FORCED WAVES OF SATURATION TYPE FOR FISHER-KPP EQUATION IN A SHIFTING ENVIRONMENT

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ABSTRACT. In this paper, we study the forced waves of the Fisher-KPP equation in a shifting environment. We introduce a new method to construct a sub-solution to derive the existence of forced waves without the monotonicity assumption on the shifting growth rate function. The forced waves we derived here are of the saturation type, in contrast to the extinction type in the literature.

1. INTRODUCTION

The study of traveling waves in reaction-diffusion equations has attracted a lot of attention in past years starting from the pioneer works [10, 13] in 1937 on a scalar equation. The Fisher-KPP equation reads

$$u_t = du_{xx} + u(1-u),\ x \in \mathbb{R},\ t > 0, \tag{1.1}$$

where $u = u(x,t)$ represents the density of a species and d is its diffusion coefficient. Here we have normalized both intrinsic growth rate and the carrying capacity to be 1. Since then, there have been a lot of interesting works on the existence, uniqueness and stability of traveling waves for many reaction-diffusion systems arising in ecology and epidemiology.

Recently, due to the threaten of global warming to the environment of ecological systems, such as sea level rise, precipitation change, and desertification, more attention is paid to the study of the effect of climate change. Among various mathematical modelings for climate change, the simplest one is to replace the constant growth rate by a function $h = h(x - st)$ in which s is a positive constant representing the shifting speed of the environment. Then the homogeneous equation (1.1) is replaced by the equation

$$u_t = du_{xx} + u[h(x-st) - u],\ x \in \mathbb{R},\ t > 0. \tag{1.2}$$

We are concerned with the spreading dynamics of solutions of (1.2). In particular, we are interested in traveling wave solutions of (1.2) in the form

$$u(x,t) = \phi(z),\ z := x - st,$$

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for some function ϕ (the wave profile), where the wave speed s is the same as the environmental shifting speed. This type of traveling wave is called a forced wave. Note that ϕ satisfies

$$d\phi''(z) + s\phi'(z) + \phi(z)[h(z) - \phi(z)] = 0,\ z \in \mathbb{R}. \tag{1.3}$$

For the study of forced waves for the scalar equation, two-species systems and a three-species system, we refer the reader to, e.g., [3, 4, 8, 12, 2, 15, 7, 14, 5, 6, 9].

Among these works, the following condition is imposed on h:

$$\begin{cases} h \text{ is bounded and continuous in } \mathbb{R} \text{ such that } h \le h(\infty) = 1 \text{ in } \mathbb{R} \\ h(z) < 0 \text{ for } z \le -L \text{ for some positive constant } L. \end{cases} \tag{1.4}$$

This implies that the devastating environment for the species is expanding as the time increases. Hence it is natural to expect that the species goes extinction in the whole habitat as $t \to \infty$. In particular, Hu and Zou [12] has derived the forced waves for (1.2) with $\phi(-\infty) = 0$ and $\phi(\infty) = 1$ for any $s > 0$, under condition (1.4) with h being assumed to be monotone. It is of *extinction* type, since $u(x,t) \to 0$ as $t \to \infty$ for all $x \in \mathbb{R}$. Moreover, forced waves were studied in [7] for a 2-species competition system and in [6] for a 1-predator-2-prey system, under condition (1.4), h is monotone and $h(z) \to 1$ as $z \to \infty$ exponentially.

However, some species, such as mosquito, is benefited by global warming. Mosquitoes grow faster with warm temperature. Of course, this is also bad to our environment, since mosquito can be a vector of transmitting diseases such as dengue fever, West Nile virus, etc. To model this phenomenon of mosquito's growth, the net growth rate $h = h(x - st)$ is assumed to satisfy

- (h1) h is bounded and continuous in $\mathbb{R}$ such that $h(-\infty) = 1$;
- (h2) $h(z) < 0$ for all $z \ge L$ for some positive constant L.

This means that the favorable habitat of mosquitoes expands in the positive x-axis.

We are looking for a forced wave solution $u(x,t) = \phi(z)$ of (1.2) such that

$$\phi(\infty) = 0 < \phi(z) < 1 = \phi(-\infty),\ \forall\, z \in \mathbb{R}. \tag{1.5}$$

Biologically, this means that with the climate change the species reaches the saturation level 1 throughout the entire habitat eventually. We shall call this wave as a forced wave of *saturation* type, since $u(x,t) \to 1$ as $t \to \infty$ for all $x \in \mathbb{R}$. In fact, for $s > 0$, under an extra assumption that h is monotone, a saturation forced wave exists if and only if $s \in (0, 2\sqrt{d})$, by a result of Fang, Lou and Wu [8, Theorem 2.1 (i)].

Without the monotonicity assumption on h, we prove

Theorem 1.1. *Under the assumptions* (h1) *and* (h2)*, there exists a solution* ϕ *of* (1.3)-(1.5) *for any* $s < \sqrt{d}$.

The proof of Theorem 1.1 is given in the next section based on the monotone iteration method. It is clear that $\overline{\phi} \equiv 1$ is a super-solution. The main difficulty of proving Theorem 1.1 is the construction of a suitable nontrivial sub-solution. To overcome this difficulty, we provide in this paper a new method to verify the constructed function is a sub-solution. See the proof of Proposition 2.2 in §2. This is one of the main contributions of this work. Note also that we do not impose the monotonicity of h and the exponential convergence of h to $h(-\infty)$ in this paper. However, the range of admissible shifting speeds, $s < \sqrt{d}$, might not be optimal. We therefore give a brief discussion on this issue in §3 and leave this as an open question for future studies.

2. Existence of forced waves

In this section, we shall study the existence of forced waves of (1.2).

First, we prove the following result.

Proposition 2.1. *For any positive solution ϕ of* (1.3)*, $\phi(\infty) = 0$.*

Proof. That $\phi(\infty) = 0$ follows from the fact $h(z) < 0$ for all $z \gg 1$, by a contradiction argument (cf. [5]). To be self-contained and for the reader's convenience, we provide the details as follows. For a contradiction, we suppose that $\phi^+ > 0$, where

$$\phi^+ := \limsup_{z\to\infty} \phi(z).$$

If ϕ is monotone ultimately at $z = \infty$, then $\phi(z) \to \phi^+$ as $z \to \infty$. Hence there is a sequence $\{z_n\}$ such that $z_n \uparrow \infty$ and $\phi'(z_n) \to 0$ as $n \to \infty$. Without loss of generality, by (h1), we may assume that $h(z) < 0$ for all $z \geq z_1$. Then by integrating (1.3) from z_1 to z_n we obtain

$$[\phi'(z_n) - \phi'(z_1)] + s[\phi(z_n) - \phi(z_1)] = \int_{z_1}^{z_n} \{\phi(z)[\phi(z) - h(z)]\}dz \geq \int_{z_1}^{z_n} \phi^2(z)dz. \tag{2.1}$$

Since the left-hand side of (2.1) is uniformly bounded for all n and the right-hand side of (2.1) tends to $+\infty$ as $n \to \infty$, we reach a contradiction.

On the other hand, if ϕ is oscillatory near $+\infty$, then there is a maximal sequence $\{z_n\}$ of ϕ such that $z_n \to \infty$ and $\phi(z_n) \to \phi^+$ as $n \to \infty$. Without loss of generality, by (h1), we may assume that $h(z) < 0$ for all $z \geq z_1$. Then

$$0 \geq \phi''(z_n) + s\phi'(z_n) = \phi(z_n)[\phi(z_n) - h(z_n)] \geq \phi^2(z_n) \to (\phi^+)^2 > 0,$$

a contradiction.

Hence we have proved that $\phi^+ = 0$. Due to $\phi \geq 0$ in $\mathbb{R}$, we therefore conclude that $\phi(\infty) = 0$. □

Next, we derive the existence of a positive solution of (1.3).

Proposition 2.2. *If $s < \sqrt{d}$, then* (1.3) *has a positive solution ϕ such that*

$$\liminf_{z\to-\infty} \phi(z) > 0.$$

Proof. First, given $\eta \in (0, 1)$, it follows from $h(-\infty) = 1$ that

$$h(z - K) \geq 1 - \eta,\ \forall\, z < 0, \tag{2.2}$$

for some positive constant $K = K(\eta)$. The constant η is to be chosen later.

We consider the function $g(z) := e^{\lambda z} - 2e^{2\lambda z}$, where $\lambda > 0$ is a constant to be determined later. Note that $g(z) > 0$ if and only if $z < z_0 := -(\ln 2)/\lambda < 0$. Also, $g(z) \le g(z_1)$ for all $z \in \mathbb{R}$, where $z_1 < z_0$ is uniquely defined by

$$e^{\lambda z_1} = 1/4.$$

Note that g has a unique maximal point at $z = z_1$ such that

$$g(z_1) = e^{\lambda z_1}(1 - 2e^{\lambda z_1}) = 1/8, \quad g(-\infty) = 0.$$

Let $\delta := e^{\lambda z_*}$ for a $z_* < z_1$ to be chosen later. Then $\delta \in (0, 1/4)$ and

$$g(z_*) = \delta(1 - 2\delta) := \gamma(\delta) = \gamma.$$

We claim that the function

$$\underline{\phi}(z) := \begin{cases} \gamma - g(z), \ z < z_*, \\ 0, \ z \ge z_*, \end{cases}$$

satisfies

$$\mathcal{L}(z) := d\underline{\phi}''(z) + s\underline{\phi}'(z) + \underline{\phi}(z)[h(z-K) - \underline{\phi}(z)] \ge 0 \ \text{ for all } z \ne z_*. \tag{2.3}$$

Clearly, (2.3) holds for $z > z_*$.

For $z < z_*$, using (2.2) we obtain

$$\begin{aligned} \mathcal{L}(z) &= -(d\lambda^2 + s\lambda)e^{\lambda z} + 4(2d\lambda^2 + s\lambda)e^{2\lambda z} + \underline{\phi}(z)[h(z-K) - \underline{\phi}(z)] \\ &\ge -(d\lambda^2 + s\lambda)e^{\lambda z} + 4(2d\lambda^2 + s\lambda)e^{2\lambda z} + \underline{\phi}(z)\{1 - \underline{\phi}(z) - \eta\} \\ &= -(d\lambda^2 + s\lambda)e^{\lambda z} + 4(2d\lambda^2 + s\lambda)e^{2\lambda z} + f(e^{\lambda z}; \delta) - \eta\underline{\phi}(z), \end{aligned}$$

where

$$f(y; \delta) = \gamma(1-\gamma) + (-1 + 2\gamma)y + (1 - 4\gamma)y^2 + 4y^3 - 4y^4.$$

Note that $y = e^{\lambda z} \in (0, \delta)$ for $z < z_*$, $f(0; \delta) = \gamma(1-\gamma) > 0$ and $f(\delta; \delta) = 0$. Moreover,

$$f'(y; \delta) = (-1 + 2\gamma) + 2(1 - 4\gamma)y + 12y^2 - 16y^3$$

gives $f'(0; \delta) = -1 + 2\gamma < 0$ and $f'(\delta; \delta) = -1 + 4\delta < 0$. Note that $\gamma < \delta < 1/4$. Furthermore,

$$f''(y; \delta) = 2(1 - 4\gamma) + 24y - 48y^2$$

so that $f''(0; \delta) = 2(1 - 4\gamma) > 0$ and $f''(\delta; \delta) = 2 + 16\gamma > 0$. This implies that $f''(y; \delta) > 0$ for all $y \in [0, \delta]$ and so $f'(y; \delta) < f'(\delta; \delta) < 0$ for all $y \in (0, \delta)$.

Now, set

$$F(y; \lambda, \delta) := -(d\lambda^2 + s\lambda)y + 4(2d\lambda^2 + s\lambda)y^2 + f(y; \delta).$$

Then

$$\begin{cases} F'(y; \lambda, \delta) = -(d\lambda^2 + s\lambda) + 8(2d\lambda^2 + s\lambda)y + f'(y; \delta), \\ F''(y; \lambda, \delta) = 8(2d\lambda^2 + s\lambda) + f''(y; \delta). \end{cases}$$

Also, we compute

$$F(\delta; \lambda, \delta) = \delta\lambda\{-(d\lambda + s) + 4(2d\lambda + s)\delta\} = \delta\lambda\{(8\delta - 1)d\lambda - (1 - 4\delta)s\}.$$

Then, to get $F(\delta;\lambda,\delta)>0$ for a positive λ, we need

$$\delta\in(1/8,1/4),\quad \lambda>\frac{(1-4\delta)s}{(8\delta-1)d}:=\lambda_m(\delta). \tag{2.4}$$

On the other hand, since

$$F'(\delta;\lambda,\delta)=-(d\lambda^2+s\lambda)+8(2d\lambda^2+s\lambda)\delta-(1-4\delta),$$

we see that $F'(\delta;\lambda,\delta)\le 0$ for some positive λ if and only if

$$\Lambda(\lambda;\delta):=(16\delta-1)d\lambda^2+(8\delta-1)s\lambda-(1-4\delta)\le 0.$$

Since the function $\Lambda(\lambda;\delta)$ is increasing in $\lambda>0$ for $\delta\in(1/8,1/4)$, we have

$$\Lambda(\lambda;\delta)>\Lambda(\lambda_m(\delta);\delta)=(1-4\delta)\left\{\left[\frac{(16\delta-1)(1-4\delta)}{(8\delta-1)^2}+1\right]\frac{s^2}{d}-1\right\},\ \forall\,\lambda>\lambda_m(\delta).$$

Note that the function

$$H(\delta):=\frac{(16\delta-1)(1-4\delta)}{(8\delta-1)^2}+1$$

is strictly decreasing in $\delta\in(1/8,1/4)$ such that $H(1/4)=1$ and $H(\delta)\to\infty$ as $\delta\to(1/8)^+$. Hence, when $s<\sqrt{d}$, there is a unique $\delta_0\in(1/8,1/4)$ such that $d=H(\delta_0)s^2$.

Next, we choose a fixed $\delta\in(\delta_0,1/4)$ such that $H(\delta)s^2<d$ so that $\Lambda(\lambda_m;\delta)<0$ for the corresponding $\lambda_m(\delta)$ defined in (2.4). Moreover, by (2.4), there exists $\lambda>\lambda_m(\delta)>0$ with $\lambda-\lambda_m(\delta)$ sufficiently small such that $F(\delta;\lambda,\delta)>0$ and $F'(\delta;\lambda,\delta)=\Lambda(\lambda;\delta)\in(\Lambda(\lambda_m,\delta),0]$. Recall that

$$F''(y;\lambda,\delta)=8(2d\lambda^2+s\lambda)+f''(y;\delta)>0,\ \forall\,y\in(0,\delta).$$

Hence $F'(y;\lambda,\delta)<0$ for all $y\in(0,\delta)$ and so

$$F(y;\lambda,\delta)>F(\delta;\lambda,\delta)>0\ \text{ for all } y\in(0,\delta). \tag{2.5}$$

Finally, we choose a constant η such that

$$0<\eta\le\frac{F(\delta;\lambda,\delta)}{\gamma}. \tag{2.6}$$

Then with the corresponding K in (2.2) to the chosen η in (2.6) it follows from (2.5) that

$$\mathcal{L}(z)\ge F(e^{\lambda z};\lambda,\delta)-\eta\underline{\phi}(z)\ge F(\delta;\lambda,\delta)-\eta\gamma\ge 0,\ \forall\,z<z_*,$$

using $\underline{\phi}(z)<\gamma$ for all $z<z_*$. Hence (2.3) is proved and this shows that $\underline{\phi}$ is a sub-solution of

$$d\hat{\phi}''(z)+s\hat{\phi}'(z)+\hat{\phi}(z)[h(z-K)-\hat{\phi}(z)]=0,\ z\in\mathbb{R}. \tag{2.7}$$

Note that

$$\lim_{z\nearrow z_*}\underline{\phi}'(z)<0=\lim_{z\searrow z_*}\underline{\phi}'(z).$$

It is clear that $\overline{\phi}(z)\equiv 1$ is a super-solution of (2.7). Hence, by the monotone iteration method, there is a solution $\hat{\phi}$ of (2.7) such that

$$\underline{\phi}(z)\le\hat{\phi}(z)\le\overline{\phi}(z),\ \forall\,z\in\mathbb{R}. \tag{2.8}$$

Then $\phi(z) := \hat{\phi}(z+K)$ is a solution of (1.3). Moreover, by (2.8) and the strong maximum principle, we see that $0 < \phi < 1$ in $\mathbb{R}$. In particular, we have

$$\phi^- := \liminf_{z\to-\infty} \phi(z) \ge \liminf_{z\to-\infty} \underline{\phi}(z+K) = \gamma > 0. \tag{2.9}$$

Therefore, the proposition is proved. □

With (2.9), we follow a method used in [11] to claim that $\phi(-\infty) = 1$. For this, we set

$$A := \{\theta \in [0,1) \mid \phi^- > l(\theta)\}, \quad l(\theta) := \theta + (1-\theta)\gamma/2.$$

Since $0 \in A$, by (2.9), the quantity $\theta_0 := \sup A$ is well-defined and $\theta_0 \in (0,1]$. Passing to the limit, we obtain that $\phi^- \ge l(\theta_0)$.

To proceed further, we assume that $\theta_0 < 1$. Then we must have $\phi^- = l(\theta_0)$, by the continuity of $l(\theta)$. Suppose that ϕ is monotone ultimately at $z = -\infty$. Then $\phi(z) \to l(\theta_0)$ as $z \to -\infty$ and there is a sequence $\{z_n\}$ such that $z_n \to -\infty$ and $\phi'(z_n) \to 0$ as $n \to \infty$. By an integration of (1.3) from z_n to 0, we obtain

$$-d[\phi'(0) - \phi'(z_n)] - s[\phi(0) - \phi(z_n)] = \int_{z_n}^{0} \{\phi(z)[h(z) - \phi(z)]\} dz, \ \forall\, n.$$

This leads to a contradiction, by letting $n \to \infty$, since

$$\lim_{z\to-\infty} \{\phi(z)[h(z) - \phi(z)]\} = l(\theta_0)[1 - l(\theta_0)] > 0.$$

On the other hand, if ϕ is oscillatory as $z \to -\infty$, then we choose a sequence of minimal points $\{z_n\}$ of ϕ with $z_n \to -\infty$ and $\phi(z_n) \to l(\theta_0)$ as $n \to \infty$. This implies

$$\begin{aligned} 0 &= d\phi''(z_n) + s\phi'(z_n) + \phi(z_n)[h(z_n) - \phi(z_n)] \\ &\ge \phi(z_n)[h(z_n) - \phi(z_n)] \to l(\theta_0)[1 - l(\theta_0)] > 0, \text{ as } n \to \infty, \end{aligned}$$

a contradiction. We conclude that $\theta_0 = 1$ and so $\phi^- \ge l(1) = 1$. Since $\phi < 1$ in $\mathbb{R}$, we have proved that $\phi(-\infty) = 1$. This completes the proof of Theorem 1.1.

3. Discussion

In this paper, we obtain a new type of forced waves, the saturation type, for the shifting speed s with $s < \sqrt{d}$. As we have seen, the existence of forced wave relies on the existence of a sub-solution $\underline{\phi}$. Here we choose

$$\underline{\phi}(z) = \begin{cases} (e^{\lambda z_*} - 2e^{2\lambda z_*}) - (e^{\lambda z} - 2e^{2\lambda z}), \ z < z_* < z_1 := -\ln 4/\lambda, \\ 0, \ z > z_*, \end{cases}$$

with a certain choice of λ to obtain the upper bound $\sqrt{d}$. We suspect that the upper bound $\sqrt{d}$ is not optimal.

A first suspicious question is what if we choose

$$\underline{\phi}(z) = \begin{cases} (e^{\lambda z_*} - qe^{(1+\nu)\lambda z_*}) - (e^{\lambda z} - qe^{(1+\nu)\lambda z}), \ z < z_* < z_1 := -\ln[(1+\nu)q]/(\lambda\nu), \\ 0, \ z > z_*, \end{cases}$$

for given constants $q > 1$ and $\nu > 0$. It turns out that the upper bound is the same as the case $q = 2$ and $\nu = 1$.

On the other hand, there might be other choices of sub-solutions which produce a wider range of admissible shifting speeds. We leave it for the interested reader to explore this possibility. Another interesting question is whether there is an upper bound for the existence of forced waves of saturation type. Recall from [1] that the spreading speed of the classical equation (1.1) is $2\sqrt{d}$ and, by comparison, a solution of (1.2) is a sub-solution of (1.1) with the same initial data. We suspect that $2\sqrt{d}$ might be the upper bound for the forced waves of saturation type. In fact, this is true under an extra assumption that h is monotone as mentioned in §1. It would be very interesting to remove the monotonicity assumption for the existence of saturation forced waves. This is left for an open question.

Note that an extinction forced wave exists for any shifting speed $s > 0$, under condition (1.4) and the monotonicity on h. Indeed, under the change of variables

$$v(x,t) = u(-x,t),\ \tilde{h}(z) = h(-z),\ \tilde{s} = -s,$$

it follows from (1.2) that v satisfies

$$v_t = dv_{xx} + v[\tilde{h}(x - \tilde{s}t) - v], \quad x \in \mathbb{R},\ t > 0,$$

and $\tilde{h}$ satisfies condition (1.4), if h satisfies (h1) and (h2). In particular, $\tilde{h}(\infty) = 1$ and $\tilde{h}(z) < 0$ for $-z \gg 1$. Then, for a forced wave $v(x,t) := \psi(x - \tilde{s}t)$ with $\psi(-\infty) = 0$ and $\psi(\infty) = 1$, we obtain $v(x,t) \to 0$ as $t \to \infty$, when $\tilde{s} > 0$ or $s < 0$. On the other hand, we obtain $v(x,t) \to 1$ as $t \to \infty$, when $\tilde{s} < 0$, i.e., $s > 0$. Hence the derivations of extinction and saturation forced waves are mathematically equivalent, except the shifting speeds in both cases have different signs. In fact, an extinction forced wave exists for any $\tilde{s} > 0$ (cf. [8, Theorem 2.1 (i)] and [12, Theorem 1.1]). We also refer the reader to [9, Proposition 3.7] for a simple proof of this result without the monotonicity on $\tilde{h}$, but requiring an exponential decay of $1 - \tilde{h}(z)$ to zero at $z = \infty$.

References

[1] D.G. Aronson, H.F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, in "Partial Differential Equations and Related Topics", J.A. Goldstein (Ed.), Lecture Notes in Math., 446, Springer, Berlin, 1975, 5-49.

[2] H. Berestycki, J. Fang, *Forced waves of the Fisher-KPP equation in a shifting environment*, J. Differential Equations, 264 (2018), 2157-2183.

[3] H. Berestycki, L. Rossi, *Reaction-diffusion equations for population dynamics with forced speed, I - the case of the whole space*, Discrete Contin. Dyn. Syst., 21 (2008), 41-67.

[4] H. Berestycki, L. Rossi, *Reaction-diffusion equations for population dynamics with forced speed, II - cylindrical type domains*, Discrete Contin. Dyn. Syst., 25 (2009), 19-61.

[5] W. Choi, T. Giletti, J.-S. Guo, *Persistence of species in a predator-prey system with climate change and either nonlocal or local dispersal*, J. Differential Equations, 302 (2021), 807-853.

[6] W. Choi, J.-S. Guo, *Forced waves of a three species predator-prey system in a shifting environment*, J. Math. Anal. Appl., 514 (2022), 126283.

[7] F.-D. Dong, B. Li, W.-T. Li, *Forced waves in a Lotka-Volterra competition-diffusion model with a shifting habitat*, J. Differential Equations, 276 (2021), 433-459.

[8] J. Fang, Y. Lou, J. Wu, *Can pathogen spread keep pace with its host invasion?*, SIAM J. Appl. Math., 76 (2016), 1633-1657.

[9] T. Giletti, J.-S. Guo, *Forced waves of a three species predator-prey system with a pair of weak-strong competing preys in a shifting environment*, Discrete Contin. Dyn. Syst. - Series B, doi:10.3934/dcdsb.2022242.

[10] R.A. Fisher, *The advance of advantageous genes*, Ann. Eugenics, 7 (1937), 355-369.

[11] J.-S. Guo, K. Guo, *Traveling waves for a three-species competition system with two weak aboriginal competitors*, Diff. Integral Equations, 35 (2022), 819-832.

[12] H. Hu, X. Zou, *Existence of an extinction wave in the Fisher equation with a shifting habitat*, Proc. Am. Math. Soc., 145 (2017) 4763-4771.

[13] A.N. Kolmogorov, I.G. Petrovsky, N.S. Piskunov, *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un probléme biologique*, Bull. Univ. Moskov. Ser. Internat. Sect. A, 1 (1937), 1-25.

[14] J.-B. Wang, C. Wu, *Forced waves and gap formations for a Lotka-Volterra competition model with nonlocal dispersal and shifting habitats*, Nonlinear Anal. Real World Appl., 58 (2021), 103208.

[15] Y. Yang, C. Wu, Z. Li, *Forced waves and their asymptotics in a Lotka-Volterra cooperative model under climate change*, Appl. Math. Comput., 353 (2019), 254-264.

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