THE SPREADING SPEED OF AN SIR EPIDEMIC MODEL WITH NONLOCAL DISPERSAL

JONG-SHENQ GUO, AMY AI LING POH, AND MASAHIKO SHIMOJO

Abstract. In this paper, we study an SIR epidemic model with nonlocal dispersal. We study the case with vital dynamics so that a renewal of the susceptible individuals is taken into account. We characterize the asymptotic spreading speed to estimate how fast the disease under consideration spreads. Due to the lack of comparison principle for the SIR model, our proof is based on a delicate analysis of related problems with nonlocal scalar equations.

1. Introduction

In epidemiology, one of the most important questions is whether a disease can spread. There are two typical classical epidemiology models, namely, the classical Kermack-McKendrick model [19] and the so-called endemic model (cf. [13]). They are differentiated by whether the vital dynamics (births and deaths) are taken into account. The spread of infectious diseases in populations have been studied extensively. We refer the reader to, for examples, [6, 9, 13, 25, 24, 30] and the references therein. On the other hand, the movements of individuals usually are not limited to a small area. Long distance effects and interactions are often presented, and this can be formulated by the nonlocal dispersal (cf. [23]).

In this paper, we study the following SIR (susceptible-infective-removed) epidemic model with nonlocal dispersal

$$S_t(x,t) = d_1\mathcal{N}_1[S(\cdot,t)](x) + \mu - \mu S(x,t) - \frac{\beta S(x,t)I(x,t)}{1+\alpha I(x,t)},\ x\in\mathbb{R},\ t>0, \tag{1.1}$$

$$I_t(x,t) = d_2\mathcal{N}_2[I(\cdot,t)](x) + \frac{\beta S(x,t)I(x,t)}{1+\alpha I(x,t)} - (\mu+\sigma)I(x,t),\ x\in\mathbb{R},\ t>0, \tag{1.2}$$

$$R_t(x,t) = d_3\mathcal{N}_3[R(\cdot,t)](x) + \sigma I(x,t) - \mu R(x,t),\ x\in\mathbb{R},\ t>0, \tag{1.3}$$

where $S(x,t), I(x,t), R(x,t)$ represent the population densities of the susceptible, infective, removed individuals at position x and time t, respectively. The parameters μ,β,α,σ are all positive constants in which μ denotes the death rates of susceptible, infective and removed populations. Also, after a suitable rescaling (cf. [21]), the inflow of newborns into the susceptible population is taken to be the same constant μ. The parameter σ is the removed/recovery

---

Date: October 6, 2019.

J.-S. Guo is partially supported by the Ministry of Science and Technology of Taiwan under the grant 105-2115-M-032-003-MY3. M. Shimojo is supported by JSPS KAKENHI Grant-in-Aid for Young Scientists (B) (No. 16K17634). A. Poh is supported by Research Grant for Encouragement of Students, Graduate School of Natural Science and Technology, Okayama University. We thank the referee for the valuable comment.

2000 Mathematics Subject Classification. Primary: 35K57, 92D30; Secondary: 34B40, 35R10.

Key words and phrases. Epidemic model, spreading speed, nonlocal dispersal.

rate, β is the infective transmission rate and α measures the saturation level ([6, 24]) in the Holling type II incidence function $\beta SI/(1+\alpha I)$.

To describe the nonlocal operators in (1.1) and (1.2), we first introduce the following class of kernels. For a given $\hat{\lambda} \in (0,\infty]$, a function $J:\mathbb{R}\to[0,\infty)$ is said to be in the class $\mathcal{P}(\hat{\lambda})$ if the following conditions hold:

(J1) The kernel J is continuous (and nonnegative);

(J2) it holds that
$$\int_{\mathbb{R}} J(y)dy = 1,\ J(y) = J(-y) \text{ for all } y\in\mathbb{R};$$

(J3) it holds that $\int_{\mathbb{R}} J(y)e^{\lambda y}dy < \infty$ for any $\lambda\in(0,\hat{\lambda})$ and
$$\int_{\mathbb{R}} J(y)e^{\lambda y}dy \to \infty \text{ as } \lambda \uparrow \hat{\lambda}.$$

Then the nonlocal diffusion operator $\mathcal{N}_i$ is defined by
$$\mathcal{N}_i[\varphi](x) := (J_i * \varphi)(x) - \varphi(x) = \int_{\mathbb{R}} J_i(x-y)\varphi(y)dy - \varphi(x),$$
where $J_i \in \mathcal{P}(\hat{\lambda}_i)$ for some constant $\hat{\lambda}_i \in (0,\infty]$ for $i = 1,2,3$.

Since equation (1.3) is decoupled from the other two equations in our SIR model, in the sequel we shall only consider the system for (1.1)-(1.2) only. The study of nonlocal evolution equations has attracted a lot of attention in past years, due to the fact that nonlocal interaction is often presented in many diffusive systems in ecology, biology, neuroscience and so on. There is a vast literature on the study of various problems with nonlocal dispersals, we refer the reader to, e.g., [15, 12, 14, 17, 7, 16, 5, 18, 2, 3, 29, 10, 1, 31, 8, 26, 11, 20] and the references cited therein.

The main purpose of this paper is to study the spreading speed of infective populations for model (1.1)-(1.2). Here the spreading speed is adopted from the definition defined by Aronson and Weinberger [4] for the scalar logistic parabolic equation. We also refer the reader to [18, 31] for the study of spreading speed for scalar nonlocal dispersal equations. For the spreading speed of local reaction diffusion systems, see, e.g., [22, 27, 28]. However, for nonlocal dispersal systems the spreading speeds have not yet been established so much.

To characterize the spreading speed of infective populations for (1.1)-(1.2), we consider the initial value problem for (1.1)-(1.2) with the following initial condition
$$S(x,0) = 1, \quad I(x,0) = I_0(x),\ x\in\mathbb{R}, \tag{1.4}$$
where I_0 is a nonnegative continuous function defined in $\mathbb{R}$ with a nonempty compact support.

Throughout this paper, we assume
$$\beta > \mu + \sigma. \tag{1.5}$$
Hereafter we set $\gamma := \mu+\sigma$ and define
$$c^* := \inf_{0<\lambda<\hat{\lambda}_2} \frac{d_2\left[\int_{\mathbb{R}} J_2(y)e^{\lambda y}dy - 1\right] + \beta - \gamma}{\lambda}. \tag{1.6}$$
Note that the constant c^* is well-defined and $c^* > 0$, since $\beta - \gamma > 0$ due to (1.5).

We now state the main theorem of this paper as follows.

Theorem 1.1. *Let (S, I) be a solution of (1.1), (1.2) and (1.4) with a nonnegative nontrivial compactly supported continuous initial data I_0. Assume the condition (1.5) is enforced. Then the constant c^* defined in (1.6) is the (asymptotic) spreading speed of I in the sense*

$$\lim_{t\to\infty} \sup_{|x|>ct} I(x,t) = 0,\ \forall\, c > c^*, \tag{1.7}$$

$$\liminf_{t\to\infty} \inf_{|x|<ct} I(x,t) > 0,\ \forall\, c \in (0, c^*). \tag{1.8}$$

To the best of our knowledge, little works are done on the spreading speeds for nonlocal reaction diffusion systems. Although our method is based on a method used in [28] for a local diffusion system, there are major differences from [28] due to the nonlocal diffusion in our system. Moreover, Holling type II incidence function makes the analysis nontrivial. One of the major difficulties in our study is the derivation of some useful a priori estimates and this is overcome by applying the fundamental solution of the nonlocal linear operator. One should note the independence of the constants to any given point in the course of deriving the key estimate (3.6) in §3. Also, the uniform continuity of I (from [18, 20]) is crucial to the proof of our main theorem.

The rest of this paper is organized as follows. In the next section, we recall some well-known theories on the scalar nonlocal diffusion equation from [17, 18, 20] and give some a priori estimates to solutions of (1.1), (1.2) and (1.4). Then Theorem 1.1 is proved in §3. Finally, a brief discussion is given in §4.

2. Preliminaries

Let (S, I) be a solution of (1.1)-(1.2) with the initial condition (1.4). The purpose of this section is to derive some a priori estimates. Hereafter we use the notation

$$\begin{aligned} X &:= \{\text{all uniformly continuous bounded functions defined in } \mathbb{R}\}, \\ X^+ &:= \{w \in X \mid w \geq 0 \ \text{ in } \mathbb{R}\}, \\ X_b &:= \{w \in X \mid 0 \leq w \leq b \ \text{ in } \mathbb{R}\},\ b > 0. \end{aligned}$$

Let $\hat\lambda \in (0, \infty]$ and a kernel $J \in \mathcal{P}(\hat\lambda)$ be given. Let $w_0 \in X^+$ and let w be a solution to

$$\begin{cases} w_t(x,t) = d\mathcal{N}[w(\cdot,t)](x),\ x \in \mathbb{R},\ t > 0, \\ w(x,0) = w_0(x),\ x \in \mathbb{R}, \end{cases}$$

where $\mathcal{N}[w] := J * w - w$. Then $w(\cdot,t) \in X^+$ for all $t > 0$, by the positivity property of the semigroup $\{\exp(td\mathcal{N})\}_{t\geq 0}$ generated by $\mathcal{N}$ (cf. [17, 18, 20]). Moreover, this positivity property also holds for

$$w_t(x,t) = d\mathcal{N}[w(\cdot,t)](x) - Lw(x,t),\ x \in \mathbb{R},\ t > 0, \tag{2.1}$$

where L is a constant. Indeed, (2.1) can be re-written as

$$[e^{Lt}w]_t(x,t) = d\mathcal{N}[e^{Lt}w(\cdot,t)](x),\ x \in \mathbb{R},\ t > 0.$$

Hence $w(\cdot,t) \in X^+$ for all $t > 0$, if $w(\cdot,0) \in X^+$.

We now derive the following a priori estimates for solutions (S, I) to system (1.1)-(1.2).

Lemma 2.1. *Let I_0 be a nonnegative nontrivial compactly supported continuous function defined in $\mathbb{R}$. Then system* (1.1)-(1.2) *with initial condition* (1.4) *admits a global solution (S, I) such that*

$$0 \le S(x,t) \le 1, \quad 0 \le I(x,t) \le \kappa, \ \forall\, x \in \mathbb{R},\ t > 0, \tag{2.2}$$

where $\kappa := \max\{b, (\beta-\gamma)/(\alpha\gamma)\}$ and $b := \|I_0\|_{L^\infty(\mathbb{R})} \in (0,\infty)$.

Proof. It is clear that a local (in time) solution (S, I) of (1.1)-(1.2) and (1.4) exists for $t < T$ for some $T \le \infty$. Hence $S(\cdot,t), I(\cdot,t) \in X$ such that $S \in \mathbb{R}$ and $I \in (-1/\alpha, \infty)$ for all $t < T$.

First, we claim that $I(x,t) \ge 0$ for all $x \in \mathbb{R}$, $t \in (0,T)$. Indeed, we first re-write (1.2) as

$$I_t(x,t) = d_2\mathcal{N}_2[I(\cdot,t)](x) + \left(\frac{\beta S(x,t)}{1+\alpha I(x,t)} - \gamma\right) I(x,t),\ x \in \mathbb{R},\ t < T.$$

Then, for any $\tau \in (0,T)$, since

$$\left(\frac{\beta S(x,t)}{1+\alpha I(x,t)} - \gamma\right) \ge -L,\ \forall\, x \in \mathbb{R},\ t < \tau,$$

we apply the comparison principle, [18, Theorem 2.3], along with the above positivity property to obtain that $I(x,t) \ge 0$ for $x \in \mathbb{R}$, $t < \tau$. Since τ is arbitrary, we conclude that $I(x,t) \ge 0$ for $x \in \mathbb{R}$, $t < T$.

Next, we claim that $S \ge 0$ in $\mathbb{R} \times [0,T)$. To see this, we observe from (1.1) that S satisfies

$$S_t(x,t) > d_1\mathcal{N}_1[S(\cdot,t)](x) - \left(\mu + \frac{\beta I(x,t)}{1+\alpha I(x,t)}\right) S(x,t),\ x \in \mathbb{R},\ t < T.$$

Hence, by the same argument as above, $S(x,t) \ge 0$ for all $x \in \mathbb{R}$, $t < T$.

To claim $S \le 1$ in $\mathbb{R} \times [0,T)$, we set $\tilde{S} := 1 - S$. Then, by (1.1), $\tilde{S}$ satisfies

$$\tilde{S}_t(x,t) \ge d_1\mathcal{N}_1[\tilde{S}(\cdot,t)]x - \mu\tilde{S}(x,t),\ x \in \mathbb{R},\ t < T.$$

This implies that $\tilde{S}(x,t) \ge 0$ for all $x \in \mathbb{R}$, $t < T$. Hence $S \le 1$ in $\mathbb{R} \times [0,T)$.

Finally, we prove $I(x,t) \le \kappa$ for $x \in \mathbb{R}$, $t < T$. To this aim, using $S \le 1$ and $I \ge 0$ in $\mathbb{R} \times [0,T)$, from (1.2) it follows that

$$\begin{aligned} I_t(x,t) &\le d_2\mathcal{N}_2[I(\cdot,t)](x) + \frac{I(x,t)}{1+\alpha I(x,t)}\{\beta - \gamma[1+\alpha I(x,t)]\} \\ &\le d_2\mathcal{N}_2[I(\cdot,t)](x) + I(x,t)[\beta - \gamma - \gamma\alpha I(x,t)],\ x \in \mathbb{R},\ t < T. \end{aligned}$$

Note that the solution $\hat{I}(t)$ to the equation

$$\hat{I}_t = \hat{I}(\beta - \gamma - \gamma\alpha\hat{I}),\ t < T,$$

with initial data $\hat{I}(0) = b$ satisfies $\hat{I}(t) \le \kappa$ for all $t < T$. Here the fact that $\alpha > 0$ was used. Then the comparison principle [18, Theorem 2.3] implies that $I(x,t) \le \kappa$ for $x \in \mathbb{R}$, $t < T$. We conclude that (2.2) holds for all $t < T$. Hence we must have $T = \infty$ and the lemma is proved. □

Next, we recall some properties of the strongly positive semigroup $\{\exp(td_1\mathcal{N}_1)\}_{t\geq 0}$ from [17, 20] as follows. First, the fundamental solution W of this semigroup can be decomposed as

$$W(x,t) = e^{-d_1 t}\delta_0(x) + K(x,t),\ x \in \mathbb{R},\ t \geq 0, \tag{2.3}$$

where K is a nonnegative smooth function satisfying the estimate

$$\int_{\mathbb{R}} K(x,t)dx \leq 2,\ \forall t \geq 0. \tag{2.4}$$

In fact, W is the solution of the problem

$$w_t(x,t) = d_1\mathcal{N}_1[w(\cdot,t)](x),\ t > 0,\ w(\cdot,0) = \delta_0,$$

where δ_0 denotes the Dirac mass at $x = 0$.

Now, consider the solution w to the problem

$$\begin{cases} w_t(x,t) = d_1\mathcal{N}_1[w(\cdot,t)](x) - kw(x,t) + f(x,t),\ x \in \mathbb{R},\ t > 0, \\ w(x,0) = w_0(x),\ x \in \mathbb{R}. \end{cases} \tag{2.5}$$

Re-writing (2.5) as

$$\begin{cases} [e^{kt}w]_t(x,t) = d_1\mathcal{N}_1[e^{kt}w(\cdot,t)](x) + e^{kt}f(x,t),\ x \in \mathbb{R},\ t > 0, \\ w(x,0) = w_0(x),\ x \in \mathbb{R}, \end{cases}$$

then, by (2.3), w can be expressed as

$$\begin{aligned} w(x,t) &= e^{-(d_1+k)t}w_0(x) + e^{-kt}\int_{\mathbb{R}} K(x-y,t)w_0(y)dy \\ &\quad + \int_0^t e^{-(d_1+k)(t-s)}f(x,s)ds + \int_0^t\int_{\mathbb{R}} e^{-k(t-s)}K(x-y,t-s)f(y,s)dyds. \end{aligned} \tag{2.6}$$

This formula will be useful in the next section.

3. Proof of Theorem 1.1

Since system (1.1)-(1.2) does not admit a comparison principle, to prove Theorem 1.1 we apply the comparison principle of the scalar nonlocal equation as follows.

Let d, a, b be given positive constants. We consider the following nonlocal logistic equation

$$\begin{cases} w_t(x,t) = d\mathcal{N}[w(\cdot,t)](x) + aw(x,t)\left[b - w(x,t)\right],\ x \in \mathbb{R},\ t > 0, \\ w(x,0) = w_0(x),\ x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where as before $\mathcal{N}[w] = J * w - w$ with $J \in \mathcal{P}(\hat{\lambda})$ for some $\hat{\lambda} \in (0,\infty]$. Then we have the following comparison principle (cf. [18, Theorem 2.3]).

Proposition 3.1. *Let u be a super-solution and v be a sub-solution of (3.1) with $u(\cdot,t) \in X_b$ and $v(\cdot,t) \in X_b$ for all $t > 0$, in the sense*

$$u_t(x,t) \geq d\mathcal{N}[u(\cdot,t)](x) + au(x,t)\left[b - u(x,t)\right],\ x \in \mathbb{R},\ t > 0,$$
$$v_t(x,t) \leq d\mathcal{N}[v(\cdot,t)](x) + av(x,t)\left[b - v(x,t)\right],\ x \in \mathbb{R},\ t > 0,$$

such that $v(x,0) \le u(x,0)$ for all $x \in \mathbb{R}$. Then $v(x,t) \le u(x,t)$ for all $x \in \mathbb{R}$, $t > 0$.

Next, we define the quantity $\hat{c}$ by

$$\hat{c} := \inf_{0<\lambda<\hat{\lambda}} \frac{d\left[\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1\right] + ab}{\lambda}.$$

Then $\hat{c}$ is well-defined and $\hat{c} > 0$ since $ab > 0$. Moreover, we have

Proposition 3.2 ([18])**.** *Let w be a solution of (3.1) with $w(\cdot,t) \in X_b$ for all $t > 0$ for a given $w_0 \in X_b$. Assume that w_0 has a nonempty compact support. Then we have*

$$\lim_{t\to\infty} \inf_{|x|<ct} w(x,t) = b \quad \text{for any } c \in (0,\hat{c}), \tag{3.2}$$

$$\lim_{t\to\infty} \sup_{|x|>ct} w(x,t) = 0 \quad \text{for any } c > \hat{c}. \tag{3.3}$$

Now, we are ready to give a proof of our main theorem as follows.

Proof of Theorem 1.1. The proof of (1.7) is trivial. We observe from (1.2) that

$$\begin{aligned} I_t(x,t) &\le d_2\mathcal{N}_2[I(\cdot,t)](x) + I(x,t)\left\{\frac{\beta}{1+\alpha I(x,t)} - \gamma[1+\alpha I(x,t)]\right\} \\ &\le d_2\mathcal{N}_2[I(\cdot,t)](x) + I(x,t)[\beta - \gamma - \gamma\alpha I(x,t)],\ x \in \mathbb{R},\ t > 0, \end{aligned}$$

using $S \le 1$ and $I \ge 0$. Hence (1.7) follows from Proposition 3.2.

To derive (1.8), we first rewrite (1.1) for $\tilde{S} := 1 - S$ as

$$\begin{aligned} \tilde{S}_t(x,t) &= d_1\mathcal{N}_1[\tilde{S}(\cdot,t)](x) - \mu\tilde{S}(x,t) + \frac{\beta S(x,t)I(x,t)}{1+\alpha I(x,t)}, \\ &\le d_1\mathcal{N}_1[\tilde{S}(\cdot,t)](x) - \mu\tilde{S}(x,t) + \beta I(x,t),\ x \in \mathbb{R},\ t > 0, \end{aligned} \tag{3.4}$$

using $S \le 1$ and $I \ge 0$ in $\mathbb{R}$. The comparison gives $\tilde{S}(x,t) \le \hat{s}(x,t)$ for all $x \in \mathbb{R}$, $t > 0$, where $\hat{s}$ satisfies

$$\hat{s}_t(x,t) = d_1\mathcal{N}_1[\hat{s}(\cdot,t)](x) - \mu\hat{s}(x,t) + \beta I(x,t),\ x \in \mathbb{R},\ t > 0,$$

and $\hat{s}(x,0) = \tilde{S}(x,0) = 0$ for all $x \in \mathbb{R}$. It follows from (2.6) that

$$\hat{s}(x,t) = \beta \int_0^t e^{-(d_1+\mu)(t-s)} I(x,s)ds + \beta \int_0^t \int_{\mathbb{R}} e^{-\mu(t-s)} K(x-y,t-s)I(y,s)dyds. \tag{3.5}$$

Next, given $\varepsilon \in (0,c^*)$. We choose a constant $\delta \in (0,\beta-\gamma)$ small enough such that

$$\inf_{0<\lambda<\hat{\lambda}_2} \frac{d_2\left[\int_{\mathbb{R}} J_2(y)e^{\lambda y}dy - 1\right] + \beta - \gamma - \delta}{\lambda} > c^* - \varepsilon.$$

For this δ, we claim that there is a sufficiently large τ such that

$$\beta\hat{s}(x,t) \le \delta + QI(x,t),\ x \in \mathbb{R},\ t \ge \tau, \tag{3.6}$$

for some positive constant Q.

To derive (3.6), we first choose

$$\tau = \tau(\delta) = \max\left\{\frac{1}{d_1+\mu}\log\left(\frac{4\beta\kappa}{(d_1+\mu)\delta}\right), \frac{1}{\mu}\log\left(\frac{8\beta\kappa}{\mu\delta}\right)\right\}.$$

Then by a simple calculation, we obtain

$$\beta\kappa\int_0^{t-\tau} e^{-(d_1+\mu)(t-s)}ds \le \frac{\delta}{4}, \quad 2\beta\kappa\int_0^{t-\tau} e^{-\mu(t-s)}ds \le \frac{\delta}{4},\ \forall\, t \ge \tau,$$

where κ is define in (2.2).

Now, given any point (x_0, t_0) with $t_0 \ge \tau$. It is trivial that (3.6) holds for (x_0, t_0), when $\beta\hat{s}(x_0, t_0) \le \delta$. Suppose that $\beta\hat{s}(x_0, t_0) > \delta$. It follows from (2.2), (2.4) and (3.5) that

$$\begin{aligned} \beta\hat{s}(x_0,t_0) \;\le\; & \beta\kappa\int_0^{t_0-\tau} e^{-(d_1+\mu)(t_0-s)}ds + 2\beta\kappa\int_0^{t_0-\tau} e^{-\mu(t_0-s)}ds \\ & +\beta\int_{t_0-\tau}^{t_0} e^{-(d_1+\mu)(t_0-s)}I(x_0,s)ds \\ & +\beta\int_{t_0-\tau}^{t_0}\int_{\mathbb{R}} e^{-\mu(t_0-s)}K(x_0-y,t_0-s)I(y,s)dyds. \end{aligned} \tag{3.7}$$

Hence we have

$$\beta\int_{t_0-\tau}^{t_0} e^{-(d_1+\mu)(t_0-s)}I(x_0,s)ds + \beta\int_{t_0-\tau}^{t_0}\int_{\mathbb{R}} e^{-\mu(t_0-s)}K(x_0-y,t_0-s)I(y,s)dyds \ge \frac{\delta}{2}.$$

Next, using (2.4) we can choose a constant R with $R \gg 1$ such that

$$\beta\kappa\int_0^{\infty}\int_{|y|\ge R} e^{-\mu t}K(y,t)dydt \le \frac{\delta}{4}.$$

Then we obtain

$$\begin{aligned} & \beta\int_{t_0-\tau}^{t_0}\int_{|x_0-y|\ge R} e^{-\mu(t_0-s)}K(x_0-y,t_0-s)I(y,s)dyds \\ \le\; & \beta\kappa\int_0^{\tau}\int_{|y|\ge R} e^{-\mu t}K(y,t)dydt \le \frac{\delta}{4} \end{aligned}$$

and so

$$\beta\int_{t_0-\tau}^{t_0} e^{-(d_1+\mu)(t_0-s)}I(x_0,s)ds + \beta\int_{t_0-\tau}^{t_0}\int_{-R}^{R} e^{-\mu(t_0-s)}K(x_0-y,t_0-s)I(y,s)dyds \ge \frac{\delta}{4}.$$

Moreover, by choosing a constant $\eta \in (0, \tau)$ small enough such that

$$\beta\kappa\left\{\int_0^{\eta} e^{-(d_1+\mu)t}dt + \int_0^{\eta}\int_{-R}^{R} e^{-\mu t}K(y,t)dydt\right\} \le \frac{\delta}{8},$$

we get

$$\begin{aligned} & \beta\int_{t_0-\tau}^{t_0-\eta} e^{-(d_1+\mu)(t_0-s)}I(x_0,s)ds \\ & +\beta\int_{t_0-\tau}^{t_0-\eta}\int_{-R}^{R} e^{-\mu(t_0-s)}K(x_0-y,t_0-s)I(y,s)dyds \ge \frac{\delta}{8}. \end{aligned} \tag{3.8}$$

It follows that there exist a positive constant ν and a point

$$(y_0, s_0) \in [x_0 - R, x_0 + R] \times [t_0 - \tau, t_0 - \eta]$$

such that $I(y_0, s_0) \ge \nu$. Note that ν is independent of (x_0, t_0).

Now, recall from [18, 20] that $I(\cdot,t)$ is uniformly continuous for each $t \geq 0$. Furthermore, by Lemma 2.1 and (1.2), the time derivative of I is bounded from $[0,\infty)$ to X. Hence I is uniformly continuous on $\mathbb{R}\times[0,\infty)$. Hence there is a positive constant ρ such that

$$I(y,s_0) \geq \frac{\nu}{2},\ \forall\, y \in [y_0-\rho, y_0+\rho].$$

Then we consider the solution z to

$$\begin{cases} z_t(y,s) = d_2\mathcal{N}_2[z(\cdot,s)](y) - \gamma z(x,t),\ y\in\mathbb{R},\ s>s_0,\\ z(y,s_0) = \underline{I}(y),\ y\in\mathbb{R}, \end{cases}$$

where $\underline{I}$ is a uniformly continuous nonnegative function defined in $\mathbb{R}$ such that $\underline{I}\leq \nu/2$ in $\mathbb{R}$ and

$$\underline{I}(y) = \nu/2,\ \forall\, y\in[y_0-\rho/2, y_0+\rho/2],\quad \underline{I}(y)=0,\ \forall\,|y-y_0|\geq\rho.$$

Note that $z(y,s)>0$ for all $y\in\mathbb{R}$ for all $s>s_0$ (cf. [18]). Hence the constant χ defined by

$$\chi := \min\{z(y,s)\mid y\in[y_0-R,y_0+R],\ s\in[s_0+\eta,s_0+\tau]\}$$

is positive. Moreover, by comparison, we have $I(y,s)\geq z(y,s)$ for $y\in\mathbb{R}$ and $s\geq s_0$, since I satisfies $I(y,s_0)\geq \underline{I}(y)$ for all $y\in\mathbb{R}$ and

$$I_t(y,s) \geq d_2\mathcal{N}_2[I(\cdot,s)](y) - \gamma I(y,s),\ y\in\mathbb{R},\ s>0.$$

In particular, we have $I(x_0,t_0)\geq z(x_0,t_0)\geq\chi$, since $(x_0,t_0)\in[y_0-R,y_0+R]\times[s_0+\eta,s_0+\tau]$. We conclude that (3.6) holds with the constant $Q := 3\beta\kappa/(\nu\chi)$, since we have

$$\begin{aligned} \beta\hat{s}(x_0,t_0) &\leq \frac{\delta}{2}+\frac{\delta}{4}+\frac{\delta}{8}+\beta\int_{t_0-\tau}^{t_0-\eta} e^{-(d_1+\mu)(t_0-s)}I(x_0,s)ds\\ &\quad +\beta\int_{t_0-\tau}^{t_0-\eta}\int_{-R}^{R} e^{-\mu(t_0-s)}K(x_0-y,t_0-s)I(y,s)dyds\\ &\leq \delta+3\beta\kappa/\nu \leq \delta + QI(x_0,t_0). \end{aligned}$$

With (3.6) at hand and using $S=1-\tilde{S}$, from (1.2) it follows that

$$\begin{aligned} I_t(x,t) &= d_2\mathcal{N}_2[I(\cdot,t)](x) + \frac{I(x,t)}{1+\alpha I(x,t)}\left\{\beta[1-\tilde{S}(x,t)]-\gamma-\alpha\gamma I(x,t)\right\}\\ &\geq d_2\mathcal{N}_2[I(\cdot,t)](x) + \frac{I(x,t)}{1+\alpha I(x,t)}\left\{\beta-\gamma-[\delta+QI(x,t)]-\alpha\gamma I(x,t)\right\} \end{aligned} \tag{3.9}$$

for $x\in\mathbb{R}$, $t\geq\tau$. Since $(1-\alpha I)\leq 1/(1+\alpha I)\leq 1$ and $\beta-\gamma-\delta>0$, we compute

$$\begin{aligned} &\frac{I(x,t)}{1+\alpha I(x,t)}\left\{\beta-\gamma-[\delta+QI(x,t)]-\alpha\gamma I(x,t)\right\}\\ =\ &\frac{I(x,t)}{1+\alpha I(x,t)}(\beta-\gamma-\delta) - \frac{I(x,t)}{1+\alpha I(x,t)}(Q+\alpha\gamma)I(x,t)\\ \geq\ & I(x,t)[1-\alpha I(x,t)](\beta-\gamma-\delta) - I(x,t)(Q+\alpha\gamma)I(x,t)\\ =\ & I(x,t)\left\{(\beta-\gamma-\delta)-MI(x,t)\right\}, \end{aligned}$$

where $M := [\alpha(\beta-\gamma-\delta)+Q+\alpha\gamma] > 0$. Hence (3.9) can be re-written as

$$I_t(x,t) \geq d_2\mathcal{N}_2[I(\cdot,t)](x) + I(x,t)\left\{(\beta-\gamma-\delta) - MI(x,t)\right\},\ x\in\mathbb{R},\ t\geq\tau. \tag{3.10}$$

Finally, applying Propositions 3.1 and 3.2, we conclude that

$$\liminf_{t\to\infty}\inf_{|x|<c^*-\varepsilon} I(x,t) \geq \lim_{t\to\infty}\inf_{|x|<c^*-\varepsilon}\hat{I}(x,t) = \frac{\beta-\gamma-\delta}{M} > 0,$$

where $\hat{I}$ is a solution to the problem

$$\begin{cases}\hat{I}_t(x,t) \geq d_2\mathcal{N}_2[\hat{I}(\cdot,t)](x) + \hat{I}(x,t)\left\{(\beta-\gamma-\delta) - M\hat{I}(x,t)\right\},\ x\in\mathbb{R},\ t>\tau,\\ \hat{I}(x,\tau) = I(x,\tau),\ x\in\mathbb{R}.\end{cases}$$

This completes the proof of Theorem 1.1. □

4. Discussion

Surprisingly, we could not find any works on the spreading speed for SIR models. Our analysis for the spreading speed on the case $\alpha>0$ relies on the global boundedness of the infective population. In fact, replacing K in (2.6) by the Gaussian kernel, Theorem 1.1 also holds for the standard diffusion case, namely, for the system

$$S_t(x,t) = d_1\Delta S(x,t) + \mu - \mu S(x,t) - \frac{\beta S(x,t)I(x,t)}{1+\alpha I(x,t)},\ x\in\mathbb{R},\ t>0, \tag{4.1}$$

$$I_t(x,t) = d_2\Delta I(x,t) + \frac{\beta S(x,t)I(x,t)}{1+\alpha I(x,t)} - (\mu+\sigma)I(x,t),\ x\in\mathbb{R},\ t>0. \tag{4.2}$$

The proof can be done by a completely similar argument as above, since we have the comparison principle and the spreading result ([4]) for the corresponding scalar equation with standard diffusion.

On the other hand, when $\alpha=0$, if we can derive the global boundedness of the infective population as that in Lemma 2.1, then our method works and Theorem 1.1 holds for both nonlocal dispersal and standard diffusion cases. In particular, when $d_2=d_1$ and $J_2=J_1$, by adding (1.1) and (1.2) we obtain

$$(S+I)_t \leq d_1\mathcal{N}_1[(S+I)(\cdot,t)](x) + \mu - \mu(S+I)(x,t).$$

From this inequality and the comparison principle it follows that

$$(S+I)(x,t) \leq 1 - (1-\|S_0+I_0\|_{L^\infty(\mathbb{R})})e^{-\mu t}$$

for all $x\in\mathbb{R}$ and $t\geq 0$. Of course, this also works for the standard diffusion case.

Finally, we remark that any solution to the ODE system

$$S_t = \mu - \mu S - \beta SI,$$
$$I_t = \beta SI - (\mu+\sigma)I,$$

is uniformly bounded and converges to the positive endemic equilibrium, because we have a Lyapunov function as is explained in [24]. However, the Lyapunov functional is not well-defined for PDE system (4.1)-(4.2) (or (1.1)-(1.2)) with $\alpha=0$. Also, the comparison principle does not hold for both systems (4.1)-(4.2) and (1.1)-(1.2). Hence the boundedness of the solution to the Cauchy problem when $\alpha = 0$ does not follow immediately. We leave the problem on the spreading speed for the endemic model with $\alpha = 0$ as an open problem.

References

[1] M. Alfaro, J. Coville, *Rapid traveling waves in the nonlocal Fisher equation connect two unstable states*, Appl. Math. Letters, 25 (2012), 2095–2099.
[2] F. Andreu-Vaillo, J. Mazón, J. Rossi, J. Toledo-Melero, *Nonlocal diffusion problems*, Mathematical Surveys and Monographs, 165, American Mathematical Society, Providence, RI, 2010.
[3] N. Apreutesei, N. Bessonov, V. Volpert, V. Vougalter, *Spatial structures and generalized travelling waves for an integro-differential equation*, Disc. Cont. Dyn. Syst. Ser. B, 13 (2010), 537–557.
[4] D.G. Aronson, H.F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, in: J.A. Goldstein(Ed.), Partial Differential Equations and Related Topics, in: Lecture Notes in Math., vol. 446, Springer, Berlin, 1975, pp. 5–49.
[5] H. Berestycki, G. Nadin, B. Perthame, L. Ryzhik, *The non-local Fisher-KPP equation: traveling waves and steady states*, Nonlinearity, 22 (2009), 2813–2844.
[6] V. Capasso, G. Serio, *A generalization of the Kermack-Mckendrick deterministic epidemic model*, Math. Biosci., 42 (1978), 41–61.
[7] J. Coville, J. Dávila, S. Martinez, *Existence and uniqueness of solutions to a nonlocal equation with monostable nonlinearity*, SIAM J. Math. Anal., 39 (2008), 1693–1709.
[8] J. Coville, J. Dávila, S. Martinez, *Pulsating fronts for nonlocal dispersion and KPP nonlinearity*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 30 (2013), 179–223.
[9] P. De Mottoni, E. Orlandi, A. Tesei, *Asymptotic behavior for a system describing epidemics with migration and spatial spread of infection*, Nonlinear Anal., 3 (1979), 663–675.
[10] J. Fang, X.-Q. Zhao, *Monotone wavefronts of the nonlocal Fisher-KPP equation*, Nonlinearity, 24 (2011), 3043–3054.
[11] G. Faye, M. Holzer, *Modulated traveling fronts for a nonlocal Fisher-KPP equation: a dynamical systems approach*, J. Diff. Equations, 258 (2015), 2257–2289.
[12] S. Genieys, V. Volpert, P. Auger, *Pattern and waves for a model in population dynamics with nonlocal consumption of resources*, Math. Model. Nat. Phenom., 1 (2006), 63–80.
[13] H.W. Hethcote, *The mathematics of infectious diseases*, SIAM Review, 42 (2000), 599–653.
[14] V. Hutson, M. Grinfeld, *Non-local dispersal and bistability*, Euro. J. Appl. Math., 17 (2006), 221–232.
[15] V. Hutson, S. Martinez, K. Mischailow, G. T. Vickers, *The evolution of dispersal*, J. Math. Biol., 47 (2003), 483–517.
[16] V. Hutson, W. Shen, G. T. Vickers, *Spectral theory for nonlocal dispersal with periodic or almost-periodic time dependence*, Rocky Mountain J. Math., 38 (2008), 1147–1175.
[17] L.I. Ignat, J.D. Rossi, *A nonlocal convection-diffusion equation*, J. Funct. Anal., 251 (2007), 399–437.
[18] Y. Jin, X.-Q. Zhao, *Spatial dynamics of a periodic population model with dispersal*, Nonlinearity, 22 (2009), 1167–1189.
[19] W. O. Kermack, A. G. McKendrick, *Contributions to the mathematical theory of epidemics, part 1*, Proc. Roy. Soc. London Ser. A, 115 (1927), 700–721.
[20] W.-T. Li, J.B. Wang, X.-Q. Zhao, *Spatial dynamics of a nonlocal dispersal population model in a shifting environment*, J. Nonlinear Sci., https://doi.org/10.1007/s00332-018-9445-2
[21] Y. Li, W.-T. Li, G. Lin, *Traveling waves of a delayed diffusive SIR epidemic model*, Comm. Pure Appl. Anal., 14 (2015), 1001–1022.
[22] G. Lin, *Spreading speeds of a Lotka-Volterra predator-prey system: the role of the predator*, Nonlinear Anal., 74 (2011), 2448–2461.
[23] F. Lutscher, E. Pachepsky, and M. A. Lewis, *The effect of dispersal patterns on stream populations*, SIAM J. Appl. Math., 65 (2005), 1305–1327.

[24] C. McCluskey, *Global stability for an SIR epidemic model with delay and nonlinear incidence*, Nonlinear Anal. Real Word Appl., 11 (2010), 3106–3109.

[25] J. Medlock, M. Kot, *Spreading disease: Integro-differential equations old and new*, Math. Biosci., 184 (2003), 201–222.

[26] G. Nadin, L. Rossi, L. Ryzhik, B. Perthame, *Wave-like solutions for nonlocal reaction-diffusion equations: a toy model*, Math. Model. Nat. Phenom., 8 (2013), 33–41.

[27] S. Pan, *Asymptotic spreading in a Lotka-Volterra predator-prey system*, J. Math. Anal. Appl., 407 (2013), 230–236.

[28] S. Pan, *Invasion speed of a predator-prey system*, Appl. Math. Lett., 74 (2017), 46–51.

[29] W. Shen, A. Zhang, *Spreading speeds for monostable equations with nonlocal dispersal in space periodic habitats*, J. Differential Equations, 15 (2010), 747–795.

[30] F.-Y. Yang, Y. Li, W.-T. Li, Z.-C. Wang, *Traveling waves in a nonlocal dispersal Kermack-McKendrick epidemic model*, Disc. Cont. Dyn. Syst., Ser. B, 18 (2013), 1969–1993.

[31] G.-B. Zhang, W.-T. Li, Z.-C. Wang, *Spreading speeds and traveling waves for nonlocal dispersal equations with degenerate monostable nonlinearity*, J. Differential Equations, 252 (2012), 5096–5124.

(J.-S. Guo) Department of Mathematics, Tamkang University, Tamsui, New Taipei City, Taiwan

Email address: jsguo@mail.tku.edu.tw

(A. Poh) Department of Mathematics, Okayama University, Okayama, Japan, and Meiji Institute for Advanced Study of Mathematical Sciences, Meiji University, Tokyo, Japan

Email address: amypoh.al@s.okayama-u.ac.jp

(M. Shimojo) Department of Applied Mathematics, Okayama University of Science, Okayama, Japan

Email address: shimojo@xmath.ous.ac.jp