A LIOUVILLE THEOREM FOR A CLASS OF REACTION-DIFFUSION SYSTEMS WITH FRACTIONAL DIFFUSION

JONG-SHENQ GUO AND MASAHIKO SHIMOJO

ABSTRACT. We prove a Liouville theorem on the positive bounded entire solution of a class of reaction-diffusion systems with fractional diffusion. Some application of this Liouville theorem is also given.

1. Introduction

In this paper, we consider the following reaction-diffusion system

$$\frac{\partial u_i}{\partial t} + d_i(-\Delta)^\alpha u_i = f_i(u_1, \ldots, u_m),\ t \in \mathbb{R},\ x \in \mathbb{R}^n,\ i = 1, \ldots, m, \tag{1.1}$$

where n, m are positive integers, $d_i > 0$ and $f_i : \mathbb{R}^m \to \mathbb{R}$ is a C^1 function for each $i = 1, \ldots, m$, and $(-\Delta)^\alpha$ represents the fractional diffusion with order $\alpha \in (0, 1)$. The fractional diffusion is one of the anomalous diffusions in which the diffusive phenomenon is described by Lévy processes allowing long jumps.

The study of fractional diffusion has attracted a lot of attentions recently and the results are very different from that of the classical diffusion. In particular, it is proved in [3] that the invading speed of stable state into the unstable one is exponentially in time for the fractional diffusion Fisher-KPP type scalar equation, in contrast to the linear rate for the classical diffusion [1]. This exponential propagation speed is also shown in [4] for a predator-prey system and in [5] for a multi-component cooperative diffusive system with at least one diffusion is of fractional type.

We are concerned with the characterization of entire solutions of (1.1). Here an entire solution is a solution defined for all $t \in \mathbb{R}$ (and for all $x \in \mathbb{R}^n$). Let $0 < k_i < K_i < \infty$ for $i = 1, \ldots, m$. Set $g(\theta) := \theta - 1 - \ln\theta$, $\theta > 0$. Note that g is a strictly convex smooth function on $(0, \infty)$ such that $g(\theta) > 0$ for all $\theta \neq 1$ and $g(1) = 0$. Throughout this paper we shall always assume that (1.1) has a unique positive constant equilibrium $u^* := (u_1^*, \ldots, u_m^*)$ such that $k_i < u_i^* < K_i$ for each i.

Our main theorem of this paper reads

Date: June 8, 2022. Corresponding Author: M. Shimojo.

This work is partially supported by the Ministry of Science and Technology of Taiwan under the grant 108-2115-M-032-006-MY3, JSPS KAKENHI Grant-in-Aid for Young Scientists (B) (No. 16K17634) and JSPS KAKENHI Grant-in-Aid for Scientific Research (C) (No. 20K03708). We thank the anonymous referee for some valuable comments.

2020 Mathematics Subject Classification. Primary: 35K55, 35K57; Secondary: 92D25, 92D40.

Key words and phrases. Fractional diffusion, entire solution, Liouville theorem, convergence.

Theorem 1.1. *Let $u = (u_1, \ldots, u_m)$ be an entire solution of (1.1) such that $u_i \in [k_i, K_i]$ for $i = 1, \ldots, m$. Suppose that the corresponding diffusion-free system of (1.1) admits a nonnegative bounded Lyapunov function in the form*

$$F(u) = \sum_{i=1}^{m} F_i(u_i),\ u = (u_1, \ldots, u_m) \in \mathbb{R}_+^m,$$

where $F_i(u_i) = c_i g(u_i/u_i^*)$ for some positive constant c_i for each i such that

$$\sum_{i=1}^{m} F_i'(u_i) f_i(u) \leq -\kappa F(u),\ u \in \prod_{i=1}^{m} [k_i, K_i], \tag{1.2}$$

for some positive constant κ. Then $u = u^*$.

In particular, we have

Corollary 1.2. *Suppose that u is a bounded entire solution of*

$$u_t + d(-\Delta)^\alpha u = ru(1-u),\ t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

where d, r are positive constant, such that $u \geq \varepsilon$ in $\mathbb{R} \times \mathbb{R}^n$ for some positive constant ε. Then $u \equiv 1$.

It is well-known that the characterization of entire solutions plays an important role in the study of asymptotic behavior of the associated reaction-diffusion systems. As a result, some applications to the study of spatial propagation, such as the spreading behaviors, of systems arising in ecology and epidemiology can be addressed. For the study of spreading dynamics of systems with the classical diffusion, we refer the reader to [1] for a scalar equation, [13, 7] for two-component systems and [14, 12, 6] for three-component systems.

In fact, Theorem 1.1 is an extension of [10, Theorem 1.1] when the standard diffusion is replaced by the fractional diffusion. Recall from [10] that there are many diffusion-free kinetic systems have Lyapunov functions which enjoy the property (1.2). Hence Theorems 1.1 holds for a large class of systems in ecology and epidemiology such as those systems mentioned in [10] with the classical diffusion replaced by the fractional diffusion.

One of the applications of Theorem 1.1 is to derive the convergence of solutions to the unique positive co-existence state u^* in the persistent (for all components) zone, by using the regularity theory of fractional diffusion equations such as the Hölder regularity [11, Theorem 1.2] and the interior regularity estimate [9, Theorem 1.1] (see also [15, Theorem 2.5]). In this aspect, we refer the reader to, e.g. [7, Theorem 2.6] and [10, Theorem 1.3] for the classical diffusion case. To our knowledge, this convergence property for multi-component reaction-diffusion systems with fractional diffusion was little addressed in literature before. In fact, Theorem 1.1 can be applied to derive the convergence of solution to the positive co-existence state behind the spreading front for the predator-prey system studied in [4].

The rest of this paper is organized as follows. First, a proof of Theorem 1.1 is given in §2, based on some delicate analysis of the associated integrations and a suitable choice of weight function (cf. [2]). Then we give another example of predator-prey system for an application of Theorem 1.1 in §3. Note that [3, Theorem 1.2 (b)] can also be proved by applying Corollary 1.2 instead of [3, Lemma 3.3].

2. Proof of Theorem 1.1

Let

$$\rho_1(z) = \begin{cases} (1+(|z|^2-1)^4)^{-\gamma/8}, & |z| \geq 1, \\ 1, & |z| \leq 1, \end{cases}$$

where the constant $\gamma \in (n, n+2\alpha)$. According to Lemma 2.1 of [2], there exists $r_0 \gg 1$ and $C_0 > 0$ such that

$$|(-\Delta_z)^\alpha \rho_1(z)| \leq \frac{C_0}{|z|^{n+2\alpha}},\ \forall\, |z| \geq r_0.$$

Thus, due to $\gamma < n+2\alpha$, there exists a constant $C_1 > 0$ such that

$$(-\Delta_z)^\alpha \rho_1(z) \geq -C_1 \rho_1(z),\ \forall\, |z| \geq r_0.$$

Hence, by taking a larger constant $C_1 > 0$ (if it is necessary), we obtain

$$(-\Delta_z)^\alpha \rho_1(z) \geq -C_1 \rho_1(z),\ \forall\, z \in \mathbb{R}^n. \tag{2.1}$$

Set $\rho_R(x) := \rho_1(x/R)$, $R > 0$. Then, from the scaling property of $(-\Delta)^\alpha$ and (2.1), we have

$$(-\Delta)^\alpha \rho_R(x) = (-\Delta_x)^\alpha \rho_1(x/R) = \frac{1}{R^{2\alpha}}(-\Delta_z)^\alpha \rho_1(z) \geq -\frac{C_1}{R^{2\alpha}}\rho_1(z) = -\frac{C_1}{R^{2\alpha}}\rho_R(x). \tag{2.2}$$

Now we introduce the functional

$$\mathcal{F}_R(t) := \int_{\mathbb{R}^n} F(u(t,x))\rho_R(x)dx,$$

where u is an entire solution of (1.1) such that

$$0 < k_i \leq u_i \leq K_i < \infty,\ i = 1, \ldots, m.$$

Note that $F(u(t,x))$ is uniformly bounded over $\mathbb{R} \times \mathbb{R}^n$ and ρ_R is integrable over $\mathbb{R}^n$, due to $\gamma > n$. Hence $\mathcal{F}_R(t)$ is well-defined and uniformly bounded for $t \in \mathbb{R}$. Then we compute

$$\frac{d}{dt}\mathcal{F}_R(t) = \sum_{i=1}^m \int_{\mathbb{R}^n} F_i'(u_i) f_i(u)\rho_R dx - \sum_{i=1}^m d_i \int_{\mathbb{R}^n} F_i'(u_i)[(-\Delta)^\alpha u_i]\rho_R\, dx.$$

It follows from (1.2) and $g'(\theta) = 1 - 1/\theta$ that

$$\frac{d}{dt}\mathcal{F}_R(t) \leq -\kappa \mathcal{F}_R(t) - \sum_{i=1}^m d_i c_i \frac{1}{u_i^*}\int_{\mathbb{R}^n}\left(1 - \frac{u_i^*}{u_i}\right)[(-\Delta)^\alpha u_i]\rho_R\, dx. \tag{2.3}$$

Recall that

$$(-\Delta)^\alpha w(x) := \int_{\mathbb{R}^n} J(x-y)[w(x) - w(y)]dy,$$

where the integral is realized as the principal value and

$$J(x) := \frac{4^\alpha \Gamma(n/2+\alpha)}{\pi^{n/2}|\Gamma(-\alpha)|}\frac{1}{|x|^{n+2\alpha}}.$$

Then for a fixed i we can calculate

$$\begin{aligned} I &:= \int_{\mathbb{R}^n} \Big(1-\frac{u_i^*}{u_i(t,x)}\Big)\,[(-\Delta)^\alpha u_i(t,x)]\,\rho_R(x)\,dx \\ &= \int_{\mathbb{R}^n} \Big(1-\frac{u_i^*}{u_i(t,x)}\Big)\bigg[\int_{\mathbb{R}^n} J(x-y)\{u_i(t,x)-u_i(t,y)\}dy\bigg]\rho_R(x)\,dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(x-y)u_i^*\bigg[-1+\frac{u_i(t,x)}{u_i^*}-\frac{u_i(t,y)}{u_i^*}+\frac{u_i(t,y)}{u_i(t,x)}\bigg]\rho_R(x)\,dydx. \end{aligned}$$

By changing the order of integration, we also have

$$I = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(x-y)u_i^*\bigg[-1+\frac{u_i(t,x)}{u_i^*}-\frac{u_i(t,y)}{u_i^*}+\frac{u_i(t,y)}{u_i(t,x)}\bigg]\rho_R(x)\,dxdy.$$

On the other hand, by exchanging the roles of x and y together with $J(x-y)=J(y-x)$, we have

$$\begin{aligned} I &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(x-y)u_i^*\bigg[-1+\frac{u_i(t,x)}{u_i^*}-\frac{u_i(t,y)}{u_i^*}+\frac{u_i(t,y)}{u_i(t,x)}\bigg]\rho_R(x)\,dydx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(y-x)u_i^*\bigg[-1+\frac{u_i(t,y)}{u_i^*}-\frac{u_i(t,x)}{u_i^*}+\frac{u_i(t,x)}{u_i(t,y)}\bigg]\rho_R(y)\,dxdy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(x-y)u_i^*\bigg[-1+\frac{u_i(t,y)}{u_i^*}-\frac{u_i(t,x)}{u_i^*}+\frac{u_i(t,x)}{u_i(t,y)}\bigg]\rho_R(y)\,dxdy. \end{aligned}$$

Hence we obtain $2I = I_1 + I_2$, where

$$\begin{aligned} I_1 &:= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(x-y)u_i^*\bigg[-\rho_R(x)-\rho_R(y)+\frac{u_i(t,x)\rho_R(y)}{u_i(t,y)}+\frac{u_i(t,y)\rho_R(x)}{u_i(t,x)}\bigg]\,dxdy \\ &= 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(x-y)u_i^*\bigg[-1+\frac{u_i(t,y)}{u_i(t,x)}\bigg]\rho_R(x)\,dxdy, \\ I_2 &:= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(x-y)\Big[u_i(t,x)\rho_R(x)-u_i(t,y)\rho_R(x)+u_i(t,y)\rho_R(y)-u_i(t,x)\rho_R(y)\Big]\,dxdy \\ &= 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(x-y)\Big[u_i(t,x)-u_i(t,y)\Big]\rho_R(x)\,dxdy. \end{aligned}$$

Here we have exchanged x and y to get the second equality for each I_1 and I_2, respectively. It follows from $I = (I_1 + I_2)/2$ that

$$\begin{aligned} I &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(x-y)\left[u_i(t,x) - u_i^* - u_i(t,y) + u_i^* \frac{u_i(t,y)}{u_i(t,x)}\right]\rho_R(x)\,dxdy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(x-y)\left[u_i(t,x) - u_i^* - u_i^*\log\left(\frac{u_i(t,x)}{u_i(t,y)}\right) - u_i(t,y)\right. \\ &\qquad\qquad \left. + u_i^*\frac{u_i(t,y)}{u_i(t,x)} - u_i^*\log\left(\frac{u_i(t,y)}{u_i(t,x)}\right)\right]\rho_R(x)\,dxdy, \\ &\geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(x-y)\left[u_i(t,x) - u_i^* - u_i^*\log\left(\frac{u_i(t,x)}{u_i(t,y)}\right) - u_i(t,y) + u_i^*\right]\rho_R(x)\,dxdy, \end{aligned}$$

using $X - \log X \geq 1$ for all $X > 0$. Thus we get

$$I \geq \frac{u_i^*}{c_i}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(x-y)\Big[F_i(u_i(t,x)) - F_i(u_i(t,y))\Big]\rho_R(x)\,dxdy. \tag{2.4}$$

Moreover, using

$$\begin{aligned} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(x-y)F_i(u_i(t,y))\rho_R(x)\,dxdy &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(y-x)F_i(u_i(t,x))\rho_R(y)\,dydx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(x-y)F_i(u_i(t,x))\rho_R(y)\,dxdy, \end{aligned}$$

it follows from (2.4) that

$$I \geq \frac{u_i^*}{c_i}\int_{\mathbb{R}^n} F_i(u_i(t,x))\left[(-\Delta)^\alpha \rho_R(x)\right]dx. \tag{2.5}$$

Finally, we apply (2.2), (2.3) and (2.5) to conclude that

$$\frac{d}{dt}\mathcal{F}_R(t) \leq -\Big(\kappa - \frac{C_2}{R^{2\alpha}}\Big)\mathcal{F}_R(t), \quad \forall\, t \in \mathbb{R},$$

where $C_2 = (\max_{1\leq i\leq m} d_i)C_1$. By choosing $R > 0$ sufficiently large such that $C_2/R^{2\alpha} \leq \kappa/2$ and integrating for time from $-\infty$ to t, we deduce that $\mathcal{F}_R(t) = 0$ for all $t \in \mathbb{R}$. Hence $F(u(t,x)) \equiv 0$ and so $u(t,x) \equiv u^*$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$. Theorem 1.1 is thereby proved. □

3. An example

Consider the following predator-prey system

$$u_t + d_1(-\Delta)^\alpha u = r_1 u(1 - u - av),\ t > 0,\ x \in \mathbb{R}^n, \tag{3.1}$$

$$v_t + d_2(-\Delta)^\alpha v = r_2 v(-1 + bu - v),\ t > 0,\ x \in \mathbb{R}^n, \tag{3.2}$$

where n is a positive integer, u and v denote the population densities of the prey and predator, d_1 and d_2 are their diffusion coefficients, $(-\Delta)^\alpha$ represents the fractional diffusion with order $\alpha \in (0,1)$. Moreover, r_1 and $-r_2$ are the intrinsic growth rates of the prey and the predator, respectively, $r_1 a$ represents the predation rate and $r_2 b$ represents the conversion rate.

Then we have the following theorem by an application of Theorem 1.1.

Theorem 3.1. *Assume that*

$$b > 1,\ a(b-1) < 1. \tag{3.3}$$

Set $s^* := [r_2(b-1)]/(n+2\alpha)$. *Let* (u,v) *be the solution to* (3.1)-(3.2) *with initial condition*

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x),\ x \in \mathbb{R}^n, \tag{3.4}$$

where the initial data is assumed to satisfy

(3.5) v_0 *is uniformly continuous with compact support,* $0 \le v_0 \le b-1$, $\chi \le u_0 \le 1$ *in* $\mathbb{R}^n$

for some positive constant χ. *Then*

$$\lim_{t\to\infty} \sup_{|x|\le e^{st}} \{|u(t,x) - u^*| + |v(t,x) - v^*|\} = 0,\ \forall\, s \in (0, s^*). \tag{3.6}$$

As mentioned in the introduction, the proof of Theorem 3.1 relies on a uniform persistent result on the zone $\{(x,t) \mid |x| \le e^{st}, t \gg 1\}$ for $s \in (0, s^*)$.

First, comparing with the corresponding ODE system, i.e., solutions independent of x, it follows from the comparison principle for the scalar equation that

$$u \ge 0,\ v \ge 0,\ u \le 1,\ v \le b-1. \tag{3.7}$$

Using $v \le b-1$, it follows from (3.1) that

$$u_t + d_1(-\Delta)^\alpha u \ge r_1 u[1 - a(b-1) - u],\ t > 0,\ x \in \mathbb{R}^n.$$

Hence, by comparison, we obtain

$$u(t,x) \ge \min\{\chi, 1 - a(b-1)\} := \beta > 0,\ \forall\, (t,x) \in (0,\infty) \times \mathbb{R}^n. \tag{3.8}$$

Next, we derive

$$\liminf_{t\to\infty} \inf_{|x|\le e^{st}} v(t,x) > 0,\ \forall\, s \in (0, s^*). \tag{3.9}$$

To show (3.9), we set $w = 1 - u$. Then $w_0 := 1 - u_0 \le 1 - \chi$ in $\mathbb{R}^n$ and, using (3.8), w satisfies

$$w_t + d_1(-\Delta)^\alpha w \le -r_1\beta w + r_1 a v,\ t > 0,\ x \in \mathbb{R}^n.$$

Let ϕ be the solution of

$$\begin{cases} \phi_t + d_1(-\Delta)^\alpha \phi = -r_1\beta\phi + r_1 a v,\ t > 0,\ x \in \mathbb{R}^n, \\ \phi(0,x) = w_0(x),\ x \in \mathbb{R}^n. \end{cases}$$

Then $w \le \phi$. Set $\Phi(t,x) := e^{r_1\beta t}\phi(t,x)$. Then Φ satisfies

$$\begin{cases} \Phi_t + d_1(-\Delta)^\alpha \Phi = r_1 a e^{r_1\beta t} v,\ t > 0,\ x \in \mathbb{R}^n, \\ \Phi(0,x) = w_0(x),\ x \in \mathbb{R}^n. \end{cases}$$

Hence, by the variation of constants formula, we obtain

$$\phi(t,x) = e^{-r_1\beta t}\left\{\int_{\mathbb{R}^n} p(t, x-y) w_0(y) dy + r_1 a \int_0^t \int_{\mathbb{R}^n} p(t-s, x-y) e^{r_1\beta s} v(s,y) dy ds\right\}, \tag{3.10}$$

where $p(t,x)$ is the kernel corresponding to $\partial_t + d_1(-\Delta)^\alpha$ (cf. [3]). Then (3.9) can be proved by a similar argument in [8], by using (3.10). We omit it here.

With (3.8) and (3.9), we finish the proof of Theorem 3.1 by the same argument as that of [10, Theorem 1.3] with the help of Theorem 1.1 and recall from [6, Lemma 4.2] that

$$F(u,v) := br_2 u^* g(u/u^*) + ar_1 v^* g(v/v^*)$$

is a Lyapunov function for (3.1)-(3.2) satisfying condition (1.2) for some positive constant κ.

Finally, we remark that the exponential spreading speed s^* depends on the fractional order α. However, the exact convergence rate to the equilibrium (u^*, v^*) in (3.6) is left open. In fact, this is not known even for the scalar Fisher-KPP equations.

References

[1] D.G. Aronson, H.F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, in: J.A. Goldstein (Ed.), Partial Differential Equations and Related Topics, in: Lecture Notes in Math. **446**, Springer, Berlin, 1975, 5–49.

[2] M. Bonforte, J.L. Vazquez, *Qualitative local and global a priori estimates for fractional nonlinear diffusion equations*, Advance in Math., 250 (2014), 242–284.

[3] X. Cabré, J.-M. Roquejoffre, *The influence of fractional diffusion in Fisher-KPP equations*, Commun. Maht. Phys., 320 (2013), 679–722.

[4] H. Cheng, R. Yuan, *The spreading property for a prey-predator reaction-diffusion system with fractional diffusion*, Fractional Calculus & Appl. Anal., 18 (2015), 565-579.

[5] A.-C. Coulon, M. Yangari, *Exponential propagation for fractional reaction-diffusion cooperative systems with fast decaying initial conditions*, J. Dyn. Diff. Equat., 29 (2017), 799–815.

[6] A. Ducrot, T. Giletti, J.-S. Guo, M. Shimojo, *Asymptotic spreading speeds for a predator-prey system with two predators and one prey*, Nonlinearity, 34 (2021), 669–704.

[7] A. Ducrot, T. Giletti, H. Matano, *Spreading speeds for multidimensional reaction-diffusion systems of the prey-predator type*, Calc. Var., 58 (2019), 1–34.

[8] A. Ducrot, J.-S. Guo, G. Lin, S. Pan, *The spreading speed and the minimal wave speed of a predator-prey system with nonlocal dispersal*, Z. Angew. Math. Phys., 70 (2019), Art. 146, 25 pp.

[9] X. Fernandez-Real, X. Ros-Oton, *Regularity theory for general stable operators: parabolic equations*, J. Funct. Anal., 272 (2017), 4165–4221.

[10] J.-S. Guo, M. Shimojo, *Stabilization to a positive equilibrium for some reaction-diffusion systems*, Nonlinear Analysis: Real World Applications, 62 (2021), Art. 103378, 12pp.

[11] M. Kassmann, R. W. Schwab, *Regularity results for nonlocal parabolic equations*, Riv. Mat. Univ. Parma, 5 (2014), 183–212.

[12] R. Mori, D. Xiao, *Spreading properties of a three-component reaction-diffusion model for the population of farmers and hunter-gatherers*, Annales de l'Institut Henri Poincaré C, Analyse non linéaire, 38 (2020), 911–951.

[13] M.A. Lewis, B. Li, H.F. Weinberger, *Spreading speed and linear determinacy for two-species competition models*, J. Math. Biology, 45 (2002), 219–233.

[14] C.-C. Wu, *The spreading speed for a predator-prey model with one predator and two preys*, Applied Mathematics Letters, 91 (2019), 9–14.

[15] Y.P. Zhang, A. Zlatos, *Optimal estimates on the propagation of reactions with fractional diffusion*, arXiv:2105.12800v1.

(J.-S. Guo) Department of Mathematics, Tamkang University, Tamsui, New Taipei City 251301, Taiwan

Email address: jsguo@mail.tku.edu.tw

(M. Shimojo) Department of Mathematical Sciences, Tokyo Metropolitan University, Hachioji, Tokyo 192-0397, Japan

Email address: shimojo@tmu.ac.jp