2 + o(\eta^2)$ as $\eta \to 0^+$.

On the other hand, applying the transformation $s = \eta t$ to (2.7) for $t \in [0,1]$, we have

$$\eta \int_0^1 \left(e^{2\lambda(\frac{1}{1-\eta} - \frac{1}{1-\eta t})} - 1\right)^{-1/2} dt = 1. \tag{2.9}$$

It is easy to see from (2.9) that $\lambda(\eta) \to 0$ as $\eta \to 1^-$. Otherwise, if $\liminf_{\eta\to 1^-} \lambda(\eta) > 0$, then there is a sequence $\eta_k \uparrow 1$ with $\lambda(\eta_k) \to \alpha$ as $k \to \infty$ for some positive constant α. This implies that the left-hand side of (2.9) with $\eta = \eta_k$ tends to 0 as $k \to \infty$, a contradiction. Hence $\lambda(1^-) = 0$.

Since λ has two zeros at $\eta = 0, 1$, we may assume λ has the form $\lambda(\eta) = \eta(1-\eta)\hat{\lambda}(\eta)$, where $\hat{\lambda}(\eta)$ is positive for $\eta \in [0,1)$. Note that $\hat{\lambda}(0) = 2$. Applying this transformation to (2.7), we have

$$\int_0^\eta \left(e^{2\hat{\lambda}\eta(\frac{\eta-s}{1-s})} - 1\right)^{-1/2} ds = 1.$$

At $\eta = 1$, it is easy to compute that $\hat{\lambda}(1) = \ln(\sqrt{2})$. Hence $\lambda(\eta)$ also has a simple zero at $\eta = 1$.

In fact, we have the following upper and lower bounds of $\lambda(\eta)$ in terms of the quadratic polynomial $\eta(1-\eta)$.

Lemma 2.2. *It holds $c_1\eta(1-\eta) \le \lambda(\eta) \le c_2\eta(1-\eta)$ for all $\eta \in (0,1)$, where $c_1 = \ln(\sqrt{2})$ and $c_2 = 2$.*

Proof. First, we show the upper bound for λ as follows. For a positive constant C, we compute

$$\begin{aligned} I(\eta, C) &:= \int_0^\eta \frac{1}{\sqrt{e^{2C\eta(1-\eta)[1/(1-\eta)-1/(1-s)]} - 1}} ds \\ &\le \int_0^\eta \frac{1}{\sqrt{2C\eta(\eta-s)/(1-s)}} ds \\ &\le \frac{1}{\sqrt{2C\eta}} \int_0^\eta \frac{1}{\sqrt{\eta - s}} ds = \sqrt{\frac{2}{C}}, \end{aligned}$$

using $e^x - 1 \ge x$ for all $x > 0$ and $1 - s \le 1$ for all $s \in (0, \eta)$. Therefore, by (2.7) and a contradiction argument, we deduce that $\lambda(\eta) \le 2\eta(1-\eta)$ for any $\eta \in (0,1)$.

For the lower bound estimate, we also argue by a contradiction and assume that there is an $\eta \in (0,1)$ such that $(\ln\sqrt{2})\eta(1-\eta) > \lambda(\eta)$. Then, by (2.7), we have $I(\eta, \ln\sqrt{2}) \le 1$. On the other hand, we observe that

$$\begin{aligned} I(\eta, \ln\sqrt{2}) &= \int_0^\eta \frac{1}{\sqrt{e^{2(\ln\sqrt{2})\eta(1-\eta)[1/(1-\eta)-1/(1-s)]}-1}}ds \\ &= \int_0^\eta \frac{1}{\sqrt{2^{\eta(\frac{\eta-s}{1-s})}-1}}ds \ge \int_0^\eta \frac{1}{\sqrt{2^{\eta^2}-1}}ds = \frac{\eta}{\sqrt{2^{\eta^2}-1}} > 1, \end{aligned}$$

a contradiction. Here we have used $0 < 2^{x^2} - 1 < x^2$ for all $x \in (0,1)$. This can be checked by observing that there is a unique critical point for the function $g(x) := 2^{x^2} - x^2 - 1$ in $(0,1)$ such that its second derivative is positive at this unique critical point. Notice that $g(0) = g(1) = 0$. Hence the lower bound estimation of $\lambda(\eta)$ is proved. □

Notice that the best estimated value of c_1 is $\ln(\sqrt{2})$, because $\lambda(1) = 0$ and $\lambda'(1) = -\ln(\sqrt{2})$. Similarly, $c_2 = 2$ is the best estimation.

Corollary 2.3. *It holds $\lambda(\eta) \le 1/2$ for all $\eta \in (0,1)$.*

Proof. By Lemma 2.2, $\lambda(\eta) \le 2\eta(1-\eta)$. Since $2\eta(1-\eta) \le 1/2$ for all $\eta \in [0,1]$. Hence the corollary follows. □

Now, we derive an implicit form of $\lambda'(\eta)$ as follows. Set

$$f(\eta, t) := \exp\{2\lambda g(\eta,t)\} - 1, \quad g(\eta,t) := \frac{1}{1-\eta} - \frac{1}{1-\eta t}.$$

Then

$$\begin{aligned} h(\eta,t) &:= \frac{\partial g(\eta,t)}{\partial \eta} = \frac{1}{(1-\eta)^2} - \frac{t}{(1-\eta t)^2}. \\ f_\eta(\eta,t) &:= \frac{\partial f(\eta,t)}{\partial \eta} = (f+1)\left[2\lambda' g(\eta,t) + 2\lambda h(\eta,t)\right], \end{aligned}$$

Differentiating (2.9) with respect to η, we obtain

$$\lambda'(\eta) = \frac{1 - \eta^2\lambda \int_0^1 \frac{(f+1)h}{f^{3/2}}dt}{\eta^2 \int_0^1 \frac{(f+1)g}{f^{3/2}}dt}. \tag{2.10}$$

Using (2.10), we are able to plot the curves of λ and λ', see Fig. 1. We observe from these curves that there is exactly one critical value λ^* in $(0,1)$. Therefore, we can expect that (1.2)-(1.3) has no solutions for $\lambda > \lambda^*$, two solutions for $\lambda < \lambda^*$, and exactly one solution when $\lambda = \lambda^*$, where the critical value

$$\lambda^* \approx 0.32302\ 45671 \text{ at } \eta = \eta^* \approx 0.35329\ 45430.$$

Our aim, as indicated in the introduction, is to derive the exact number of solutions of problem (1.2)-(1.3) for any given $\lambda > 0$. One way to reach this goal is to prove that $\lambda''(\eta) < 0$ whenever $\lambda'(\eta) = 0$. However, this is not the case from the numerical simulation (as in Fig. 1)

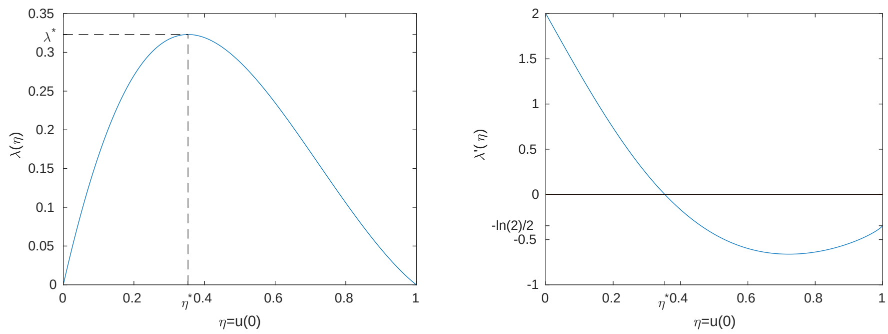

FIGURE 1. (Left) Curve of $\lambda(\eta)$ derived from (2.7). (Right) Curve of $\lambda'(\eta)$ from the implicit relation in (2.10).

when η is close to 1. Due to this difficulty, we need to derive a more precise estimate for the range of η such that $\lambda'(\eta) = 0$.

For this purpose, we shall establish a better bound estimation of λ in terms of the following cubic polynomial

$$H(\eta) := [(\ln\sqrt{2})\eta + 2(1-\eta)]\eta(1-\eta). \tag{2.11}$$

In the sequel, we let c_3 be the positive root of the quadratic equation

$$\frac{4(\ln 2)(e^M - 1 - M)}{(4-\ln 2)M^2}x^2 + (\ln 2)x - 1 = 0, \; M := 0.86 \cdot \frac{256}{27(4-\ln 2)^2}. \tag{2.12}$$

Note that $c_3 = \frac{-\ln(2)+\sqrt{(\ln(2))^2+4m}}{2m} \approx 0.8599799$, where $m := \frac{4(\ln 2)(e^M-1-M)}{(4-\ln 2)M^2}$.

Lemma 2.4. *It holds $\lambda(\eta) \geq c_3 H(\eta)$ for all $\eta \in (0,1)$.*

Proof. For $0 < s < \eta < 1$, let

$$a(\eta, s) := 2c_3 H(\eta)[\frac{1}{1-\eta} - \frac{1}{1-s}] = 2C[\ln(\sqrt{2})\eta + 2(1-\eta)]\eta\frac{\eta - s}{1-s} > 0,$$

where $C = c_3$. Since

$$0 < 2[\ln(\sqrt{2})\eta + 2(1-\eta)]\eta^2 \leq \frac{256}{27(4-\ln 2)^2}, \; \forall \eta \in (0,1),$$

we obtain that

$$0 < a(\eta, s) \leq 2C[\ln(\sqrt{2})\eta + 2(1-\eta)]\eta^2 \leq 0.86 \cdot \frac{256}{27(4-\ln 2)^2} = M.$$

We also observe the inequality

$$e^x - 1 \leq x + \frac{e^M - 1 - M}{M^2}x^2, \; \forall x \in [0, M].$$

This inequality can be seen by observing that $(e^x-1)/x$ is a convex function for $x\in(0,\infty)$ and so the inequality

$$\frac{e^x-1}{x}\le 1+\frac{e^M-1-M}{M^2}x$$

holds for $x\in(0,M]$.

It is clear that

$$0<[\ln(2)x+4(1-x)]x\le\frac{4}{4-\ln(2)}\quad\text{for } x\in(0,1).$$

Then we obtain

$$\begin{aligned}\int_0^\eta\frac{1}{\sqrt{e^a-1}}ds &\ge \int_0^\eta\frac{1}{\sqrt{a+\frac{e^M-1-M}{M^2}a^2}}ds\\ &\ge \frac{1}{\sqrt{C(\ln(2)\eta+4(1-\eta))\eta}}\int_0^\eta\frac{1}{\sqrt{\frac{\eta-s}{1-s}+\frac{e^M-1-M}{M^2}C\frac{4}{4-ln(2)}\left(\frac{\eta-s}{1-s}\right)^2}}ds \qquad (2.13)\\ &:= \frac{1}{\sqrt{C(\ln(2)\eta+4(1-\eta))\eta}}I(\eta),\end{aligned}$$

where

$$I(\eta):=\int_0^\eta\frac{1}{\sqrt{\left(\frac{\eta-s}{1-s}\right)+D\left(\frac{\eta-s}{1-s}\right)^2}}\,ds$$

and the constant $D:=C\frac{e^M-1-M}{M^2}\frac{4}{4-ln(2)}\approx 0.67759$.

By Property 5.4 (in Appendix), we have

$$\begin{aligned}I(\eta) &= \frac{\left(-\ln(1-\eta)+2\ln(\sqrt{D\eta+1}+\sqrt{D\eta+\eta})\right)(D+\frac{1}{2})(1-\eta)}{(D+1)^{3/2}}\\ &\quad+\frac{\sqrt{(D\eta+1)(D\eta+\eta)}}{(D+1)^{3/2}}.\end{aligned}$$

Let

$$G(\eta):=\frac{(D+1)^{3/2}I(\eta)}{\sqrt{(\ln(2)\eta+4(1-\eta))\eta}}.$$

Then

$$\begin{aligned}G(\eta) &= \frac{\left(-\ln(1-\eta)+2\ln(\sqrt{D\eta+1}+\sqrt{D\eta+\eta})\right)(D+\frac{1}{2})(1-\eta)}{\sqrt{(\ln(2)\eta+4(1-\eta))\eta}}\\ &\quad+\frac{\sqrt{(D\eta+1)(D\eta+\eta)}}{\sqrt{(\ln(2)\eta+4(1-\eta))\eta}}.\end{aligned}$$

We shall show that $G(\eta)$ is decreasing on $(0,1)$.

For this, we first compute

$$
\begin{aligned}
G'(\eta) &= \Big(-\ln(1-\eta) + 2\ln(\sqrt{D\eta+1}+\sqrt{D\eta+\eta})\Big)\left(\frac{(D+\frac{1}{2})(1-\eta)}{\sqrt{(\ln(2)\eta+4(1-\eta))\eta}}\right)' \\
&+ \Big(-\ln(1-\eta) + 2\ln(\sqrt{D\eta+1}+\sqrt{D\eta+\eta})\Big)'\frac{(D+\frac{1}{2})(1-\eta)}{\sqrt{(\ln(2)\eta+4(1-\eta))\eta}} \\
&+ \left(\frac{\sqrt{(D\eta+1)(D\eta+\eta)}}{\sqrt{(\ln(2)\eta+4(1-\eta))\eta}}\right)' \\
= &- \frac{(D+\frac{1}{2})\left(-\ln(1-\eta)+2\ln(\sqrt{D\eta+1}+\sqrt{D\eta+\eta}\right)((\ln(2)-2)\eta+2)}{((\ln(2)\eta+4(1-\eta))\eta)^{3/2}} \\
&+ \sqrt{D\eta+\eta}\frac{D(\ln(2)-2)\eta+(4D+2)}{\sqrt{D\eta+1}((\ln(2)\eta+4(1-\eta))\eta)^{3/2}}.
\end{aligned}
$$

To show $G'>0$, we consider the function

$$k(\eta) := -\frac{((\ln(2)\eta+4(1-\eta))\eta)^{3/2}}{((\ln(2)-2)\eta+2)}G'(\eta).$$

Then

$$
\begin{aligned}
k(\eta) &= (D+\frac{1}{2})\Big(-\ln(1-\eta)+2\ln(\sqrt{D\eta+1}+\sqrt{D\eta+\eta}\Big) \\
&\quad -\sqrt{D\eta+\eta}\frac{D(\ln(2)-2)\eta+(4D+2)}{((\ln(2)-2)\eta+2)\sqrt{D\eta+1}}, \forall \eta\in(0,1),
\end{aligned}
$$

and $G'(\eta)<0$ is equivalent to $k(\eta)>0$.

To show $k>0$, we compute

$$
\begin{aligned}
k'(\eta) &= \frac{D+\frac{1}{2}}{1-\eta}\frac{D+1}{\sqrt{D\eta+1}\sqrt{D\eta+\eta}} \\
&\quad -\sqrt{D+1}\frac{4(2D+1)+2(1-D)(2-\ln(2))\eta+D(2-\ln(2))(4D-\ln(2)+6)\eta^2}{2\sqrt{\eta}(D\eta+1)^{3/2}((\ln(2)-2)\eta+2)^2} \\
&= \frac{(D+1)^{3/2}\eta(a_0+a_1\eta+a_2\eta^2)}{2(1-\eta)(D\eta+1)^{3/2}\sqrt{\eta}((\ln(2)-2)\eta+2)^2},
\end{aligned}
$$

where

$$
\begin{aligned}
a_0 &:= 8D+6\ln(2)-8 \approx 1.57962, \\
a_1 &:= (12\ln 2-24)D+8-6\ln 2+(\ln 2)^2 \approx -6.30458, \\
a_2 &:= [2(\ln(2))^2-12\ln(2)+16]D \approx 5.85652.
\end{aligned}
$$

On $(0,1)$, $k'(\eta)=0$ only at $\eta=\frac{-a_1\pm\sqrt{a_1^2-4a_2a_0}}{2a_2}\approx 0.39684, 0.67966$. It follows that $k(\eta)$ is decreasing on $(\frac{-a_1-\sqrt{a_1^2-4a_2a_0}}{2a_2}, \frac{-a_1+\sqrt{a_1^2-4a_2a_0}}{2a_2})$ and increasing on $(0, \frac{-a_1-\sqrt{a_1^2-4a_2a_0}}{2a_2})$,

$(\frac{-a_1+\sqrt{a_1^2-4a_2a_0}}{2a_2}, 1)$. Since $k(0) = 0$ and $k(\frac{-a_1+\sqrt{a_1^2-4a_2a_0}}{2a_2}) \approx 0.01271 > 0$, we deduce that $k(\eta) > 0$ for all $\eta \in (0, 1)$. Hence $G'(\eta) < 0$ for all $\eta \in (0, 1)$.

Then we have $G(\eta) \geq G(1)$. From (2.13) and (2.12), it follows that

$$\begin{aligned}\int_0^\eta \frac{1}{\sqrt{e^a - 1}} ds &\geq \frac{1}{\sqrt{C}(D+1)^{3/2}} G(1) = \frac{1}{\sqrt{C(D+1)\ln(2)}} \\ &= \left(C(C\frac{e^M - 1 - M}{M^2}\frac{4}{4 - ln(2)} + 1)\ln(2) \right)^{-1/2} \\ &= \left(\frac{4\ln(2)(e^M - 1 - M)}{(4 - \ln(2))M^2} C^2 + \ln(2)C \right)^{-1/2} = 1.\end{aligned}$$

Therefore, by (2.7), the lower estimation is derived. □

The following lemma is one of the crucial estimates in this paper. If we only require a coarser bound (a larger constant c_4), then the proof can be much simpler.

Lemma 2.5. *It holds $\lambda(\eta) \leq c_4 H(\eta)$ for all $\eta \in (0, 1)$, where $c_4 = 1.1$.*

Proof. As before, for $0 < s < \eta < 1$, we let

$$a(\eta, s) := 2c_4 H(\eta)[\frac{1}{1-\eta} - \frac{1}{1-s}] = 2C[\ln(\sqrt{2})\eta + 2(1-\eta)]\eta\frac{\eta - s}{1 - s} > 0,$$

where $C = c_4 = 1.1$. To estimate the upper bound of λ in terms of $H(\eta)$, we observe the following inequality

$$\begin{aligned}\int_0^\eta \frac{1}{\sqrt{e^a - 1}} ds &\leq \int_0^\eta \frac{1}{\sqrt{a + \frac{1}{2}a^2}} ds \\ &= \frac{1}{\sqrt{C(\ln(2)\eta + 4(1-\eta))\eta}} \int_0^\eta \frac{1}{\sqrt{\left(\frac{\eta - s}{1-s}\right) + D\left(\frac{\eta - s}{1-s}\right)^2}} ds \\ &:= \frac{1}{\sqrt{2D}} J(\eta, D),\end{aligned}$$

where

$$D := D(\eta) = \frac{1}{2}C(\ln(2)\eta + 4(1-\eta))\eta > 0, \ \forall\, \eta \in (0, 1).$$

By Property 5.4 (in Appendix), we have

$$\begin{aligned}J(\eta, D) &= \frac{\left(-\ln(1-\eta) + 2\ln(\sqrt{D\eta + 1} + \sqrt{D\eta + \eta})\right)(D + \frac{1}{2})(1-\eta)}{(D+1)^{3/2}} \\ &\quad + \frac{\sqrt{(D\eta + 1)(D\eta + \eta)}}{(D+1)^{3/2}}.\end{aligned}$$

We consider the function

$$G(\eta) := \frac{(D+1)^{3/2}}{(D + \frac{1}{2})(1-\eta)}(\sqrt{2D} - J(\eta, D)), \ \eta \in (0, 1).$$

Then we have

$$G(\eta)=\frac{(D+1)^{3/2}\sqrt{2D}-\sqrt{(D\eta+1)(D\eta+\eta)}}{(D+\frac{1}{2})(1-\eta)}-\ln(\frac{\left(\sqrt{D\eta+1}+\sqrt{D\eta+\eta}\right)^2}{1-\eta}).$$

Note that $G(0)=0$ and $G(1^-)=\infty$.

Now, we show that $G'>0$ on $(0,1)$. For this, we compute the following three derivatives.

Firstly, we compute

$$\begin{aligned}\frac{d}{d\eta}\ln(\frac{\left(\sqrt{D\eta+1}+\sqrt{D\eta+\eta}\right)^2}{1-\eta}) &= \frac{\frac{D+\eta D'}{\sqrt{D\eta+1}}+\frac{1+D+\eta D'}{\sqrt{D\eta+\eta}}}{\sqrt{D\eta+1}+\sqrt{D\eta+\eta}}+\frac{1}{1-\eta}\\ &= \frac{(\frac{D+\eta D'}{\sqrt{D\eta+1}}+\frac{1+D+\eta D'}{\sqrt{D\eta+\eta}})(\sqrt{D\eta+1}-\sqrt{D\eta+\eta})}{1-\eta}+\frac{1}{1-\eta}\\ &= \frac{-(D\eta+\eta)(D+\eta D')+(D\eta+1)(1+D+\eta D')}{(1-\eta)\sqrt{D\eta+1}\sqrt{D\eta+\eta}}\\ &:= \frac{P_3(\eta)}{(1-\eta)\sqrt{D\eta+1}\sqrt{D\eta+\eta}},\end{aligned}$$

where the cubic polynomial P_3 is defined by

$$P_3(\eta):=(1+D)+\eta(1-\eta)D'=1+4C\eta+(-8+\frac{3\ln(2)}{2})C\eta^2+(4-\ln(2))C\eta^3.$$

Secondly, we have

$$\begin{aligned}\frac{d}{d\eta}\frac{\sqrt{(D\eta+1)(D\eta+\eta)}}{(D+\frac{1}{2})(1-\eta)} &= \frac{1+D+2\eta(D+D^2)+(\eta+\eta^2(1+2D))D'}{2\sqrt{(D\eta+1)(D\eta+\eta)}(D+\frac{1}{2})(1-\eta)}\\ &\quad -\frac{\sqrt{(D\eta+1)(D\eta+\eta)}(-\frac{1}{2}-D+(1-\eta)D')}{(D+\frac{1}{2})^2(1-\eta)^2}\\ &= \frac{P_7(\eta)}{2\sqrt{D\eta+1}\sqrt{D\eta+\eta}(D+\frac{1}{2})^2(1-\eta)^2},\end{aligned}$$

where P_7 is a polynomial of degree seven defined by

$$\begin{aligned}P_7(\eta) &:= (1+\eta+2\eta D)(D+1)(D+\frac{1}{2})+\eta(1-\eta)(-\frac{3}{2}+\frac{1}{2}\eta-D)D'\\ &= \frac{1}{2}+\frac{1}{2}\eta+\left(12-\frac{3\ln 2}{4}\right)C\eta^2+\left[-14+\frac{13\ln 2}{4}+(24-\ln 2)C\right]C\eta^3\\ &\quad+\left[2-\frac{\ln 2}{2}+\left(-48+13\ln 2-\frac{(\ln 2)^2}{4}\right)C+16C^2\right]C\eta^4\\ &\quad+\left[24-12\ln 2+\frac{3(\ln 2)^2}{2}+(-48+12\ln 2)C\right]C^2\eta^5\\ &\quad+\left[48-24\ln 2+3(\ln 2)^2\right]C^3\eta^6+\left[-16+12\ln 2-3(\ln 2)^2+\frac{(\ln 2)^3}{4}\right]C^3\eta^7.\end{aligned}$$

Thirdly, we compute

$$\begin{aligned}\frac{d}{d\eta}\frac{\sqrt{(D+1)^3 2D}}{(D+\frac{1}{2})(1-\eta)} &= \frac{\frac{(4D+1)(D+1)^2D'}{\sqrt{(D+1)^3D}}}{\sqrt{2}(D+\frac{1}{2})(1-\eta)} - \frac{\sqrt{2}\sqrt{(D+1)^3D}(-\frac{1}{2}-D+(1-\eta)D')}{(D+\frac{1}{2})^2(1-\eta)^2}\\ &:= \frac{\sqrt{D+1}P_6(\eta)}{\sqrt{2}\sqrt{D}(D+\frac{1}{2})^2(1-\eta)^2},\end{aligned}$$

where $P_6(\eta)$ is a polynomial of degree six defined by

$$\begin{aligned}& P_6(\eta) := 2D(D+\frac{1}{2})(D+1)+(1-\eta)(\frac{1}{2}+D+2D^2)D'\\ = \; & C+(-1+\frac{\ln(2)}{2}+4C)C\eta+(-4+3\ln(2)+16C)C^2\eta^2\\ & +(-4-\ln(2)+\frac{(\ln(2))^2}{2}+(-64+16\ln(2))C)C^2\eta^3\\ & +(4-2\ln(2)+\frac{(\ln(2))^2}{4}+(96-44\ln(2)+5(\ln(2))^2)C)C^2\eta^4\\ & +(-64+40\ln(2)-8(\ln(2))^2+\frac{(\ln(2))^3}{2})C^3\eta^5\\ & +(16-12\ln(2)+3(\ln(2))^2-\frac{(\ln(2))^3}{4})C^3\eta^6.\end{aligned}$$

We deduce that

$$\begin{aligned}G'(\eta) &= \frac{\sqrt{2}\sqrt{D\eta+1}\sqrt{\eta}(D+1)P_6-\sqrt{D}P_7-2\sqrt{D}(D+\frac{1}{2})^2(1-\eta)P_3}{2\sqrt{D}\sqrt{D\eta+1}\sqrt{D\eta+\eta}(D+\frac{1}{2})^2(1-\eta)^2}\\ &= \frac{\sqrt{2}\sqrt{D\eta+1}\sqrt{\eta}(D+1)P_6-\sqrt{D}\left(P_7+2(D+\frac{1}{2})^2(1-\eta)P_3\right)}{2\sqrt{D}\sqrt{D\eta+1}\sqrt{D\eta+\eta}(D+\frac{1}{2})^2(1-\eta)^2}\end{aligned}$$

Therefore, for $0<\eta<1$, $G'(\eta)=0$ if and only if

$$\begin{aligned}& \sqrt{2}\sqrt{D\eta+1}\sqrt{\eta}(D+1)P_6 = \sqrt{D}\left(P_7+2(D+\frac{1}{2})^2(1-\eta)P_3\right)\\ \Leftrightarrow \quad & 4(D\eta+1)(D+1)^2(P_6)^2 = C(\ln(2)\eta+4(1-\eta))\left(P_7+2(D+\frac{1}{2})^2(1-\eta)P_3\right)^2.\end{aligned}$$

Hence we are reduced to consider the following real polynomial of degree 19

$$P_{19} := 4(D\eta+1)(D+1)^2(P_6)^2 - C(\ln(2)\eta+4(1-\eta))\left(P_7+2(D+\frac{1}{2})^2(1-\eta)P_3\right)^2.$$

Using Matlab, we know that P_{19} has 5 real roots and 14 complex roots ≈ -0.35207, -0.35207, 1.48106, 1.56168, 1.56168, $-0.10171\pm 0.14272i$, $-0.19230\pm 0.16501i$, $0.04204\pm 0.15679i$, $0.90295\pm 0.96819i$, $0.90704\pm 0.02790i$, $1.27197\pm 0.04656i$, $1.45350\pm 0.45664i$. Hence $G'(\eta)\neq 0$ on $(0,1)$. Because $G'(\eta)$ has no zero point on $(0,1)$, and $G'(\frac{1}{2})>0$, we deduce that

$$G' > 0 \text{ on } (0,1).$$

Since G is increasing on $(0,1)$ and $G(0)=0$, so $G(\eta)\geq 0, \forall\eta\in(0,1)$. It follows that $\sqrt{2D}\geq J(\eta,D), \forall\eta\in(0,1)$. This implies that

$$\int_0^\eta \frac{1}{\sqrt{e^a-1}}ds \leq \frac{1}{\sqrt{2D}}J(\eta,D)\leq 1, \forall\eta\in(0,1).$$

Hence the upper bound estimation is proved. □

3. Uniqueness of critical point

This section is devoted to the proof of our main result, namely, there are exactly 2 solutions to (1.2)-(1.3) for each $\lambda\in(0,\lambda^*)$. This is equivalent to show that $\lambda(\eta)$ has only a critical point η in $(0,1)$.

First, we recall from (2.7) that

$$\int_0^\eta \frac{e^{\lambda[r(s)-r(\eta)]}}{\sqrt{1-e^{2\lambda[r(s)-r(\eta)]}}}ds=1, \quad r(s):=\frac{1}{1-s}. \tag{3.1}$$

Hereafter we always have $\lambda=\lambda(\eta)$. Note also that

$$e^{\lambda[r(s)-r(\eta)]}<1, \quad \forall\, s\in(0,\eta).$$

Observe that for any integer n we have

$$\begin{aligned}&\frac{d}{ds}\{\arcsin(e^{\lambda[r(s)-r(\eta)]})(1-s)^{n+2}\}\\ =\;&\lambda\frac{e^{\lambda[r(s)-r(\eta)]}}{\sqrt{1-e^{2\lambda[r(s)-r(\eta)]}}}(1-s)^n-(n+2)\arcsin(e^{\lambda[r(s)-r(\eta)]})(1-s)^{n+1}.\end{aligned}$$

This implies that

$$\begin{aligned}&\lambda\int_0^\eta \frac{e^{\lambda[r(s)-r(\eta)]}}{\sqrt{1-e^{2\lambda[r(s)-r(\eta)]}}}(1-s)^n ds\\ =\;&\frac{\pi}{2}(1-\eta)^{n+2}-\arcsin(e^{-\lambda\eta r(\eta)})+(n+2)\int_0^\eta \arcsin(e^{\lambda[r(s)-r(\eta)]})(1-s)^{n+1}ds.\end{aligned} \tag{3.2}$$

In particular, taking $n=0$ and using (3.1), we deduce from (3.2) that

$$\lambda=\frac{\pi}{2}(1-\eta)^2-\arcsin(e^{-\lambda\eta r(\eta)})+2\int_0^\eta \arcsin(e^{\lambda[r(s)-r(\eta)]})(1-s)ds. \tag{3.3}$$

Next, differentiating (3.3) with respect to η and using (3.1), for $0 < \eta < 1$, we obtain

$$\begin{aligned}(3.4)\qquad \lambda' &= -\pi(1-\eta) - \frac{e^{-\lambda\eta r(\eta)}}{\sqrt{1-e^{-2\lambda\eta r(\eta)}}}[-\lambda(1-\eta)^{-2} - \lambda'\eta(1-\eta)^{-1}] + \pi(1-\eta) \\ &\quad +2\int_0^\eta \frac{e^{\lambda[r(s)-r(\eta)]}}{\sqrt{1-e^{2\lambda[r(s)-r(\eta)]}}}\{-\lambda(1-\eta)^{-2} + \lambda'[r(s)-r(\eta)]\}(1-s)ds \\ &= \frac{e^{-\lambda\eta r(\eta)}}{\sqrt{1-e^{-2\lambda\eta r(\eta)}}}[\lambda(1-\eta)^{-2} + \lambda'\eta(1-\eta)^{-1}] + 2\lambda' \\ &\quad -2[\lambda(1-\eta)^{-2} + \lambda'(1-\eta)^{-1}]\int_0^\eta \frac{e^{\lambda[r(s)-r(\eta)]}}{\sqrt{1-e^{2\lambda[r(s)-r(\eta)]}}}(1-s)ds.\end{aligned}$$

For convenience, we define

$$\begin{aligned} Q(\eta) &:= \frac{e^{-\lambda\eta r(\eta)}}{\sqrt{1-e^{-2\lambda\eta r(\eta)}}} = \frac{1}{\sqrt{e^{2\lambda\eta r(\eta)}-1}}, \\ I_n(\eta) &:= \int_0^\eta \frac{e^{\lambda[r(s)-r(\eta)]}}{\sqrt{1-e^{2\lambda[r(s)-r(\eta)]}}}(1-s)^n ds,\ n\in\mathbb{Z}, \\ J_n(\eta) &:= \int_0^\eta \arcsin(e^{\lambda[r(s)-r(\eta)]})(1-s)^n ds,\ n\in\mathbb{Z}.\end{aligned}$$

Note that J_n is positive and

$$(3.5)\qquad J_n(\eta) \le \frac{\pi}{2}\frac{1-(1-\eta)^{n+1}}{n+1},\ \text{ if } n\neq -1;\quad -\frac{\pi}{2}\ln(1-\eta),\ \text{ if } n=-1.$$

Hence, by (3.2), $I_n(\eta)$ is well-defined and

$$(3.6)\qquad I_n(\eta) = \lambda^{-1}\left\{\frac{\pi}{2}(1-\eta)^{n+2} - \arcsin(e^{-\lambda\frac{\eta}{1-\eta}}) + (n+2)J_{n+1}(\eta)\right\},\ \forall\,\eta\in(0,1).$$

On the other hand, rewriting (3.4) as

$$\lambda' = [\lambda(1-\eta)^{-2} + \lambda'\eta(1-\eta)^{-1}]Q(\eta) + 2\lambda' - 2[\lambda(1-\eta)^{-2} + \lambda'(1-\eta)^{-1}]I_1(\eta),$$

or equivalently,

$$(3.7)\qquad [2I_1(\eta) - \eta Q(\eta) - (1-\eta)]\lambda'(\eta) = \frac{\lambda(\eta)}{1-\eta}[Q(\eta) - 2I_1(\eta)],$$

we obtain the following formula for the first derivative of λ

$$(3.8)\qquad \lambda' = \lambda'(\eta) = \frac{\lambda(\eta)}{1-\eta}\cdot\frac{-2I_1(\eta)+Q(\eta)}{2I_1(\eta) - \eta Q(\eta) - (1-\eta)},$$

if $2I_1(\eta) - \eta Q(\eta) - (1-\eta) \neq 0$. In particular, when $2I_1(\eta) - \eta Q(\eta) - (1-\eta) \neq 0$,

$$(3.9)\qquad \lambda'(\eta) = 0 \ \text{ if and only if } \ Q(\eta) = 2I_1(\eta).$$

Note that

$$I_n(\eta) = \int_0^\eta \frac{(1-s)^n}{\sqrt{e^{2\lambda[r(\eta)-r(s)]}-1}}ds,\ n\in\mathbb{Z}.$$

Using that the function $y/(e^y-1)$ is decreasing in y for $y>0$ and observing that

$$r(\eta)-r(s)=\frac{\eta-s}{(1-\eta)(1-s)}<\frac{\eta}{(1-\eta)},\ s\in(0,\eta),\ \eta\in(0,1),$$

we obtain

$$\begin{aligned}I_1(\eta) &= \int_0^\eta \frac{1-s}{\sqrt{e^{2\lambda[r(\eta)-r(s)]}-1}}ds=\int_0^\eta\sqrt{\frac{2\lambda[r(\eta)-r(s)]}{e^{2\lambda[r(\eta)-r(s)]}-1}}\frac{1-s}{\sqrt{2\lambda[r(\eta)-r(s)]}}ds\\ &\geq \sqrt{\frac{2\lambda\eta r(\eta)}{e^{2\lambda\eta r(\eta)}-1}}\int_0^\eta\frac{1-s}{\sqrt{2\lambda[r(\eta)-r(s)]}}ds=\sqrt{\eta}Q(\eta)\int_0^\eta\frac{(1-s)^{3/2}}{\sqrt{\eta-s}}ds.\end{aligned}$$

From Property 5.2 (see Appendix)

$$\int_0^\eta\frac{(1-s)^{3/2}}{\sqrt{\eta-s}}ds=\frac{1}{4}[(5-3\eta)\sqrt{\eta}+3(1-\eta)^2\ln(\frac{1+\sqrt{\eta}}{\sqrt{1-\eta}})],$$

we deduce that

$$I_1(\eta)\geq Q(\eta)\frac{1}{4}[(5-3\eta)\eta+3\sqrt{\eta}(1-\eta)^2\ln(\frac{1+\sqrt{\eta}}{\sqrt{1-\eta}})]. \tag{3.10}$$

Using (3.10), we now prove the following important lemma.

Lemma 3.1. *It holds $2I_1(\eta)-\eta Q(\eta)-(1-\eta)>0$ for all $\eta\in(0,1)$.*

Proof. First, we consider the quartic function (polynomial of degree 4)

$$P_4(x)=[(1+\frac{x}{3})(1-x)+1]^2-[(\ln 2)x+4(1-x)],\ x\in\mathbb{R}.$$

We compute

$$P_4'(x)=\frac{4}{9}\{(x+1)^3-7(x+1)+9(1-\frac{\ln(2)}{4})\}.$$

Then, using Cardano's formula with $p=-7$ and $q=9(1-\frac{\ln(2)}{4})$, the cubic polynomial $P_4'(x)$ has only one (real) root at

$$x=-1+\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}-\frac{343}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}-\frac{343}{27}}}<-1.$$

Hence $P_4(x)$ is increasing for $x\in[0,1]$. Since $P_4(0)=0$, we have $P_4(x)>0$ for $x\in(0,1)$. Therefore, we have

$$\frac{\sqrt{(\ln 2)x+4(1-x)}-1}{1-x}<1+\frac{x}{3},\ x\in(0,1). \tag{3.11}$$

On the other hand, when $x\in(0,1)$, using

$$\ln\left(\frac{1+\sqrt{x}}{\sqrt{1-x}}\right)=\sum_{k=0}^{\infty}x^{\frac{1}{2}}\frac{x^k}{2k+1}, \tag{3.12}$$

we get

$$\frac{1}{\sqrt{x}}\ln\left(\frac{1+\sqrt{x}}{\sqrt{1-x}}\right)=1+\frac{x}{3}+\sum_{k=2}^{\infty}\frac{x^k}{2k+1}>1+\frac{x}{3}.$$

Therefore, we obtain from (3.11) that

$$\frac{1}{\sqrt{x}}\ln\left(\frac{1+\sqrt{x}}{\sqrt{1-x}}\right)>\frac{\sqrt{(\ln 2)x+4(1-x)}-1}{1-x},\ x\in(0,1). \tag{3.13}$$

Next, it is easy to see that $2y>e^y-1$ when $0<y\le 1$. Since

$$\max_{x\in[0,1]}\{[(\ln 2)x+4(1-x)]x^2\}=\frac{256}{27(4-\ln 2)^2},$$

we have

$$y(x):=c_4[(\ln 2)x+4(1-x)]x^2\le c_4\cdot\frac{256}{27(4-\ln 2)^2}<1,\ \forall x\in[0,1].$$

Here the last inequality follows due to the constant $c_4=1.1$ derived in Lemma 2.5. So

$$2c_4[(\ln 2)x+4(1-x)]x^2>e^{c_4[(\ln 2)x+4(1-x)]x^2}-1,\ 0<x<1.$$

Then we have

$$\begin{aligned}&2\sqrt{e^{c_4[(\ln 2)x+4(1-x)]x^2}-1}\\<\ &2x\sqrt{2c_4[(\ln 2)x+4(1-x)]}=\sqrt{8c_4}x\sqrt{(\ln 2)x+4(1-x)}\\<\ &3x\sqrt{(\ln 2)x+4(1-x)}<3x+3\sqrt{x}(1-x)\ln(\frac{1+\sqrt{x}}{\sqrt{1-x}})\end{aligned}$$

for $x\in(0,1)$, using $8c_4<9$ and (3.13). From (3.10) and the above estimation, it follows that

$$\begin{aligned}&2I_1(\eta)-\eta Q(\eta)-(1-\eta)\\\ge\ &Q(\eta)\frac{3}{2}[(1-\eta)\eta+\sqrt{\eta}(1-\eta)^2\ln(\frac{1+\sqrt{\eta}}{\sqrt{1-\eta}})]-(1-\eta)\\=\ &Q(\eta)\frac{1-\eta}{2}\left(3\eta+3\sqrt{\eta}(1-\eta)\ln(\frac{1+\sqrt{\eta}}{\sqrt{1-\eta}})-2\sqrt{e^{2\lambda\eta r(\eta)}-1}\right)\\>\ &Q(\eta)(1-\eta)\left(\sqrt{e^{c_4[(\ln 2)\eta+4(1-\eta)]\eta^2}-1}-\sqrt{e^{2\lambda\eta r(\eta)}-1}\right)\ge 0,\end{aligned}$$

using Lemma 2.5. The lemma is proved. □

From Lemma 3.1 and (3.9), we see that $\lambda'(\eta)=0$ if and only if $Q(\eta)=2I_1(\eta)$.

The following lemma gives an important fact that any critical point of λ is away from 1.

Lemma 3.2. *If $\lambda'(\eta)=0$, then $\eta\le r_2$, where $r_2\in(0.38,0.39)$ is the unique root of*

$$2=(5-3\eta)\eta+3\sqrt{\eta}(1-\eta)^2\ln\left(\frac{1+\sqrt{\eta}}{\sqrt{1-\eta}}\right)$$

in $(0,1)$.

Proof. Set

$$a(x) := 3\sqrt{x}\ln\left(\frac{1+\sqrt{x}}{\sqrt{1-x}}\right), \quad b(x) := \frac{2-3x}{1-x}.$$

It is easy to check that $a(x)$ is increasing on $(0,1)$, $b(x)$ is decreasing on $(0,1)$, $a(0) = 0 < 2 = b(0)$, $a(1^-) = +\infty$ and $b(1^-) = -\infty$. Hence there is exactly one intersection of $a(x)$ and $b(x)$ on $(0,1)$, say r_2 ($\approx 0.38834\ 67189$), such that

$$(5-3x)x + 3\sqrt{x}(1-x)^2\ln\left(\frac{1+\sqrt{x}}{\sqrt{1-x}}\right) < 2 \text{ for } x \in (0, r_2), \tag{3.14}$$

$$(5-3x)x + 3\sqrt{x}(1-x)^2\ln\left(\frac{1+\sqrt{x}}{\sqrt{1-x}}\right) > 2 \text{ for } x \in (r_2, 1). \tag{3.15}$$

It follows from (3.10) and (3.15) that $I_1(\eta) > Q(\eta)/2$ for $\eta \in (r_2, 1)$. Hence the lemma is proved by using (3.9). □

Our goal is to derive that λ has a unique critical point in $(0,1)$. This can be done if we can derive that $\lambda''(\eta) < 0$ when $\lambda'(\eta) = 0$. In the sequel, we let $\bar\eta \in (0, r_2]$ be any point such that $\lambda'(\bar\eta) = 0$ and set $\bar\lambda = \lambda(\bar\eta)$. At $\bar\eta$, since $2I_1(\bar\eta) = Q(\bar\eta)$, it follows from Lemma 3.1 and

$$2I_1(\bar\eta) - \bar\eta Q(\bar\eta) - (1-\bar\eta) = [Q(\bar\eta) - 1](1-\bar\eta)$$

that $Q(\bar\eta) > 1$.

To compute $\lambda''(\bar\eta)$, using $\lambda'(\bar\eta) = 0$, (3.8) and (3.9), we can easily derive that

$$\lambda''(\bar\eta) = \frac{\bar\lambda}{1-\bar\eta} \cdot \frac{-2I_1'(\bar\eta) + Q'(\bar\eta)}{2I_1(\bar\eta) - \bar\eta Q(\bar\eta) - (1-\bar\eta)}. \tag{3.16}$$

To proceed further, we use (3.6) to compute the first derivative of $I_1(\eta)$ as follows. First, we can easily compute from the definition of J_2 that

$$J_2'(\bar\eta) = \frac{\pi}{2}(1-\bar\eta)^2 - \frac{\bar\lambda}{(1-\bar\eta)^2} I_2(\bar\eta).$$

Then, by (3.6) with $n = 1$, we obtain

$$I_1'(\bar\eta) = \frac{1}{\bar\lambda}\left\{-\frac{3}{2}\pi(1-\bar\eta)^2 + \frac{\bar\lambda}{(1-\bar\eta)^2}Q(\bar\eta) + 3J_2'(\bar\eta)\right\} = \frac{1}{(1-\bar\eta)^2}[Q(\bar\eta) - 3I_2(\bar\eta)].$$

On the other hand, we have

$$Q'(\bar\eta) = -\frac{\bar\lambda}{(1-\bar\eta)^2}\frac{1}{1-e^{-2\bar\lambda\bar\eta/(1-\bar\eta)}}Q(\bar\eta) < 0.$$

Hence we deduce that

$$-2I_1'(\bar\eta) + Q'(\bar\eta) = \frac{1}{(1-\bar\eta)^2}\left\{6I_2(\bar\eta) - \left[2 + \frac{\bar\lambda}{1-e^{-2\bar\lambda\bar\eta/(1-\bar\eta)}}\right]Q(\bar\eta)\right\}. \tag{3.17}$$

Finally, we prove

Lemma 3.3. *It holds*

$$6I_2(\eta) - \left[2 + \frac{\lambda}{1 - e^{-2\lambda\eta/(1-\eta)}}\right] Q(\eta) < 0$$

for $\eta \in (0, r_2]$.

Proof. First, since $e^x - 1 > x$ for all positive x and

$$\int_0^x \frac{(1-s)^{5/2}}{\sqrt{x-s}} ds = \frac{1}{24}[\sqrt{x}(33 - 40x + 15x^2) + 15(1-x)^3 \ln(\frac{1+\sqrt{x}}{\sqrt{1-x}})]$$

for all $x \in (0, 1)$ (see Property 5.3 in Appendix), it follows that

$$\begin{aligned} (3.18) \qquad I_2(\eta) &= \int_0^\eta \frac{1}{\sqrt{e^{2\lambda[r(\eta)-r(s)]} - 1}}(1-s)^2 ds \\ &\le \frac{1}{\sqrt{2\lambda r(\eta)}} \int_0^\eta \frac{(1-s)^{5/2}}{(\eta - s)^{1/2}} ds \\ &= \frac{1}{\sqrt{2\lambda r(\eta)}} \frac{1}{24}[\sqrt{\eta}(33 - 40\eta + 15\eta^2) + 15(1-\eta)^3 \ln(\frac{1+\sqrt{\eta}}{\sqrt{1-\eta}})] \\ &= \frac{1}{\sqrt{2\lambda\eta r(\eta)}} \frac{1}{24}[\eta(33 - 40\eta + 15\eta^2) + 15\sqrt{\eta}(1-\eta)^3 \ln(\frac{1+\sqrt{\eta}}{\sqrt{1-\eta}})]. \end{aligned}$$

To proceed further, we consider the following two auxiliary functions for $x \in (0, 1)$

$$\begin{aligned} a(x) &:= x(33 - 40x + 15x^2) + 15\sqrt{x}(1-x)^3 \ln(\frac{1+\sqrt{x}}{\sqrt{1-x}}), \\ b(x) &:= 2\left[4 + \frac{1-x}{x} \frac{y}{e^y - 1}\right] \frac{\sqrt{y}}{\sqrt{e^y - 1}}, \ y = y(x) = c_4[(\ln 2)x + 4(1-x)]x^2. \end{aligned}$$

Set $x_0 := 8/[3(4 - \ln 2)] \approx 0.8$. Since $y(x)$ is increasing for $0 \le x \le x_0$ and $z/(e^z - 1)$ is decreasing for positive z, so the composition $y(x)/[e^{y(x)} - 1]$ is decreasing for $0 \le x \le x_0$. Moreover, $(1-x)/x$ is also decreasing for $x \in (0, 1)$. Therefore, $b(x)$ is decreasing for $0 < x \le x_0$, where $x_0 > r_2$.

For $a(x)$, we compute

$$(3.19) \qquad a'(x) = \frac{1}{2}\left(105x^2 - 190x + 81 + 15(1-x)^2(1-7x)\frac{\ln\left(\frac{1+\sqrt{x}}{\sqrt{1-x}}\right)}{\sqrt{x}}\right)$$

By (3.12), $\ln(\frac{1+\sqrt{x}}{\sqrt{1-x}}) > \sqrt{x}$ for $x \in (0, 1)$. Then we have $a'(x) > p(x)/2$ for $x \in (0, 1/7)$, where

$$p(x) := 105x^2 - 190x + 81 + 15(1-x)^2(1-7x).$$

Since $p'(x) = -5(63x^2 - 132x + 65)$ which has two zeros at $\frac{22 \pm \sqrt{29}}{21} \approx 0.79, 1.30$, $p(x)$ is decreasing on $(0, 1/7)$. Hence $p(x) > p(1/7) > 0$ and so $a(x)$ is increasing for $x \in (0, 1/7)$.

On the other hand, for $x \in [1/7, r_2]$, it follows from (3.14) and (3.19) that

$$a'(x) \geq \frac{1}{2}\left[105x^2 - 190x + 81 + \frac{5(1-7x)}{\sqrt{x}} \cdot \frac{2-(5-3x)x}{\sqrt{x}}\right] = \frac{1}{2x}(10 - 14x).$$

Hence $a(x)$ is also increasing for $x \in [1/7, r_2]$.

Using the definition of r_2 in Lemma 3.2, we compute

$$\begin{aligned} a(r_2) &= r_2(33 - 40r_2 + 15r_2^2) + 5(1-r_2) \cdot 3\sqrt{r_2}(1-r_2)^2 \ln\left(\frac{1+\sqrt{r_2}}{\sqrt{1-r_2}}\right) \\ &= r_2(33 - 40r_2 + 15r_2^2) + 5(1-r_2) \cdot [2 - (5-3r_2)r_2] = 10 - 2r_2. \end{aligned}$$

Moreover, it is easy to check that $a(r_2) < b(r_2)$. We conclude that

$$a(x) < b(x) \quad \text{for } x \in [0, r_2], \tag{3.20}$$

since $a(x)$ is increasing and $b(x)$ is decreasing for $x \in [0, r_2]$.

Now, it follows from (3.18), (3.20), and the decreasing property of $y/(e^y - 1)$ with $y \geq 2\lambda\eta r(\eta)$ that

$$\begin{aligned} 6I_2(\eta) &< \frac{1}{\sqrt{2\lambda\eta r(\eta)}} \cdot \frac{1}{2} \cdot \left[4 + \frac{1-\eta}{\eta}\frac{2\lambda\eta r(\eta)}{e^{2\lambda\eta r(\eta)} - 1}\right] \frac{\sqrt{2\lambda\eta r(\eta)}}{\sqrt{e^{2\lambda\eta r(\eta)} - 1}} \\ &= \left[2 + \frac{\lambda}{e^{2\lambda\eta r(\eta)} - 1}\right] Q(\eta) = \left[2 + \frac{\lambda e^{-2\lambda\eta r(\eta)}}{1 - e^{-2\lambda\eta r(\eta)}}\right] Q(\eta) \\ &\leq \left[2 + \frac{\lambda}{1 - e^{-2\lambda\eta r(\eta)}}\right] Q(\eta) \end{aligned}$$

for $\eta \in (0, r_2]$. Hence the lemma follows.

□

Now we are ready to prove the following uniqueness theorem on the critical points of $\lambda(\eta)$.

Theorem 3.4. *There is a unique point η^* on $(0,1)$ such that $\lambda'(\eta) > 0$ for $0 < \eta < \eta^*$, $\lambda'(\eta^*) = 0$, and $\lambda'(\eta) < 0$ for $\eta^* < \eta < 1$.*

Proof. Since $\lambda'(0^+) > 0$ and $\lambda'(1^-) < 0$, there exists a point $\eta^* \in (0,1)$ such that $\lambda'(\eta^*) = 0$. By Lemma 3.2, $\eta^* \in (0, r_2]$. From (3.16), (3.17) and Lemma 3.3 it follows that $\lambda''(\eta^*) < 0$. Hence $\lambda(\eta)$ cannot have another critical point than η^* and the theorem is proved. □

4. Stability analysis

In this section, we study the following initial boundary value problem

$$u_t - u_{xx} = \lambda\frac{1 + u_x^2}{(1-u)^2}, \; x \in (-1,1), \; t > 0, \tag{4.1}$$

$$u(x,0) = u_0(x), \; x \in [-1,1], \quad u(\pm 1, t) = 0. \tag{4.2}$$

We shall derive the stability of some stationary solutions of (4.1)-(4.2).

First, we prove

Lemma 4.1. *For each $\lambda \in (0, \lambda^*)$, the corresponding two solutions u_1, u_2 of* (1.2)-(1.3) *are ordered in the sense that $u_1 < u_2$ in $(-1, 1)$ if $u_1(0) < u_2(0)$.*

Proof. Fix $\lambda \in (0, \lambda^*)$. Let u_1 and u_2 be two solutions of (1.2)-(1.3) with $0 < \eta_1 < \eta_2 < 1$, where $\eta_i := u_i(0)$, $i = 1, 2$. We claim that $u_1 < u_2$ in $(-1, 1)$.

For contradiction, we assume that there is a $\bar{x} \in (0, 1)$ such that $u_1(\bar{x}) \geq u_2(\bar{x})$. Without loss of generality we may assume that $u_1(\bar{x}) = u_2(\bar{x})$. From (2.6), at $x = \bar{x}$, we have

$$\int_0^{u_1(\bar{x})} \left[(e^{2\lambda(\frac{1}{1-\eta_1} - \frac{1}{1-s})} - 1)^{-1/2} - (e^{2\lambda(\frac{1}{1-\eta_2} - \frac{1}{1-s})} - 1)^{-1/2} \right] ds = 0.$$

Since $\frac{1}{1-\eta_1} < \frac{1}{1-\eta_2}$, we have

$$\left[(e^{2\lambda(\frac{1}{1-\eta_1} - \frac{1}{1-s})} - 1)^{-1/2} - (e^{2\lambda(\frac{1}{1-\eta_2} - \frac{1}{1-s})} - 1)^{-1/2} \right] > 0$$

for all $s \in [0, u_1(\bar{x})]$, which leads a contradiction. Hence the lemma is proved. □

Corollary 4.2. *For $0 < \eta_1 < \eta_2 < 1$, set $\lambda_1 = \lambda(\eta_1)$ and $\lambda_2 = \lambda(\eta_2)$. Let u_i be the solution of* (1.2)-(1.3) *with $\lambda = \lambda_i$ and $u_i(0) = \eta_i$, $i = 1, 2$. If $\frac{\lambda_1 \eta_1}{1-\eta_1} < \frac{\lambda_2 \eta_2}{1-\eta_2}$, then we have $u_1 < u_2$ in $(-1, 1)$.*

Proof. We claim that the following inequality holds for $s \in (0, 1)$

$$(e^{2\lambda_1(\frac{1}{1-\eta_1} - \frac{1}{1-s})} - 1)^{-1/2} > (e^{2\lambda_2(\frac{1}{1-\eta_2} - \frac{1}{1-s})} - 1)^{-1/2}.$$

If the claim is true, then the proof of this corollary is similar to that for Lemma 4.1.

If $\lambda_1 \leq \lambda_2$, then it is easy to show that the claim holds using $\eta_1 < \eta_2$. Suppose that $\lambda_1 > \lambda_2$. Then we have, at $s = 0$,

$$(e^{2\lambda_1(\frac{1}{1-\eta_1} - 1)} - 1)^{-1/2} > (e^{2\lambda_2(\frac{1}{1-\eta_2} - 1)} - 1)^{-1/2},$$

since $\frac{\lambda_1 \eta_1}{1-\eta_1} < \frac{\lambda_2 \eta_2}{1-\eta_2}$. If the claim is not true, then there exists a $w \in (0, 1)$ such that

$$(e^{2\lambda_1(\frac{1}{1-\eta_1} - \frac{1}{1-w})} - 1)^{-1/2} = (e^{2\lambda_2(\frac{1}{1-\eta_2} - \frac{1}{1-w})} - 1)^{-1/2}.$$

It follows that

$$w = 1 + \frac{\lambda_1 - \lambda_2}{\lambda_2/(1-\eta_2) - \lambda_1/(1-\eta_1)} > 1,$$

a contradiction with $w \in (0, 1)$. Hence the corollary follows. □

From (3.8), it follows that

$$\left(\frac{\lambda \eta}{1-\eta} \right)' = \lambda \frac{2I_1(\eta) - 1}{[2I_1(\eta) - \eta Q(\eta) - (1-\eta)](1-\eta)}. \tag{4.3}$$

Using this identity, we prove

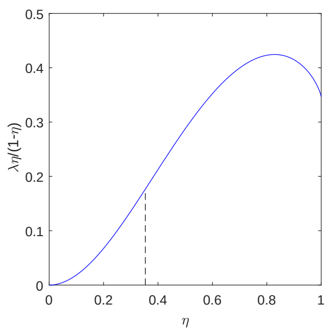

FIGURE 2. Curve of $\frac{\lambda\eta}{1-\eta}$.

Lemma 4.3. *It holds $\frac{\lambda^*\eta^*}{1-\eta^*} < \frac{\lambda\eta}{1-\eta}$ for $\eta^* < \eta < 1$, where $\lambda = \lambda(\eta)$ and $\lambda^* = \lambda(\eta^*)$.*

Proof. First, we claim that $\frac{\lambda\eta}{1-\eta}$ is increasing for $\eta \in (0, 1/2)$. When $0 < s < \eta < 1/2$, since $s < 1 - s$, we have

$$I_1(\eta) = \int_0^\eta \frac{1-s}{\sqrt{e^{2\lambda[r(\eta)-r(s)]}-1}}ds > \int_0^\eta \frac{s}{\sqrt{e^{2\lambda[r(\eta)-r(s)]}-1}}ds.$$

By (2.7), we know that

$$\int_0^\eta \frac{1-s}{\sqrt{e^{2\lambda[r(\eta)-r(s)]}-1}}ds + \int_0^\eta \frac{s}{\sqrt{e^{2\lambda[r(\eta)-r(s)]}-1}}ds = 1.$$

Hence $I_1(\eta) > 1/2$. Recall Lemma 3.1. Then, using (4.3), $\frac{\lambda\eta}{1-\eta}$ is increasing for $\eta \in (0, 1/2)$ and so the lemma holds for $\eta^* < \eta < 1/2$.

Next, we set

$$q(x) := [\ln(\sqrt{2})x + 2(1-x)]x^2,\ x \in [1/2, 1].$$

It is easy to see that $q(x)$ is increasing on $(0, x_0)$, $x_0 := 8/[3(4-\ln 2)] \approx 0.8$, and decreasing on $(x_0, 1)$. Then we have $q(x) \geq q(1/2)$ for all $x \in [1/2, 1]$, since $q(1/2) < q(1)$. Recalling from Lemma 3.2 that $\eta^* \in (0, r_2)$ and $r_2 \in (0.38, 0.39)$. Then $q(0.39) > q(\eta^*)$ and it is easy to check that $c_3 q(1/2) > c_4 q(0.39)$. By using Lemmas 2.4 and 2.5, we deduce that

$$\frac{\lambda\eta}{1-\eta} \geq c_3 q(\eta) \geq c_3 q(1/2) > c_4 q(0.39) > c_4 q(\eta^*) \geq \frac{\lambda^*\eta^*}{1-\eta^*}$$

for $\eta \in [1/2, 1)$. This complete the proof of the lemma. □

Remark 4.4. The above lemma can be observed numerically (see Figure 2). This figure also implies that u_i solutions in Corollary 4.2 are ordered for $\eta_i \in (0, 0.82]$.

For convenience, we let $U_+^\lambda(x)$ be the larger stationary solution for $\lambda \in (0, \lambda^*)$, $U_-^\lambda(x)$ be the smaller one for $\lambda \in (0, \lambda^*)$, and $U^*(x)$ be the unique stationary solution for $\lambda = \lambda^*$.

From Lemma 4.1, Corollary 4.2, and Lemma 4.3, we have proved the following theorem.

Theorem 4.5. *It holds $U_-^\lambda(x) < U^*(x) < U_+^\lambda(x)$ for all $\lambda \in (0, \lambda^*)$.*

Now, for the initial boundary value problem (4.1)-(4.2), we have

Lemma 4.6 (Comparison Principle)**.** *If u and v are solutions of (4.1)-(4.2) with initial data u_0 and v_0, respectively, such that $u_0 \geq v_0$, then $u(x,t) \geq v(x,t)$ for $x \in [-1.1]$, $t > 0$.*

Proof. Set $w = u - v$. Then w satisfies $w(\pm 1, t) = 0$, $w(x, 0) \geq 0$ for $x \in [-1, 1]$ and

$$w_t - w_{xx} = \lambda(aw_x + bw),\ x \in (-1, 1),\ t > 0,$$

where

$$a := \frac{1}{(1-u)^2(1-v)^2}\{[u_x(1-v) + v_x(1-u)](1-v)\},$$
$$b := \frac{1}{(1-u)^2(1-v)^2}\{(2-v-u) + [u_x(1-v) + v_x(1-u)]v_x\}.$$

By the maximum principle, we have $w \geq 0$. Hence the lemma is proved. ☐

Note that the existence of local (in time) solution to (4.1)-(4.2) with smooth initial data u_0 satisfying $0 \leq u_0 < 1$ in $[-1, 1]$ follows easily from the standard argument with the parabolic regularity theory. As a consequence of Lemma 4.6, there exists a unique solution to (4.1)-(4.2) as long as $u < 1$.

Theorem 4.7. *Let $\lambda \in (0, \lambda^*)$ and let u be the corresponding solution of (4.1)-(4.2) with initial data u_0 satisfying $0 \leq u_0(x) \leq U^*(x)$ for all $x \in [-1, 1]$. Then u converges to U_-^λ as $t \to \infty$.*

Proof. First, we let w be the solution of (4.1)-(4.2) with initial data U^* for a given $\lambda \in (0, \lambda^*)$. Set $z = w_t$. Then z satisfies

$$z_t = z_{xx} + \lambda \frac{2w_x}{(1-w)^2} z_x + 2\lambda \frac{1 + w_x^2}{(1-w)^3} z,\ x \in (-1, 1),\ t > 0.$$

Since

$$z(x, 0) = U_{xx}^*(x) + \lambda \frac{1 + [U_x^*(x)]^2}{[1 - U^*(x)]^2} < U_{xx}^*(x) + \lambda^* \frac{1 + [U_x^*(x)]^2}{[1 - U^*(x)]^2} = 0,\ \forall\, x \in (-1, 1),$$

it follows from the maximum principle that $z < 0$ for all $t > 0$. Therefore, the limit $W(x) := \lim_{t\to\infty} w(x,t)$ exists for all $x \in [-1, 1]$. Note that W is a stationary solution of (4.1)-(4.2) for the given λ, since $w_t(x,t) \to 0$ as $t \to \infty$ for all $x \in [-1, 1]$. Hence we have either $W = U_-^\lambda$ or $W = U_+^\lambda$. However, $U^*(0) < U_+^\lambda(0)$. Hence we must have $W = U_-^\lambda$.

Next, we consider the solution v of (4.1)-(4.2) with zero initial data. Note that $v_t(x, 0) \equiv \lambda > 0$ for all $x \in (-1, 1)$. It follows from the maximum principle that $v_t > 0$ for all $t > 0$. Hence the limit $V(x) := \lim_{t\to\infty} v(x,t)$ exists for all $x \in [-1, 1]$. As above, V is also a stationary solution of (4.1)-(4.2) for the given λ. This gives that $V = U_-^\lambda$.

Finally, since $0 \le u_0 \le U^*$, we have $v \le u \le w$, by Lemma 4.6. We conclude that $u \to U_-^\lambda$ as $t \to \infty$. The proof is complete. □

The stability of U_-^λ follows immediately from Theorem 4.7.

Corollary 4.8. *For $\lambda \in (0, \lambda^*)$, U_-^λ is stable.*

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5. Appendix

In this appendix, for reader's convenience, we provide the derivation of some non-standard integrals used in this paper.

Property 5.1. It holds

$$\int_0^\eta \frac{(1-s)^{1/2}}{\sqrt{\eta-s}}ds = \sqrt{\eta} + (1-\eta)\ln(\frac{1+\sqrt{\eta}}{\sqrt{1-\eta}})$$

for all $\eta \in (0,1)$.

Proof. For $0 < s < \eta < 1$, set $t = \sqrt{\frac{\eta-s}{1-s}}$. Then

$$dt = -\frac{1}{2}\sqrt{\frac{1-s}{\eta-s}}\frac{1-\eta}{(1-s)^2}ds$$

and we have

$$\begin{aligned}
\int_0^\eta \frac{(1-s)^{1/2}}{\sqrt{\eta-s}}ds &= 2(1-\eta)\int_0^{\sqrt{\eta}} \frac{1}{(1-t^2)^2}dt \\
&= 2(1-\eta)\left(\frac{1}{4}\ln(1+t) - \frac{1}{4(1+t)} - \frac{1}{4}\ln(1-t) + \frac{1}{4(1-t)}\right)\Big|_0^{\sqrt{\eta}} \\
&= 2(1-\eta)\left(\frac{1}{4}\ln(1+\sqrt{\eta}) - \frac{1}{4\left(1+\sqrt{\eta}\right)} - \frac{1}{4}\ln(1-\sqrt{\eta}) + \frac{1}{4\left(1-\sqrt{\eta}\right)}\right) \\
&= \sqrt{\eta} + (1-\eta)\ln(\sqrt{\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}}) \\
&= \sqrt{\eta} + (1-\eta)\ln(\frac{1+\sqrt{\eta}}{\sqrt{1-\eta}}).
\end{aligned}$$

The property is derived. □

Property 5.2. We have

$$\int_0^\eta \frac{(1-s)^{3/2}}{\sqrt{\eta - s}} ds = \frac{1}{4}\left[(5-3\eta)\sqrt{\eta} + 3(1-\eta)^2 \ln(\frac{1+\sqrt{\eta}}{\sqrt{1-\eta}})\right]$$

for all $\eta \in (0,1)$

Proof. Similar to the proof of the previous property, set $t = \sqrt{\frac{\eta-s}{1-s}}$, for $0 < s < \eta < 1$. Then we have

$$\begin{aligned}
&\int_0^\eta \frac{(1-s)^{3/2}}{\sqrt{\eta - s}} ds = 2(1-\eta)^2 \int_0^{\sqrt{\eta}} \frac{1}{(1-t^2)^3} dt \\
= \;& 2(1-\eta)^2 \Big(\frac{3}{16}\ln(1+t) - \frac{3}{16(1+t)} - \frac{1}{16(1+t)^2} \\
& \qquad - \frac{3}{16}\ln(1-t) + \frac{3}{16(1-t)} + \frac{1}{16(1-t)^2} \Big)\Big|_0^{\sqrt{\eta}} \\
= \;& 2(1-\eta)^2 \left(\frac{3}{16}\ln(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}) + \frac{3}{16}(\frac{1}{1-\sqrt{\eta}} - \frac{1}{1+\sqrt{\eta}}) + \frac{1}{16}(\frac{1}{(1-\sqrt{\eta})^2} - \frac{1}{(1+\sqrt{\eta})^2}) \right) \\
= \;& \frac{1}{4}\left[(5-3\eta)\sqrt{\eta} + 3(1-\eta)^2 \ln(\sqrt{\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}})\right] \\
= \;& \frac{1}{4}\left[(5-3\eta)\sqrt{\eta} + 3(1-\eta)^2 \ln(\frac{1+\sqrt{\eta}}{\sqrt{1-\eta}})\right].
\end{aligned}$$

The property is proved. □

Similar technique can be used to obtain the following property.

Property 5.3. It holds

$$\int_0^\eta \frac{(1-s)^{5/2}}{\sqrt{\eta - s}} ds = \frac{1}{24}\left[\sqrt{\eta}(33 - 40\eta + 15\eta^2) + 15(1-\eta)^3 \ln(\frac{1+\sqrt{\eta}}{\sqrt{1-\eta}})\right]$$

for all $\eta \in (0,1)$.

Finally, we prove

Property 5.4. For $D > 0$ and $\eta \in (0,1)$, it holds that

$$\begin{aligned}
& J(\eta, D) := \int_0^\eta \frac{1}{\sqrt{\left(\frac{\eta-s}{1-s}\right) + D\left(\frac{\eta-s}{1-s}\right)^2}} ds \\
= \;& \frac{(1-\eta)(2D+1)}{2(D+1)^{3/2}} \left\{ -\ln(1-\eta) + 2\ln(\sqrt{D\eta+1} + \sqrt{D\eta+\eta}) \right\} + \frac{\sqrt{D\eta^2+\eta}}{D+1}.
\end{aligned}$$

Proof. Set

$$t = \frac{1-s}{\eta - s}, \; s \in (0,\eta), \; \eta \in (0,1).$$

Then $t \in (1/\eta, \infty)$ and

$$ds = (1-\eta)\frac{dt}{(t-1)^2}.$$

We obtain

$$\begin{aligned}
J(\eta, D) &= (1-\eta)\int_{1/\eta}^{\infty} \frac{1}{\sqrt{D+t}}\frac{t}{(t-1)^2}dt \\
&= (1-\eta)\left\{\int_{1/\eta}^{\infty} \frac{1}{\sqrt{D+t}(t-1)}dt + \int_{1/\eta}^{\infty} \frac{1}{\sqrt{D+t}(t-1)^2}dt\right\} \\
&:= (1-\eta)(J_1 + J_2).
\end{aligned}$$

To compute J_i, $i = 1, 2$, we first set $\sqrt{D+t} = u$ and then set $u = \sqrt{D+1}\sec\theta$. This gives (without the upper and lower bounds of the integrals)

$$\begin{aligned}
J_1 &= \frac{2}{\sqrt{D+1}}\int \csc\theta d\theta = -\frac{2}{\sqrt{D+1}}\ln\left(\frac{\sqrt{D+t}}{\sqrt{t-1}} + \frac{\sqrt{D+1}}{\sqrt{t-1}}\right) + C, \\
J_2 &= \frac{2}{(D+1)^{3/2}}\int \csc\theta\cot^2\theta d\theta \\
&= -\frac{\sqrt{D+t}}{(D+1)(t-1)} + \frac{1}{(D+1)^{3/2}}\ln\left(\frac{\sqrt{D+t}}{\sqrt{t-1}} + \frac{\sqrt{D+1}}{\sqrt{t-1}}\right) + C,
\end{aligned}$$

where C is a constant. Here we have used the following standard integrals

$$\int \csc\theta d\theta = -\ln(|\csc\theta + \cot\theta|) + C,$$

$$\int \csc\theta\cot^2\theta d\theta = \frac{1}{2}\{-\csc\theta\cot\theta + \ln(|\csc\theta + \cot\theta|)\} + C.$$

Then the property is derived. □

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