STABILITY OF MONOSTABLE TRAVELING WAVES IN DIFFUSIVE THREE-SPECIES COMPETITION SYSTEMS

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ABSTRACT. In this note, we derive the stability of various monostable traveling waves in two different classes of three-species competition systems. This includes cases of three weak competitors and two-weak-one-strong competitors.

1. Introduction

In this note, we consider the following three-species diffusive competition system

$$(u_i)_t = (u_i)_{xx} + u_i g_i(u_1, u_2, u_3),\ x \in \mathbb{R},\ t > 0,\ i = 1, 2, 3, \tag{1.1}$$

where

$$\begin{cases} g_1(u_1, u_2, u_3) := r_1(1 - u_1 - a_2 u_2 - a_3 u_3), \\ g_2(u_1, u_2, u_3) := r_2(1 - b_1 u_1 - u_2 - b_3 u_3), \\ g_3(u_1, u_2, u_3) := r_3(1 - c_1 u_1 - c_2 u_2 - u_3) \end{cases} \tag{1.2}$$

in which the positive parameters r_i, $i = 1, 2, 3$, denote the intrinsic growth rates, and a_j, b_k, c_l are the inter-specific competition coefficients of three species with densities u_i, $i = 1, 2, 3$. Here we assume all species have the same diffusivities 1 and the carrying capacity of each species is normalized to be 1 which can be done by taking a suitable scaling.

Suppose that (1.1) has two different constant equilibria $\{E_\mp\}$ such that E_- is unstable and E_+ is stable in the ODE sense. A monostable traveling wave of (1.1) connecting E_- and E_+ is a solution u of (1.1) such that

$$u_i(x, t) = \phi_i(z),\ z := x + st,\ i = 1, 2, 3,$$

for some *positive* constant s (the wave speed) and function $\Phi := (\phi_1, \phi_2, \phi_3)$ (the wave profile) satisfying

$$\Phi(-\infty) = E_-,\ \Phi(\infty) = E_+. \tag{1.3}$$

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Then it is easy to see that $\{s, \Phi\}$ satisfies

$$\phi_i''(z) - s\phi_i'(z) + \phi_i g_i(\Phi(z)) = 0,\ z \in \mathbb{R},\ i = 1, 2, 3. \tag{1.4}$$

For the existence of traveling waves in competition systems and non-cooperative systems, we refer the reader to, e.g., [2, 3, 4, 5]. In particular, in the case of three weak competitors such that

$$a_2 + a_3 < 1,\ b_1 + b_3 < 1,\ c_1 + c_2 < 1, \tag{1.5}$$

there exist a unique positive co-existence state $E_* := (u_*, v_*, w_*)$, with $u_*, v_*, w_* \in (0, 1)$, the semi-coexistence state $E_c := (0, v_c, w_c) = (0, (1 - b_3)/(1 - b_3c_2), (1 - c_2)/(1 - b_3c_2))$, and the state $E_3 := (0, 0, 1)$. It is easy to see that both E_3 and E_c are unstable, while E_* is stable in the ODE sense. Recall from [3] that (1.1) has a positive traveling wave connecting $E_- = E_3$ and $E_+ = E_*$ if and only if $s \geq 2\sqrt{r_1(1 - a_3)}$, provided

$$r_2(1 - b_3) = r_1(1 - a_3). \tag{1.6}$$

While, (1.1) has a positive traveling wave connecting $E_- = E_c$ and $E_+ = E_*$ if and only if $s \geq 2\sqrt{r_1\beta}$, $\beta := 1 - a_2v_c - a_3w_c$, provided

$$r_1\beta \geq \max\{r_2(b_1 + b_3c_2v_c), r_3[c_1 + c_2(1 - v_c)]\}. \tag{1.7}$$

On the other hand, for the case of two weak and one strong competitors, under the assumption

$$a_2 + a_3 < 1,\ b_3, c_2 < 1,\ b_1 \geq 1/a_2,\ c_1 \geq 1/a_3, \tag{1.8}$$

by [2, 3], (1.1) has a positive traveling wave connecting $E_- = E_3 = (0, 0, 1)$ and $E_+ = E_1 := (1, 0, 0)$ if and only if $s \geq 2\sqrt{r_1(1 - a_3)}$, provided

$$r_2(1 - b_3) = r_1(1 - a_3) \geq r_3(c_1 + c_2 - 1)_+. \tag{1.9}$$

While, (1.1) has a positive traveling wave connecting $E_- = E_c$ and $E_+ = E_1$ if and only if $s \geq 2\sqrt{r_1\beta}$, provided (1.7) is enforced.

The aim of this paper is to derive the stability of these traveling waves. For the study of stability of traveling waves in non-cooperative systems using the Evans function's approach, we refer the reader to, e.g., [7, 8], and the references cited in [6]. Recently, by a very fundamental method, a stability theorem was proved in [6] for general reaction-diffusion systems with equal diffusivities. In [6], an application of this general theorem to derive the stability of traveling waves in various predator-prey systems was given.

Let $\mathcal{I} := [0, 1]^3$. Suppose that there exists a set of positive constants $\{\sigma_1, \sigma_2, \sigma_3\}$ such that

$$\sum_{i=1}^{3} \sigma_i (u_i - v_i) \{g_i(u_1, u_2, u_3) - g_i(v_1, v_2, v_3)\} \leq 0,\ \forall\, (u_1, u_2, u_3), (v_1, v_2, v_3) \in \mathcal{I}. \tag{1.10}$$

Then a *distance* function

$$\mathcal{K}[U] := \sum_{i=1}^{3} \sigma_i \mathcal{K}_i[U_i], \quad \mathcal{K}_i[U_i] := U_i - \phi_i - \phi_i \ln \frac{U_i}{\phi_i}, \tag{1.11}$$

is defined for any *positive* function $U = (U_1, U_2, U_3) : \mathbb{R} \mapsto (0, \infty)^3$. Note that $\mathcal{K}[U](z) \geq 0$ for all $z \in \mathbb{R}$ and $\mathcal{K}[U](z) = 0$ if and only if $U(z) = \Phi(z)$ for some $z \in \mathbb{R}$. For a positive constant R, we let

$$\lambda = \lambda(s; R) := \frac{s - \sqrt{s^2 - 4R}}{2}, \; s \geq 2\sqrt{R}. \tag{1.12}$$

Also, using the moving coordinate $z = x + st$, (1.1) is re-written as

$$(u_i)_t = (u_i)_{zz} - s(u_i)_z + u_i g_i(u), \; z \in \mathbb{R}, \; t > 0, \; i = 1, 2, 3, \tag{1.13}$$

hereafter $u = u(z, t) := (u_1(z, t), u_2(z, t), u_3(z, t))$.

Then we have the following two stability theorems.

Theorem 1. *Assume, in addition to* (1.5), *that either* $a_2 b_3 c_1 = a_3 b_1 c_2$ *or*

$$a_2 + b_1 + a_3 + c_1 \leq 2, \; a_2 + b_1 + b_3 + c_2 \leq 2, \; a_3 + c_1 + b_3 + c_2 \leq 2.$$

Let Φ *be a positive traveling wave of* (1.1) *connecting* $E_- \in \{(0, 0, 1), (0, v_c, w_c)\}$ *and* $E_+ = E_*$ *with wave speed* $s \geq 2\sqrt{R}$, *where* $R := \max\{r_1, r_2, r_3\}$. *Then* $\{s, \Phi\}$ *is stable in the sense that* $u(z, t) \to \Phi(z)$ *as* $t \to \infty$ *locally uniformly for* $z \in \mathbb{R}$ *for any solution* u *of* (1.13) *with initial data* u_0 *at* $t = 0$ *satisfying* $e^{-\lambda z}\mathcal{K}[u_0] \in L^1(\mathbb{R})$, *where* $\lambda = \lambda(s; R)$ *is defined in* (1.12) *and the constants* $\{\sigma_1, \sigma_2, \sigma_3\}$ *in the definition of* $\mathcal{K}$ *in* (1.11) *are chosen so that* (1.10) *holds.*

Theorem 2. *Assume, in addition to* (1.8), *that* $a_2 b_1 = 1 = a_3 c_1$ *and*

$$2a_2 a_3 = a_2^2 b_3 + a_3^2 c_2.$$

Let Φ *be a positive traveling wave of* (1.1) *connecting* $E_- \in \{(0, 0, 1), (0, v_c, w_c)\}$ *and* $E_+ = E_1$ *with wave speed* $s \geq 2\sqrt{R}$, *where* $R := \max\{r_1, r_2, r_3\}$. *Then* $\{s, \Phi\}$ *is stable in the sense described in Theorem 1.*

In fact, Theorems 1 and 2 follow from an application of the general theorem [6, Theorem 1.1]. We make some remarks on the perturbation of initial data as follows. At the stable tail of traveling wave, i.e., $z = \infty$, the perturbation is allowed to be arbitrarily large due to condition $e^{-\lambda z}\mathcal{K}[u_0] \in L^1(\mathbb{R})$. Note that $\lambda > 0$. However, at the unstable tail ($z = -\infty$), the perturbation can only be made with decay rate faster than $e^{\lambda z}$. This is a typical phenomenon in the stability of monostable waves in many reaction-diffusion systems, including their discrete analogues. Note that the exponent λ is a function of the wave speed s.

In applying [6, Theorem 1.1], we first note that the constant R is defined by

$$\max_{1 \leq i \leq 3} \sup_{z \in \mathbb{R}} \{g_i(\Phi(z))\} \leq R.$$

Hence $R = \max\{r_1, r_2, r_3\}$ by using $\phi_i \geq 0$ for all i. Therefore, to prove Theorems 1 and 2, all we need to do is to find some conditions on the parameters in (1.2) along with a suitable set of constants $\{\sigma_1, \sigma_2, \sigma_3\}$ such that (1.10) holds. The verification of (1.10) is given in the next section. The main contribution of this note is providing some conditions on the parameters in (1.2) to ensure the stability of traveling waves in competition systems. The importance of this work is that these stability results can be proved by a very simple method without any hard work on spectral analysis.

Note that $R > r_1(1-a_3)$ and $R > r_1\beta$. The stability of wave with speed $s < 2\sqrt{R}$ is still left open. In fact, the stability of wave with speed $s < 2\sqrt{R}$ can be proved if one can derive sharper lower bounds for wave profiles. We also remark that the conditions on the parameters in (1.2) imposed in Theorems 1 and 2 may not be optimal. The stability and instability of traveling waves for the other cases are still left open.

2. Verification of (1.10)

For notational convenience, we let

$$\begin{cases} u := (u_1, u_2, u_3),\ v := (v_1, v_2, v_3),\ I := -\sum_{i=1}^{3}\{\sigma_i(u_i - v_i)[g_i(u) - g_i(v)]\}, \\ X := u_1 - v_1,\ Y := u_2 - v_2,\ Z := u_3 - v_3. \end{cases}$$

Then I is computed as

$$\begin{aligned} I &= [X, Y, Z]B \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \sigma_1 r_1 X^2 + \sigma_2 r_2 Y^2 + \sigma_3 r_3 Z^2 \\ &\quad + (\sigma_1 r_1 a_2 + \sigma_2 r_2 b_1)XY + (\sigma_2 r_2 b_3 + \sigma_3 r_3 c_2)YZ + (\sigma_1 r_1 a_3 + \sigma_3 r_3 c_1)XZ, \end{aligned} \tag{2.1}$$

where

$$B := \begin{bmatrix} \sigma_1 r_1 & (\sigma_1 r_1 a_2 + \sigma_2 r_2 b_1)/2 & (\sigma_1 r_1 a_3 + \sigma_3 r_3 c_1)/2 \\ (\sigma_1 r_1 a_2 + \sigma_2 r_2 b_1)/2 & \sigma_2 r_2 & (\sigma_2 r_2 b_3 + \sigma_3 r_3 c_2)/2 \\ (\sigma_1 r_1 a_3 + \sigma_3 r_3 c_1)/2 & (\sigma_2 r_2 b_3 + \sigma_3 r_3 c_2)/2 & \sigma_3 r_3 \end{bmatrix}.$$

Our task is to find some conditions on parameters in (1.2) and a suitable set of $\{\sigma_i\}$ so that (1.10) holds, i.e., $I \geq 0$. This is also equivalent to find some conditions on parameters in (1.2) and positive constants $\{\sigma_i\}$ such that the matrix B is *symmetric positive semi-definite.*

2.1. **Three weak competitors.**

Case 1. $a_2 b_3 c_1 = a_3 b_1 c_2$, i.e.,

$$\frac{a_2 b_3}{b_1} = \frac{a_3 c_2}{c_1}. \tag{2.2}$$

With (1.5) and the choice

$$\sigma_1 = \frac{1}{r_1},\ \sigma_2 = \frac{a_2}{r_2 b_1},\ \sigma_3 = \frac{a_3}{r_3 c_1},$$

we obtain from (2.1) and (2.2) that $I \geq 0$, since by using (2.2) we can write

$$\begin{aligned} I &= (1 - a_2 - a_3)X^2 + \frac{a_2}{b_1}(1 - b_1 - b_3)Y^2 + \frac{a_3}{c_1}(1 - c_1 - c_2)Z^2 \\ &\quad + a_2(X+Y)^2 + a_3(X+Z)^2 + \frac{a_2 b_3}{b_1}(Y+Z)^2. \end{aligned}$$

Case 2. condition

$$a_2 + b_1 + a_3 + c_1 \leq 2,\ a_2 + b_1 + b_3 + c_2 \leq 2,\ a_3 + c_1 + b_3 + c_2 \leq 2 \tag{2.3}$$

is enforced. Setting $\sigma_i = 1/r_i$, $i = 1, 2, 3$, we obtain from (2.1) that

$$\begin{aligned} I &= X^2 + Y^2 + Z^2 + (a_2 + b_1)XY + (a_3 + c_1)XZ + (b_3 + c_2)YZ, \\ B &= \begin{bmatrix} 1 & (a_2+b_1)/2 & (a_3+c_1)/2 \\ (a_2+b_1)/2 & 1 & (b_3+c_2)/2 \\ (a_3+c_1)/2 & (b_3+c_2)/2 & 1 \end{bmatrix}. \end{aligned}$$

Hence $I \geq 0$ under condition (2.3), by Gerschgorin's Theorem [1]. Indeed, we can write I as

$$\begin{aligned} I &= \frac{a_2+b_1}{2}(X+Y)^2 + \frac{a_3+c_1}{2}(X+Z)^2 + \frac{b_3+c_2}{2}(Y+Z)^2 + \left[1 - \frac{a_2+b_1+a_3+c_1}{2}\right]X^2 \\ &\quad + \left[1 - \frac{a_2+b_1+b_3+c_2}{2}\right]Y^2 + \left[1 - \frac{a_3+c_1+b_3+c_2}{2}\right]Z^2 \end{aligned}$$

so that $I \geq 0$, if condition (2.3) holds. Thereby, Theorem 1 is proved. □

2.2. **Two weak and one strong competitors.**

This case is more delicate. Recall from the fundamental theory of linear algebra that B is symmetric positive semi-definite if and only if all principal minors of B are nonnegative.

First, we take $\sigma_1 = 1/r_1$. Then the symmetric positive semi-definiteness of B requires

$$r_2^2 b_1^2 \sigma_2^2 - 2r_2(2 - a_2 b_1)\sigma_2 + a_2^2 \leq 0. \tag{2.4}$$

Thus, in order to have some $\sigma_2 > 0$ such that (2.4) holds, we need

$$2 - a_2 b_1 > 0 \text{ and } (2 - a_2 b_1)^2 \geq a_2^2 b_1^2. \tag{2.5}$$

Since (2.5) holds if and only if $a_2 b_1 \leq 1$, we obtain from condition $a_2 b_1 \geq 1$ in (1.8) that $a_2 b_1 = 1$ and so, by (2.4),

$$\sigma_2 r_2 = \frac{a_2}{b_1} = a_2^2.$$

Next, since the symmetric positive semi-definiteness of B also requires

$$r_3^2c_1^2\sigma_3^2 - 2r_3(2-a_3c_1)\sigma_3 + a_3^2 \le 0,$$

a similar argument as before implies that we must have $a_3c_1 = 1$ and

$$\sigma_3 r_3 = \frac{a_3}{c_1} = a_3^2.$$

It follows that

$$B = \begin{bmatrix} 1 & a_2 & a_3 \\ a_2 & a_2^2 & (a_2^2b_3 + a_3^2c_2)/2 \\ a_3 & (a_2^2b_3 + a_3^2c_2)/2 & a_3^2 \end{bmatrix}.$$

Note that

$$\det(B) = -[a_2a_3 - (a_2^2b_3 + a_3^2c_2)/2]^2.$$

Hence all eigenvalues of B are nonnegative if and only if

$$2a_2a_3 = a_2^2b_3 + a_3^2c_2. \tag{2.6}$$

Recall from (1.8) that $b_3, c_2 \in (0,1)$. To see (2.6) is admissible for some $\{b_3, c_2\} \in (0,1)$, we consider, e.g., $a_3 = 2a_2$. Then (2.6) is equivalent to $b_3 + 4c_2 = 4$, which can be achieved when we choose $c_2 \in (3/4, 1)$ and $b_3 = 4(1-c_2)$.

Now, using (2.6), B can be re-written as

$$B = \begin{bmatrix} 1 & a_2 & a_3 \\ a_2 & a_2^2 & a_2a_3 \\ a_3 & a_2a_3 & a_3^2 \end{bmatrix}.$$

It is easy to see that B has the eigenvalues $\{0, 0, 1+a_2^2+a_3^2\}$ with the corresponding orthogonal eigenvectors

$$[0, -a_3, a_2]^T,\ [-(a_2^2+a_3^2), a_2, a_3]^T,\ [1, a_2, a_3]^T.$$

We conclude that $I \ge 0$, under the assumptions (1.8), $a_2b_3 = 1 = a_3c_1$ and (2.6), with

$$\sigma_1 = 1/r_1,\ \sigma_2 = a_2^2/r_2,\ \sigma_3 = a_3^2/r_3.$$

In fact, we have $I = (X + a_2Y + a_3Z)^2$. This proves Theorem 2. □

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