CONVERGENCE TO TRAVELING WAVES IN REACTION-DIFFUSION SYSTEMS WITH EQUAL DIFFUSIVITIES

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Abstract. In this paper, we first derive a theorem on the convergence of solutions to traveling waves in reaction-diffusion systems with equal diffusivities. Then we apply this theorem to some specific examples of predator-prey models which were studied recently in the literature. This gives a stability result for these traveling waves in the corresponding predator-prey system under certain perturbations of initial data.

1. Introduction

In this paper, we consider the following equal-diffusive reaction-diffusion system

$$(u_i)_t = (u_i)_{xx} + u_i g_i(u),\ x \in \mathbb{R},\ t > 0,\ i = 1, \ldots, m, \tag{1.1}$$

where $u := (u_1, \ldots, u_m)$, $g_i \in C^1((0,\infty)^m)$, $i = 1, \ldots, m$, and m is a positive integer. Suppose that (1.1) has two different constant equilibria $\{E_\mp\}$ such that E_- is unstable and E_+ is stable in the ODE sense. A traveling wave of (1.1) connecting E_- and E_+ is a solution u of (1.1) in the form

$$u_i(x,t) = \phi_i(z),\ z := x + st,\ i = 1, \ldots, m,$$

for some constant $s \in \mathbb{R}$ and function $\Phi := (\phi_1, \ldots, \phi_m)$ such that

$$\Phi(-\infty) = E_-,\ \Phi(\infty) = E_+. \tag{1.2}$$

Then $\{s, \Phi\}$ satisfies

$$\phi_i''(z) - s\phi_i'(z) + \phi_i g_i(\Phi(z)) = 0,\ z \in \mathbb{R},\ i = 1, \ldots, m. \tag{1.3}$$

The study of traveling waves in reaction-diffusion systems has attracted a lot of attentions in recent years, due to its importance in describing the spatial-temporal behavior of solutions. In particular, for the existence of traveling waves in predator-prey systems, we refer the reader

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to works [14, 19, 18, 20, 23, 13, 24, 37, 35, 36, 5, 25, 38, 17, 2, 3] done in past years. On the other hand, great attention is paid to the stability analysis of traveling waves in parabolic equations and systems, we refer the reader to, e.g., [9]-[12] and [28, 29, 33, 1, 15, 26, 27, 32, 21, 22, 31, 34]. Here the stability is equivalent to the convergence to the traveling wave for solutions of (1.1) with initial data under certain perturbations. The main purpose of this paper is to present a simple approach to tackle this stability problem.

Throughout this paper, we assume that system (1.1) has a bounded invariant set $\mathcal{I}$ in the sense that a solution u of (1.1) stays in $\mathcal{I}$ if its initial data $u(\cdot,0)=u_0$ lies in $\mathcal{I}$. Moreover, we assume that there is a positive constant s^* such that system (1.1) has a traveling wave solution $\{s,\Phi\}$ connecting E_- and E_+ with $\Phi:=(\phi_1,\ldots,\phi_m)$ positive if and only if $s\geq s^*$. Hereafter a vector-valued function is called positive if each component of this function is positive.

Let u be a positive solution of (1.1). Then, using the moving coordinate $z=x+st$, $\{u_i=u_i(z,t)\}$ satisfies

$$(u_i)_t=(u_i)_{zz}-s(u_i)_z+u_ig_i(u),\ z\in\mathbb{R},\ t>0,\ i=1,\ldots,m. \tag{1.4}$$

Note that traveling wave profile Φ is a stationary solution of (1.4). Suppose that there exists a set of positive constants $\{\sigma_i\}$ such that

$$I:=\sum_{i=1}^{m}\sigma_i\,(u_i-v_i)\,\{g_i(u)-g_i(v)\}\leq 0,\ \forall\,u,v\in\mathcal{I}. \tag{1.5}$$

We introduce the following distance function

$$\mathcal{K}[U]:=\sum_{i=1}^{m}\sigma_i\mathcal{K}_i[U_i],\quad \mathcal{K}_i[U_i]:=U_i-\phi_i-\phi_i\ln\frac{U_i}{\phi_i}, \tag{1.6}$$

for a positive function $U=(U_1,\ldots,U_m)$ defined in $\mathbb{R}$. Note that $\mathcal{K}[U](z)\geq 0$ for all $z\in\mathbb{R}$ and $\mathcal{K}[U](z)=0$ if and only if $U(z)=\Phi(z)$ for some $z\in\mathbb{R}$. Then we define the related entropy function of u by

$$\Psi(z,t):=\mathcal{K}[u(\cdot,t)](z),\ z\in\mathbb{R},\ t>0. \tag{1.7}$$

Now, we are ready to state the following convergence theorem for system (1.4).

Theorem 1.1. *Assume that system* (1.1) *has a bounded invariant set* $\mathcal{I}$*. Let* R *be a positive constant,* $\{s,\Phi\}$ *be a positive traveling wave solution of* (1.1) *for some* $s\geq 2\sqrt{R}$ *and let* u *be a solution of* (1.4) *with initial data* u_0*. Assume that condition* (1.5) *is enforced. Suppose that the related entropy function* Ψ *of* u *satisfies*

$$\Psi_t-\Psi_{zz}+s\Psi_z\leq R\Psi,\ z\in\mathbb{R},\ t>0. \tag{1.8}$$

Let $\mu := \{s - \sqrt{s^2 - 4R}\}/2$. *If* $e^{-\mu z}\mathcal{K}[u_0] \in L^1(\mathbb{R})$, *then* $u(z,t) \to \Phi(z)$ *as* $t \to \infty$ *locally uniformly for* z *in* $\mathbb{R}$.

Theorem 1.1 provides a stability of traveling wave in a certain sense. The key of this stability is to derive the inequality (1.8). We shall show in §2 that, in addition to (1.5), it requires that $g_i(\Phi(z)) \le R$ for all $z \in \mathbb{R}$ for all $i = 1, \dots, m$.

The rest of this paper is organized as follows. In §2, we provide a proof of Theorem 1.1. Then applications of Theorem 1.1 to three specific predator-prey models studied in [4, 5, 17] are given in §3. Here the constant R can be chosen so that $s^* = 2\sqrt{R}$ and so traveling waves with all admissible wave speeds are stable (in the above sense). However, in some cases, we were unable to obtain such a sharp result and we present other two examples (cf. [2, 3]) in §4 for which stability of traveling waves can be derived only for wave speeds large enough. Therefore, there remains an open problem for the stability of traveling waves with smaller admissible wave speeds.

We add some remarks at the end of this section as follows. In most of the above-mentioned works, one of the key ingredients is to analyze the spectrum of the associated linearized operator of the studied system. In particular, the method developed by Evans [9]-[12] has been widely used in the literature. However, Evans' method requires a heavy and complicated spectral analysis of the related operator (see, e.g., [1, 21, 22]). In some cases, a singular perturbation method for a system involving a small parameter is adopted ([15]). On the other hand, in some works on stability, some conditions on the spectrum are assumed in order to derive the desired stability (e.g., [28, 29]). However, those spectrum conditions are not easy to be verified for some specific systems.

Note that, in applying the Evans function method to obtain a stability of traveling waves in predator-prey systems studied in this paper, certain steps are needed. The first step is to analyze the essential spectrum of the associated asymptotic systems. This is done by calculating the Fredholm borders which is not too hard to be carried out, even for systems with non-equal diffusivies. From this information on the essential spectrum, we see immediately that the traveling waves can only be stable in a suitable weighted function space. Secondly, in a suitably chosen weighted space, we need to analyze the point spectrum, including the verification that 0 is a simple eigenvalue. This is extremely difficult to achieve and we were unable to carry it out by this method.

We are not sure whether the conclusion of this paper can be obtained by the classical methods. However, in comparing with the existing literature, our method does not need to derive the spectrum of the associated linearized operator. Moreover, it can be applied to reaction-diffusion equations (such as the classical Fisher-KPP equation) and systems of any number of components, as long as we can find a suitable set of positive constants $\{\sigma_i\}$ such that (1.5) holds. One should note that the existence of traveling waves in [2, 3, 4, 5, 17] does not require the equal diffusivities condition. However, our method of deriving the convergence to traveling waves requires the equal diffusivities condition. The non-equal diffusivities case is delicate and we leave it as an open question.

2. Proof of Theorem 1.1 and the derivation of (1.8)

In this section, we first prove Theorem 1.1 as follows.

Proof of Theorem 1.1. Let $s \geq 2\sqrt{R}$ and let Ψ be the related entropy function of a solution u of (1.4) with initial data u_0. Consider the following linear heat equations

$$\begin{cases} W_t = W_{zz} - sW_z + RW, & z \in \mathbb{R},\, t > 0, \\ W(z,0) = \Psi(z,0), & z \in \mathbb{R}. \end{cases}$$

Define $V(z,t) := e^{-\mu z} W(z,t)$. Then $V(z,t)$ satisfies

$$V_t = V_{zz} + (2\mu - s)V_z,$$

using the identity $\mu^2 - s\mu + R = 0$. The assumption $e^{-\mu z}\mathcal{K}[u_0] \in L^1(\mathbb{R})$ implies $V(\cdot,0) \in L^1(\mathbb{R})$. Thus

$$0 \leq V(z,t) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \exp\Big\{ - \frac{(z-(2\mu-s)t-y)^2}{4t} \Big\} V(y,0)\, dy \leq \frac{\|V(\cdot,0)\|_{L^1(\mathbb{R})}}{\sqrt{4\pi t}} \to 0$$

as $t \to \infty$. Hence $W(z,t) = e^{\mu z}V(z,t)$ converges to zero locally uniformly for $z \in \mathbb{R}$ as $t \to +\infty$. By the comparison principle, $\Psi \leq W$. Therefore, the theorem is proved. $\square$

Next, we provide a general calculation to derive (1.8).

By a simple calculation, we have

$$\Psi_t = \sum_{i=1}^{m} \sigma_i (u_i)_t \left(1 - \frac{\phi_i}{u_i}\right), \quad \Psi_z = \sum_{i=1}^{m} \left\{ \sigma_i (u_i)_z \left(1 - \frac{\phi_i}{u_i}\right) - \sigma_i \phi_i' \ln \frac{u_i}{\phi_i} \right\},$$

and

$$\Psi_{zz} = \sum_{i=1}^{m} \left\{ \sigma_i (u_i)_{zz} \left(1 - \frac{\phi_i}{u_i}\right) - \sigma_i \phi_i'' \ln \frac{u_i}{\phi_i} - \sigma_i \left[(u_i)_z \left(\frac{\phi_i}{u_i}\right)_z + \frac{\phi_i \phi_i'}{u_i} \left(\frac{u_i}{\phi_i}\right)_z \right] \right\}$$

Since $\sigma_i > 0$ for each i and

$$(u_i)_z \left(\frac{\phi_i}{u_i}\right)_z + \frac{\phi_i \phi_i'}{u_i} \left(\frac{u_i}{\phi_i}\right)_z = -\phi_i \left[\frac{(u_i)_z}{u_i} - \frac{\phi_i'}{\phi_i} \right]^2 \leq 0, \ \forall\, i,$$

it follows from (1.3) and (1.4) that

$$\begin{aligned}
\Psi_t - \Psi_{zz} + s\Psi_z &\leq \sum_{i=1}^{m} \left\{ \sigma_i \left(u_i - \phi_i\right) g_i(u) - \sigma_i \phi_i g_i(\Phi) \ln \frac{u_i}{\phi_i} \right\} \\
&= \sum_{i=1}^{m} \sigma_i \left(u_i - \phi_i\right) g_i(u) + \sum_{i=1}^{m} \sigma_i g_i(\Phi) \left\{ \mathcal{K}_i[u_i] - (u_i - \phi_i) \right\} \\
&= \sum_{i=1}^{m} \sigma_i \left(u_i - \phi_i\right) \left\{ g_i(u) - g_i(\Phi) \right\} + \sum_{i=1}^{m} \sigma_i g_i(\Phi) \mathcal{K}_i \left[u_i\right],
\end{aligned}$$

using

$$\phi_i \ln \frac{u_i}{\phi_i} = \mathcal{K}_i[u_i] - (u_i - \phi_i).$$

Therefore, if (1.5) holds for a suitable set of positive constants $\{\sigma_i\}$ and if

$$\max_{1 \leq i \leq m} \{\|g_i(\Phi)\|_{L^\infty(\mathbb{R})}\} \leq R,$$

then we can get the desired inequality (1.8) so that Theorem 1.1 can be applied.

3. Some examples of predator-prey models

In this section, we apply Theorem 1.1 to derive the asymptotic stability of traveling waves in some specific predator-prey models.

3.1. A singular predator-prey model.

First, we consider the following singular predator-prey system

$$\begin{cases} u_t = u_{xx} + au(1-u) - v, & x \in \mathbb{R},\ t > 0, \\ v_t = v_{xx} + bv\left(1 - v/u\right), & x \in \mathbb{R},\ t > 0, \end{cases} \tag{3.1}$$

where $a > 0$ and $b > 0$. System (3.1) arises from a reduced model of the control of introduced rabbits to protect native birds from introduced cat predation in an island [6, 7, 8, 4]. We call this system singular, since the prey density u may reach zero in some finite time so that the reaction term in the predator v equation becomes singular.

If $a > 1$, then (3.1) has a positive constant state $E^* = (u^*, v^*)$, where $u^* = v^* = 1 - 1/a$. We are interested in whether an alien predator can invade the existing prey so that both predator and prey can live together in the habitat. This is equivalent to the existence of traveling wave $\{s, \phi_1, \phi_2\}$ connecting the predator-free state $(1, 0)$ and the co-existence state (u^*, v^*) in the following sense

$$\lim_{z \to -\infty} (\phi_1, \phi_2) = (1, 0), \quad \lim_{z \to \infty} (\phi_1, \phi_2) = (u^*, v^*). \tag{3.2}$$

Note that the wave profiles $\{\phi_1, \phi_2\}$ satisfy

$$\begin{cases} \phi_1''(z) - s\phi_1'(z) + a\phi_1(z)\left[1 - \phi_1(z)\right] - \phi_2(z) = 0,\ z \in \mathbb{R}, \\ \phi_2''(z) - s\phi_2'(z) + b\phi_2(z)\left[1 - \frac{\phi_2(z)}{\phi_1(z)}\right] = 0,\ z \in \mathbb{R}. \end{cases} \tag{3.3}$$

Suppose that $a \geq 4$. Recall from [4] that for each $s \geq s^* := 2\sqrt{b}$ there exists a positive solution (ϕ_1, ϕ_2) of (3.3) such that condition (3.2) holds.

For $a \geq 4$, let

$$\underline{a} := \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{a}},\ \overline{a} := \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{a}}.$$

Then it is easy to check that $\mathcal{I}_\alpha := [\alpha, 1] \times [0, 1]$ is an invariant domain for the corresponding ordinary differential system of (3.1) for any $\alpha \in [\underline{a}, \overline{a}]$. Hence, by the standard invariant domain theory [30], any solution (u, v) of (3.1) with $(u, v)(\cdot, 0) = (u_0, v_0)$ exists globally in time such that $(u, v)(x, t) \in \mathcal{I}_\alpha$ for all $(x, t) \in \mathbb{R} \times (0, +\infty)$, if $(u_0, v_0) \in \mathcal{I}_\alpha$ in $\mathbb{R}$.

Note that traveling waves (ϕ_1, ϕ_2) constructed in [4] satisfy $\phi_1 \geq 1/2$. We then define the following admissible set

$$\mathcal{A} := \{(a, \alpha) \in [4, \infty) \times [\underline{a}, \overline{a}] \mid a\alpha \geq 4\}.$$

It is easy to check that $\{(a, 1/2) \mid a \geq 8\} \subset \mathcal{A}$. Note also that with the moving coordinate z, (3.1) is reduced to

$$\begin{cases} u_t = u_{zz} - su_z + au(1-u) - v,\ z \in \mathbb{R},\ t > 0, \\ v_t = v_{zz} - sv_z + bv\left(1 - \frac{v}{u}\right),\ z \in \mathbb{R},\ t > 0, \end{cases} \tag{3.4}$$

for $u = u(z, t)$ and $v = v(z, t)$.

For a (positive) solution (u, v) of (3.4), we define its related entropy function by

$$\begin{cases} \Psi(z, t) := \mathcal{K}[u(\cdot, t), v(\cdot, t)](z) = b\mathcal{K}_1[u(\cdot, t)](z) + \mathcal{K}_2[v(\cdot, t)](z), \\ \mathcal{K}_1[u] := u - \phi_1 - \phi_1 \ln(u/\phi_1),\ \mathcal{K}_2[v] := v - \phi_2 - \phi_2 \ln(v/\phi_2). \end{cases} \tag{3.5}$$

Then we have the following lemma.

Lemma 3.1. *Given $s \geq s^* = 2\sqrt{b}$. Assume $(a, \alpha) \in \mathcal{A}$ and $b \geq a/2$. Then for a solution (u, v) of (3.4) with initial data $(u_0, v_0) \in \mathcal{I}_\alpha$, its related entropy function Ψ satisfies*

$$\Psi_t - \Psi_{zz} + s\Psi_z \leq b\Psi, \quad z \in \mathbb{R},\ t \in (0, \infty). \tag{3.6}$$

Proof. Note that $(u(x, t), v(x, t)) \in \mathcal{I}_\alpha$ for all $x \in \mathbb{R}$ and $t > 0$, since $(u_0, v_0) \in \mathcal{I}_\alpha$.

For this problem, we have

$$g_1(u, v) = a(1-u) - v/u, \quad g_2 = b(1 - v/u).$$

Then the quantity I in (1.5) is computed as

$$I = -b\left\{a(u-\phi_1)^2 + [(u-\phi_1)+(v-\phi_2)]\left(\frac{v}{u}-\frac{\phi_2}{\phi_1}\right)\right\}.$$

Using

$$\frac{v}{u}-\frac{\phi_2}{\phi_1} = \frac{1}{u}(v-\phi_2) - \frac{\phi_2}{u\phi_1}(u-\phi_1),$$

we obtain

$$I = -b\left[\left(a-\frac{\phi_2}{u\phi_1}\right)(u-\phi_1)^2 + \frac{\phi_1-\phi_2}{u\phi_1}(u-\phi_1)(v-\phi_2) + \frac{1}{u}(v-\phi_2)^2\right].$$

Note that the determinant of the above bilinear form

$$\left(\frac{\phi_1-\phi_2}{u\phi_1}\right)^2 - \frac{4}{u}\left(a-\frac{\phi_2}{u\phi_1}\right) \leq 0$$

if and only if

$$a \geq \frac{(\phi_1-\phi_2)^2}{4u\phi_1^2} + \frac{\phi_2}{u\phi_1} = \frac{(\phi_1+\phi_2)^2}{4u\phi_1^2} := Q.$$

Since the coefficient of $(v-\phi_2)^2$, $1/u$, is positive and $Q \leq 4/\alpha$, due to $\phi_2 \in [0,1]$, $u \in [\alpha, 1]$ and $\phi_1 \in [1/2, 1]$, it follows from the assumption $a\alpha \geq 4$ that $I \leq 0$.

On the other hand, since $\phi_1 \geq 1/2$, $\phi_2 \geq 0$ and $b \geq a/2$, we have

$$g_1(\Phi) = a(1-\phi_1) - \frac{\phi_2}{\phi_1} \leq a/2 \leq b, \quad g_2(\Phi) = b(1-\phi_2/\phi_1) \leq b.$$

Thus the lemma is proved. □

Applying Theorem 1.1, we have the following stability theorem for (3.1).

Theorem 3.2. *Suppose that $s \geq s^* = 2\sqrt{b}$. Assume $(a,\alpha) \in \mathcal{A}$ and $b \geq a/2$. Let (u,v) be a solution of system* (3.4) *with initial data $(u_0,v_0) \in \mathcal{I}_\alpha$ such that $e^{-\mu z}\mathcal{K}[(u_0,v_0)] \in L^1(\mathbb{R})$, where $\mu := (s-\sqrt{s^2-4b})/2$. Then $(u,v)(z,t)$ converges to $(\phi_1,\phi_2)(z)$ as $t \to +\infty$ locally uniformly for z in $\mathbb{R}$, where $\{s,(\phi_1,\phi_2)\}$ is a traveling wave obtained in* [4].

3.2. **Regular predator-prey system.**

The second example is the following (regular) predator-prey system

$$\begin{cases} u_t = u_{xx} + au(1-u-kv), \ x \in \mathbb{R},\ t>0, \\ v_t = v_{xx} + bv(1-v/u), \ x \in \mathbb{R},\ t>0, \end{cases} \tag{3.7}$$

with nonnegative nontrivial initial data (u_0,v_0) at $t=0$, where a,b,k are positive constants. Note that u never vanishes for positive times and so the ratio term v/u never becomes singular.

Recall from [5] that, given a constant $k \in (0,1)$, for each $s \geq s^* := 2\sqrt{b}$ there is a positive solution (ϕ_1, ϕ_2) of

$$\begin{cases} \phi_1''(z) - s\phi_1'(z) + a\phi_1(z)[1 - \phi_1(z) - k\phi_2(z)] = 0, \ z \in \mathbb{R}, \\ \phi_2''(z) - s\phi_2'(z) + b\phi_2(z)\left[1 - \frac{\phi_2(z)}{\phi_1(z)}\right] = 0, \ z \in \mathbb{R}, \end{cases} \tag{3.8}$$

satisfying

$$\lim_{z \to -\infty} (\phi_1, \phi_2) = (1, 0), \quad \lim_{z \to \infty} (\phi_1, \phi_2) = \left(\frac{1}{1+k}, \frac{1}{1+k} \right).$$

As for system (3.1), it is easy to check that the set $\mathcal{I}_k := [1-k, 1] \times [0,1]$ is an invariant set for system (3.7). Note that the traveling waves (ϕ_1, ϕ_2) constructed in [5] satisfy $\phi_1 \geq 1-k$.

Using the moving coordinate z, (3.7) is re-written as

$$\begin{cases} u_t = u_{zz} - su_z + au(1 - u - kv), \ z \in \mathbb{R}, \, t > 0, \\ v_t = v_{zz} - sv_z + bv(1 - v/u), \ z \in \mathbb{R}, \, t > 0. \end{cases} \tag{3.9}$$

For a solution (u, v) of (3.9), we define its related entropy function Ψ and distance function $\mathcal{K}$ as that for (3.4). Then we have the following lemma.

Lemma 3.3. *Given $s \geq s^* = 2\sqrt{b}$ and $a > 1/16$. Assume $k \in (0,1)$ such that*

$$\frac{1}{a(1-k)^2} - \frac{2}{\sqrt{a}} \leq k \leq \frac{2}{\sqrt{a}}, \tag{3.10}$$

and $b \geq ak$. Then its related entropy function Ψ satisfies (3.6) *for a solution (u, v) of* (3.9) *with initial data $(u_0, v_0) \in \mathcal{I}_k$.*

Proof. The proof is similar to that of Lemma 3.1. Here we have

$$g_1(u, v) = a(1 - u - kv), \quad g_2 = b(1 - v/u).$$

First, we compute

$$I = -b\Big[a(u - \phi_1)^2 + \left(ak - \frac{\phi_2}{u\phi_1}\right)(u - \phi_1)(v - \phi_2) + \frac{1}{u}(v - \phi_2)^2\Big].$$

Note that $I \leq 0$ if and only if

$$\left(ak - \frac{\phi_2}{u\phi_1}\right)^2 \leq \frac{4a}{u}$$

which is equivalent to

$$K_1 := \frac{\phi_2}{au\phi_1} - \frac{2}{\sqrt{a}\sqrt{u}} \leq k \leq \frac{\phi_2}{au\phi_1} + \frac{2}{\sqrt{a}\sqrt{u}} := K_2. \tag{3.11}$$

It is easy to check that

$$\max K_1 = \frac{1}{a(1-k)^2} - \frac{2}{\sqrt{a}}, \quad \min K_2 = \frac{2}{\sqrt{a}},$$

by using $u, \phi_1 \in [1-k, 1]$ and $0 \le \phi_2 \le 1$. We conclude that $I \le 0$, if condition (3.10) is enforced.

On the other hand, we compute

$$g_1(\Phi) = a(1-\phi_1 - k\phi_2) \le ak \le b, \quad g_2(\Phi) = b(1-\phi_2/\phi_1) \le b,$$

using $\phi_1 \ge 1-k$, $\phi_2 \ge 0$ and $b \ge ak$. Thus the lemma is proved. □

Then we have the following stability theorem for system (3.7).

Theorem 3.4. *Given $s \ge s^* = 2\sqrt{b}$ and $a > 1/16$. Assume* (3.10) *holds for some $k \in (0,1)$ and $b \ge ak$. Let (u,v) be a solution of system* (3.9) *with initial data $(u_0, v_0) \in \mathcal{I}_k$ such that $e^{-\mu z}\mathcal{K}[(u_0,v_0)] \in L^1(\mathbb{R})$, where $\mu := (s - \sqrt{s^2-4b})/2$. Then $(u,v)(z,t)$ converges to $(\phi_1,\phi_2)(z)$ as $t \to +\infty$ locally uniformly for z in $\mathbb{R}$, where $\{s, (\phi_1,\phi_2)\}$ is a traveling wave obtained in* [5].

Remark 3.5. We show here that condition (3.10) is not void. Note that the set

$$B := \left\{ k \in (0,1) \,\Big|\, \frac{1}{a(1-k)^2} - \frac{2}{\sqrt{a}} < \frac{2}{\sqrt{a}} \right\} \ne \emptyset,$$

if $a > 1/16$. Furthermore, the condition

$$\frac{1}{a(1-k)^2} - \frac{2}{\sqrt{a}} \le k, \tag{3.12}$$

is equivalent to $g(k) \ge 1$, where

$$g(k) := (ak + 2\sqrt{a})(1-k)^2.$$

Note that $g(0) = 2\sqrt{a}$ and $g(1) = 0$. We compute

$$g'(k) = (1-k)[a(1-k) - 2(ak+2\sqrt{a})] = a(1-k)(1 - 4/\sqrt{a} - 3k) = 0$$

if and only if $k = 1$ or $k = k_0 := (1-4/\sqrt{a})/3 \in (0,1)$. Since

$$g''(k) = a(-4 + 4/\sqrt{a} + 6k),$$

we obtain from $g''(1) > 0$ and $g''(k_0) < 0$ that

$$\max_{k\in(0,1)} g(k) = g(k_0) = (ak_0 + 2\sqrt{a})(1-k_0)^2 = \frac{4}{27\sqrt{a}}(2+\sqrt{a})^3 := h(a).$$

It is easy to check that $h(a) \ge h(1) = 4$ for all $a > 0$. We conclude that condition (3.12) holds for some $k \in (0,1)$, if we assume $a > 1/16$.

3.3. **Two-predator-one-prey system.**

Finally, we consider the following two-predator-one-prey system

$$\begin{cases} u_t = u_{xx} + r_1 u(-1 - u - hv + aw), \ x \in \mathbb{R}, \ t > 0, \\ v_t = v_{xx} + r_2 v(-1 - ku - v + aw), \ x \in \mathbb{R}, \ t > 0, \\ w_t = w_{xx} + r_3 w(1 - bu - bv - w), \ x \in \mathbb{R}, \ t > 0, \end{cases} \tag{3.13}$$

where

$$0 < h, k < 1, \ a > 1, \ b < \min\left\{ \frac{1}{2(a-1)}, \frac{1-h}{2a}, \frac{1-k}{2a} \right\}. \tag{3.14}$$

Note that the set

$$\mathcal{I}_0 := \{0 \le u, v \le a - 1, 0 \le w \le 1\}$$

is an invariant set of system (3.13).

Recall from [17] that for each $s \ge s^* := \max\{2\sqrt{r_1(a-1)}, 2\sqrt{r_2(a-1)}\}$ system (3.13) admits a traveling wave solution

$$(u, v, w)(x, t) = (\phi_1, \phi_2, \phi_3)(x + st)$$

for some functions (wave profiles) (ϕ_1, ϕ_2, ϕ_3) such that

$$(\phi_1, \phi_2, \phi_3)(-\infty) = (0, 0, 1), \ (\phi_1, \phi_2, \phi_3)(+\infty) = (u_*, v_*, w_*),$$

where

$$w_* := \frac{(1 - hk) + b(2 - h - k)}{(1 - hk) + ab(2 - h - k)}, \ v_* := \frac{1 - h}{1 - hk}(aw^* - 1), \ u_* := \frac{1 - k}{1 - hk}(aw^* - 1).$$

Without loss of generality, we may assume that $r_1 \ge r_2$. Then $s^* = 2\sqrt{r_1(a-1)}$.

Using the moving coordinate $z := x + st$, system (3.13) is equivalent to

$$\begin{cases} u_t = u_{zz} - su_z + r_1 u(-1 - u - hv + aw), \ z \in \mathbb{R}, \ t > 0, \\ v_t = v_{zz} - sv_z + r_2 v(-1 - ku - v + aw), \ z \in \mathbb{R}, \ t > 0, \\ w_t = w_{zz} - sw_z + r_3 w(1 - bu - bv - w), \ z \in \mathbb{R}, \ t > 0. \end{cases} \tag{3.15}$$

For a solution (u, v, w) of (3.15), we set

$$\begin{cases} \Psi = \sigma_1 \Psi_1 + \sigma_2 \Psi_2 + \sigma_3 \Psi_3, \ \Psi_i(z, t) := \mathcal{K}_i[u(\cdot, t), v(\cdot, t), w(\cdot, t)](z), \\ \mathcal{K}_1 := (u - \phi_1) - \phi_1 \ln \frac{u}{\phi_1}, \ \mathcal{K}_2 := (v - \phi_2) - \phi_2 \ln \frac{v}{\phi_2}, \ \mathcal{K}_3 := (w - \phi_3) - \phi_3 \ln \frac{w}{\phi_3}, \end{cases}$$

where $\sigma_1 := 1$, $\sigma_2 := r_1/r_2$ and $\sigma_3 := r_1 a/(r_3 b)$. Then we have

Lemma 3.6. *Given* $s \geq s^* = 2\sqrt{r_1(a-1)}$. *Suppose that, in addition to* (3.14),

$$r_2 \leq r_1,\ r_3 \leq r_1(a-1). \tag{3.16}$$

Then the function Ψ *satisfies*

$$\Psi_t - \Psi_{zz} + s\Psi_z \leq r_1(a-1)\Psi,\ z \in \mathbb{R},\ t > 0, \tag{3.17}$$

if the initial data $(u, v, w)(z, 0)$ *of a solution* (u, v, w) *of* (3.15) *lies in* $\mathcal{I}_0$.

Proof. Note that $(u, v, w)(z, t) \in \mathcal{I}_0$ for all $z \in \mathbb{R}$ and $t > 0$. Here we set

$$\begin{cases} g_1(u, v, w) = r_1(-1 - u - hv + aw),\ g_2(u, v, w) = r_2(-1 - ku - v + aw), \\ g_3(u, v, w) = r_3(1 - bu - bv - w). \end{cases}$$

Then the quantity I in (1.5) is computed as

$$I = -[r_1(u-\phi_1)^2 + \sigma_2 r_2(v-\phi_2)^2 + \sigma_3 r_3(w-\phi_3)^2 + (r_1 h + \sigma_2 r_2 k)(u-\phi_1)(v-\phi_2)].$$

It is easy to check that $I \leq 0$, since $0 < h, k < 1$. Moreover, using $0 \leq \phi_i \leq 1$, $i = 1, 2, 3$, and (3.16), we obtain

$$g_1(\Phi) \leq r_1(a-1),\ g_2(\Phi) \leq r_2(a-1) \leq r_1(a-1),\ g_3(\Phi) \leq r_3 \leq r_1(a-1).$$

Hence the lemma is proved. □

From Lemma 3.6 and applying Theorem 1.1, we have the following stability theorem for system (3.13).

Theorem 3.7. *Suppose that* $s \geq s^* = 2\sqrt{r_1(a-1)}$. *Let* (3.14) *and* (3.16) *be enforced. Let* (u, v, w) *be a solution of system* (3.15) *with initial data* $(u_0, v_0, w_0) \in \mathcal{I}_0$ *such that* $e^{-\mu z}\mathcal{K}[(u_0, v_0, w_0)] \in L^1(\mathbb{R})$, *where* $\mu := [s - \sqrt{s^2 - 4r_1(a-1)}]/2$. *Then* $(u, v, w)(z, t)$ *converges to* $(\phi_1, \phi_2, \phi_3)(z)$ *as* $t \to +\infty$ *locally uniformly for* z *in* $\mathbb{R}$, *where* $\{s, (\phi_1, \phi_2, \phi_3)\}$ *is a traveling wave obtained in* [17].

4. Two more examples of predator-prey systems

In this section, we shall provide two more examples of predator-prey systems studied in [2, 3]. In the following examples, under certain conditions there is a unique positive coexistence state $E_c = (u_c, v_c, w_c)$ in each system. For simplicity of presentation, we shall not mention the specific conditions and refer the reader to the references [2, 3]. As we shall see, by applying Theorem 1.1, stability can be assured only for those traveling waves with large enough wave speeds.

First, in [3], (1.1) for a system of two weak competing preys and one predator was studied in which

$$\begin{cases} g_1(u_1,u_2,u_3) = r_1(1-u_1-ku_2-bu_3), \\ g_2(u_1,u_2,u_3) = r_2(1-hu_1-u_2-bu_3), \\ g_3(u_1,u_2,u_3) = r_3(-1+au_1+au_2-u_3), \end{cases}$$

where

$$0<h,k<1,\ a>\frac{2}{2-h-k},\ 0<b<\min\left\{\frac{1-k}{2a-1},\frac{1-h}{2a-1},\frac{a(2-h-k)-2}{2a(2a-1)}\right\}.$$

It is clear that $\mathcal{I}=[0,1]\times[0,1]\times[0,2a-1]$ is an invariant set for this case. Moreover, with $\sigma_1 r_1=1$, $\sigma_2 r_2 = 1$ and $\sigma_3 r_3 = b/a$, we have

$$\begin{aligned} & -\sum_{i=1}^{3}\{\sigma_i(u_i-v_i)[g_i(u)-g_i(v)]\} \\ = \ & \sigma_1 r_1(u_1-v_1)^2+\sigma_2 r_2(u_2-v_2)^2+\sigma_3 r_3(u_3-v_3)^2+(\sigma_1 k r_1+\sigma_2 h r_2)(u_1-v_1)(u_2-v_2) \\ & +(\sigma_2 b r_2-\sigma_3 a r_3)(u_2-v_2)(u_3-v_3)+(\sigma_1 b r_1-\sigma_3 a r_3)(u_1-v_1)(u_3-v_3) \\ = \ & \{(u_1-v_1)^2+(h+k)(u_1-v_1)(u_2-v_2)+(u_2-v_2)^2\}+\frac{b}{a}(u_3-v_3)^2\geq 0 \end{aligned}$$

for any $u:=(u_1,u_2,u_3)$ and $v:=(v_1,v_2,v_3)$ in $\mathbb{R}^3$, since $h+k<2$.

Set $E_-=(u_p,v_p,0)$ and $E_+=E_c$, where

$$u_p:=\frac{1-k}{1-hk},\ v_p:=\frac{1-h}{1-hk}.$$

Let $\beta := a(u_p+v_p)-1>0$. The minimal wave speed for waves connecting $\{E_\mp\}$ is characterized by $s^*:=2\sqrt{r_3\beta}$ in [3]. However, since we have

$$g_1(\Phi)\leq r_1,\ g_2(\Phi)\leq r_2,\ g_3(\Phi)\leq r_3(2a-1),$$

we can only obtain the stability for waves with speed $s\geq 2\sqrt{R}$, where

$$R:=\max\{r_1,r_2,r_3(2a-1)\}\geq r_3(2a-1)>r_3\beta,$$

by applying Theorem 1.1.

Secondly, in [2], (1.1) for a system of a pair of strong-weak competing preys and one predator was studied in which

$$\begin{cases} g_1(u_1,u_2,u_3) = r_1(1-u_1-ku_2-b_1u_3), \\ g_2(u_1,u_2,u_3) = r_2(1-hu_1-u_2-b_2u_3), \\ g_3(u_1,u_2,u_3) = r_3(-1+au_1+au_2-u_3), \end{cases}$$

where $0<h<1<k$ and $a>1$. It is clear that $\mathcal{I}=[0,1]\times[0,1]\times[0,2a-1]$ is also an invariant set for this case.

Note that, with $\sigma_1 = 1/r_1$, $\sigma_2 = b_1/(b_2 r_2)$ and $\sigma_3 = b_1/(a r_3)$, we have

$$\begin{aligned}
& -\sum_{i=1}^{3}\{\sigma_i(u_i - v_i)[g_i(u) - g_i(v)]\} \\
= \;& \sigma_1 r_1 (u_1 - v_1)^2 + \sigma_2 r_2 (u_2 - v_2)^2 + \sigma_3 r_3 (u_3 - v_3)^2 + (\sigma_1 k r_1 + \sigma_2 h r_2)(u_1 - v_1)(u_2 - v_2) \\
& +(\sigma_2 b_2 r_2 - \sigma_3 a r_3)(u_2 - v_2)(u_3 - v_3) + (\sigma_1 b_1 r_1 - \sigma_3 a r_3)(u_1 - v_1)(u_3 - v_3) \\
= \;& \{(u_1 - v_1)^2 + (k + h b_1/b_2)(u_1 - v_1)(u_2 - v_2) + \frac{b_1}{b_2}(u_2 - v_2)^2\} + \frac{b_1}{a}(u_3 - v_3)^2 \geq 0
\end{aligned}$$

for any $u := (u_1, u_2, u_3)$ and $v := (v_1, v_2, v_3)$ in $\mathbb{R}^3$, provided that

$$\left(k + h\frac{b_1}{b_2}\right)^2 \leq 4\frac{b_1}{b_2}. \tag{4.1}$$

See also [2, Lemma 4.7].

Let $E^* := (u^*, 0, w^*)$ and $E_* = (0, v_*, w_*)$, where

$$u^* := \frac{1 + b_1}{1 + a b_1},\ w^* := \frac{a - 1}{1 + a b_1}; \quad v_* := \frac{1 + b_2}{1 + a b_2},\ w_* := \frac{a - 1}{1 + a b_2}.$$

Set

$$\beta^* = 1 - h u^* - b_2 w^*,\ \beta_* = 1 - k v_* - b_1 w_*.$$

Recall from [2] that both E^* and E_* are unstable (i.e., $\beta^* > 0$ and $\beta_* > 0$) and E_c is stable, if E_c exists and

$$\left(k + h\frac{b_1}{b_2}\right)^2 < 4\frac{b_1}{b_2}. \tag{4.2}$$

Case 1. Recall from [2] that there is a positive traveling wave with $E_- = E^*$ for any $s \geq 2\sqrt{r_2 \beta^*}$, if

$$r_2 \beta^* \geq \max\{r_1[k + b_1(2a - 1)], r_3\}. \tag{4.3}$$

Moreover, these waves satisfy $E_+ = E_c$, if E_c exists and condition (4.2) is enforced. However, since

$$g_1(\Phi) \leq r_1,\ g_2(\Phi) \leq r_2,\ g_3(\Phi) \leq r_3(2a - 1),$$

we can only imply the stability of those waves with speeds $s \geq 2\sqrt{R}$, where

$$R = \max\{r_1, r_2, r_3(2a - 1)\} \geq r_2 > r_2 \beta^*.$$

Case 2. Similar result for waves connecting E_* and E_c to Case 1 holds, if we replace (4.3) by

$$r_1 \beta_* \geq \max\{r_2[h + b_2(2a - 1)], r_3\} \tag{4.4}$$

and the minimal wave speed by $2\sqrt{r_1\beta_*}$. Then we have the stability of those waves with speeds $s \geq 2\sqrt{R}$, where

$$R = \max\{r_1, r_2, r_3(2a-1)\} \geq r_1 > r_1\beta_*.$$

Case 3. $E_- = E^*$ and $E_+ = E_*$. In this case, we have $\beta^* > 0$ and $\beta_* < 0$. Moreover, we assume, in addition to (4.3), that

$$a(1-h) > 1,\ b_2 < \frac{a(1-h)-1}{a(2a-1)}.$$

Then the minimal wave speed is given by $s^* = 2\sqrt{r_2\beta^*}$ (cf. [2]).

For (1.5), if we choose $\sigma_i = 1/r_i$ for $i = 1, 2, 3$, then

$$\begin{aligned} &-\sum_{i=1}^{3}\{\sigma_i(u_i - v_i)[g_i(u) - g_i(v)]\} \\ = \quad & X^2 + Y^2 + Z^2 + (h+k)XY + (b_1 - a)XZ + (b_2 - a)YZ := [X, Y, Z]\, B \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \end{aligned}$$

where

$$\begin{cases} X := u_1 - v_1,\ Y := u_2 - v_2,\ Z := u_3 - v_3, \\ B := \begin{bmatrix} 1 & (h+k)/2 & (b_1 - a)/2 \\ (h+k)/2 & 1 & (b_2 - a)/2 \\ (b_1 - a)/2 & (b_2 - a)/2 & 1 \end{bmatrix}. \end{cases}$$

Hence (1.5) can be ensured if

$$h + k + b_1 - a \leq 2,\ h + k + b_2 - a \leq 2,\ b_1 + b_2 - 2a \leq 2, \tag{4.5}$$

by Gershgorin's Theorem [16].

On the other hand, by taking

$$\sigma_1 = \frac{1}{r_1},\ \sigma_2 = \frac{b_1}{b_2 r_2},\ \sigma_3 = \frac{b_1}{a r_3},$$

we have

$$-\sum_{i=1}^{3}\{\sigma_i(u_i - v_i)[g_i(u) - g_i(v)]\} = X^2 + (hb_1/b_2 + k)XY + \frac{b_1}{b_2}Y^2 + \frac{b_1}{a}Z^2.$$

Hence (4.5) is changed to

$$hb_1 + kb_2 \leq 2b_2,\ hb_1 + kb_2 \leq 2b_1 \tag{4.6}$$

so that (1.5) is assured.

Recall from [2] that $\beta^* > 0$ if and only if

$$b_2 < \frac{a-h}{a-1} b_1 + \frac{1-h}{a-1}. \tag{4.7}$$

Moreover, $\beta_* < 0$ if and only if

$$b_2(a-k) < (a-1)b_1 + (k-1). \tag{4.8}$$

Note that (4.8) holds, if either $1 < a \le k$ or

$$a > k,\ b_2 < \frac{a-1}{a-k} b_1 + \frac{k-1}{a-k}. \tag{4.9}$$

From these conditions, (4.5) and (4.6) can be achieved under certain conditions on parameters $\{h, k, a, b_1, b_2\}$. We leave the detailed characterization of the parameters range to the interested reader. We conclude from Theorem 1.1 that the waves connecting $\{E^*, E_*\}$ are stable in the sense described in Theorem 1.1 for those speeds $s \ge 2\sqrt{R}$, where

$$R = \max\{r_1, r_2, r_3(2a-1)\} \ge r_2 > r_2\beta^*.$$

References

[1] J. Alexander, R. Gardner, C. Jones, *A topological invariant arising in the stability analysis of travelling waves*, J. Reine Angew. Math., 410 (1990), 167-212.

[2] Y.-S. Chen, T. Giletti, J.-S. Guo, *Persistence of preys in a diffusive three species predator-prey system with a pair of strong-weak competing preys*, J. Differential Equations, 281 (2021), 341-378.

[3] Y.-S. Chen and J.-S. Guo, *Traveling wave solutions for a three-species predator-prey model with two aborigine preys*, Japan J. Industrial and Applied Mathematics, 38 (2021), 455-471.

[4] Y.-S. Chen, J.-S. Guo, M. Shimojo, *Recent developments on a singular predator-prey model*, Discrete Contin. Dyn. Syst. Ser. B, 26 (2021), 1797–1809.

[5] Y.-Y. Chen, J.-S. Guo, C.-H. Yao, *Traveling wave solutions for a continuous and discrete diffusive predator-prey model*, J. Math. Anal. Appl., 445 (2017), 212–239.

[6] F. Courchamp, M. Langlais, G. Sugihara, *Controls of rabbits to protect birds from cat predation*, Biological Conservations, 89 (1999), 219–225.

[7] F. Courchamp, G. Sugihara, *Modelling the biological control of an alien predator to protect island species from extinction*, Ecological Applications, 9 (1999), 112–123.

[8] A. Ducrot, J.-S. Guo, *Quenching behavior for a singular predator-prey model*, Nonlinearity, 25 (2012), 2059–2073.

[9] J. W. Evans, *Nerve axon equations: I linear approximation*, Indiana Univ. Math. J., 21 (1972), 877-885.

[10] J. W. Evans, *Nerve axon equations: II stability at rest*, Indiana Univ. Math. J., 22 (1972), 75-90.

[11] J. W. Evans, *Nerve axon equations: III stability of the nerve impluse*, Indiana Univ. Math. J., 22 (1972), 577-593.

[12] J. W. Evans, *Nerve axon equations: VI the stable and the unstable impulse*, Indiana Univ. Math. J., 24 (1975), 1169-1190.

[13] S.-C. Fu, J.-C. Tsai, *Wave propagation in predator-prey systems*, Nonlinearity, 28 (2015), 4389-4423.

[14] R.A. Gardner, *Existence of travelling wave solutions of predator-prey systems via the connection index*, SIAM J. Appl. Math., 44 (1984), 56-79.

[15] R. Gardner, C. Jones, *Stability of travelling wave solutions of diffusive predator-prey systems*, Transactions Amer. Math. Soc., 327 (1991), 465-524.

[16] S. Gerschgorin, *Uber die Abgrenzung der Eigenwerte einer Matrix*, Izv. Akad. Nauk. USSR Otd. Fiz.-Mat. Nauk, 6 (1931), 749-754.

[17] J.-S. Guo, K.-I. Nakamura, T. Ogiwara and C.-C. Wu, *Traveling wave solutions for a predator-prey system with two predators and one prey*, Nonlinear Analysis: Real World Applications, 54 (2020), Art. 103111.

[18] C. Hsu, C. Yang, T. Yang, T. Yang, *Existence of traveling wave solutions for diffusive predator-prey type systems*, J. Differential Equations, 252 (2012), 3040-3075.

[19] W. Huang, *Traveling wave solutions for a class of predator-prey systems*, J. Dyn. Diff. Equat., 24 (2012), 633-644.

[20] Y.L. Huang, G. Lin, *Traveling wave solutions in a diffusive system with two preys and one predator*, J. Math. Anal. Appl., 418 (2014), 163-184.

[21] Y. Li, Y. Wu, *Stability of traveling front solutions with algebraic spatial decay for some autocatalytic chemical reaction systems*, SIAM J. Math. Anal., 44 (2012), 1474-1521.

[22] Y. Li, Y.Wu, *Existence and stability of travelling front solutions for general auto-catalytic chemical reaction systems*, Math. Model. Nat. Phenom., 8 (2013), 104-132.

[23] G. Lin, *Invasion traveling wave solutions of a predator-prey system*, Nonlinear Anal., 96 (2014), 47-58.

[24] J.-J. Lin, W. Wang, C. Zhao, T.-H. Yang, *Global dynamics and traveling wave solutions of two-predators-one-prey models*, Discrete Contin. Dynam. Syst. Ser. B, 20 (2015), 1135-1154.

[25] J.-J. Lin, T.-H. Yang, *Traveling wave solutions for a diffusive three-species intraguild predation model*, Int. J. Biomath., 11 (2018), Art. 1850022.

[26] R. Pego, M. Weinstein, *Eigenvalues and instabilities of solitary waves*, Phil. Trans. Royal Soc. London, 340 (1992), 47-94.

[27] B. Sandstede, Stability of Travelling Waves, Handbook of Dynamical Systems, Volume 2, Chapter 18 (2002), 983–1055.

[28] D.H. Sattinger, *On the stability of waves of nonlinear parabolic systems*, Ad. Math., 22 (1976), 312-355.

[29] D.H. Sattinger, *Weighted norms for the stability of travelling waves*, J. Differential Equations, 25 (1977), 179-201.

[30] H. Weinberger, *Invariant sets for weakly coupled parabolic and elliptic systems*, Rend. Mat., 8 (1975), 295-310.

[31] Y. Wu, N. Yan, *Stability of travelling waves for autocatalytic reaction systems with strong decay*, Discrete Contin. Dynam. Syst. Ser. B, 22 (2017), 1601-1633.

[32] Y. Wu, X. Zhao, *The existence and stability of travelling waves with transition layers for some singular cross-diffusion systems*, Phys. D, 200 (2005), 325-358.

[33] E. Yanagida, K. Maginu, *Stability of double-pulse solutions in nerve axon equations*, SIAM J. Appl.Math., 49 (1989),1158–1173.

[34] H. Zhang, H. Izuhara, Y. Wu, *Asymptotic stability of two types of traveling waves for some predator-prey models*, Discrete Contin. Dynam. Syst. Ser. B, 26 (2021), 2323-2342.

[35] T. Zhang, *Minimal wave speed for a class of non-cooperative reaction-diffusion systems of three equations*, J. Differential Equations, 262 (2017), 4724-4770.

[36] T. Zhang, Y. Jin, *Traveling waves for a reaction-diffusion-advection predator-prey model*, Nonlinear Analysis: Real World Applications, 36 (2017), 203-232.

[37] T. Zhang, W. Wang, K. Wang, *Minimal wave speed for a class of non-cooperative diffusion-reaction system*, J. Differential Equations, 260 (2016), 2763-2791.

[38] W. Zuo 1, J. Shi, *Traveling wave solutions of a diffusive ratio-dependent Holling-Tanner system with distributed delay*, Comm. Pure. Appl. Anal., 17 (2018), 1179–1200.

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