PRECISE ASYMPTOTIC SPREADING BEHAVIOR FOR AN EPIDEMIC MODEL WITH NONLOCAL DISPERSAL

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ABSTRACT. This paper is to derive the precise asymptotic spreading behavior for an epidemic model with nonlocal dispersal. The proof is based on a Liouville type theorem on the positive bounded entire solutions. This Liouville theorem holds for a general class of reaction-diffusion systems with nonlocal dispersal which can be useful for reaction-diffusion systems arising in ecology and epidemiology.

1. INTRODUCTION

In this paper, we consider the following SIR (susceptible-infective-removed) epidemic model with nonlocal dispersal

$$S_t(x,t) = d_1\mathcal{N}_1[S(\cdot,t)](x) + \mu - \mu S(x,t) - \frac{\beta S(x,t)I(x,t)}{1+\alpha I(x,t)},\ x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$I_t(x,t) = d_2\mathcal{N}_2[I(\cdot,t)](x) + \frac{\beta S(x,t)I(x,t)}{1+\alpha I(x,t)} - (\mu+\sigma)I(x,t),\ x \in \mathbb{R},\ t > 0, \tag{1.2}$$

$$R_t(x,t) = d_3\mathcal{N}_3[R(\cdot,t)](x) + \sigma I(x,t) - \mu R(x,t),\ x \in \mathbb{R},\ t > 0, \tag{1.3}$$

where $S(x,t), I(x,t), R(x,t)$ represent the population densities of the susceptible, infective, removed individuals at position x and time t, respectively. The parameters $d_1, d_2, d_3, \mu, \beta, \sigma$ are all positive constants in which d_i is the diffusion coefficient, $i = 1, 2, 3$, and μ denotes the same death rates of susceptible, infective and removed populations. Also, after a suitable rescaling (cf. [21]), the inflow of newborns into the susceptible population is taken to be the same constant μ. The parameter σ is the removed/recovery rate and β is the infective transmission rate. While the *nonnegative* constant α measures the saturation level ([5, 23]) in the Holling type II incidence function $\beta SI/(1+\alpha I)$.

Moreover, the nonlocal dispersal $\mathcal{N}_i$ is an operator acting on a function φ defined by

$$\mathcal{N}_i[\varphi](x) := (J_i * \varphi)(x) - \varphi(x) = \int_{\mathbb{R}} J_i(x-y)\varphi(y)dy - \varphi(x),\ x \in \mathbb{R},$$

where the kernel J_i is a probability density function, $i = 1, 2, 3$. Throughout this paper, we adopt the following class of kernels. For a given $\hat{\lambda} \in (0, \infty]$, a function $J : \mathbb{R} \to [0, \infty)$ is said to be in the class $\mathcal{P}(\hat{\lambda})$ if the following conditions hold:

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(J1) The kernel J is nonnegative and continuous;

(J2) it holds that

$$\int_{\mathbb{R}} J(y)dy = 1,\ J(y) = J(-y) \text{ for all } y \in \mathbb{R};$$

(J3) it holds that $\int_{\mathbb{R}} J(y)e^{\lambda|y|}dy < \infty$ for any $\lambda \in (0, \hat{\lambda})$ and

$$\int_{\mathbb{R}} J(y)e^{\lambda|y|}dy \to \infty \text{ as } \lambda \uparrow \hat{\lambda}.$$

Unlike the classical diffusion modelling the random movements, the mechanism of nonlocal dispersal describes the individuals moving freely to have a long-range diffusion effect [22]. This nonlocal interaction nature is often presented in many diffusive systems in ecology, biology, neuroscience etc. Therefore, the study of nonlocal evolution equations has attracted a lot of attention in past years, we refer the reader to, e.g., [15, 10, 14, 17, 6, 16, 4, 18, 2, 3, 25, 8, 1, 27, 26, 7, 24, 9, 20, 11] and the references cited therein.

We are concerned with the precise asymptotic spreading behaviors of solutions to system (1.1)-(1.3). Since (1.3) is decoupled from the other two equations in our SIR model, in the sequel we shall only consider the system (1.1)-(1.2). In particular, we are interested in the initial value problem for (1.1)-(1.2) with the initial condition

$$S(x,0) = 1, \quad I(x,0) = I_0(x),\ x \in \mathbb{R}, \tag{1.4}$$

where I_0 is a nonnegative continuous function defined in $\mathbb{R}$ with a nonempty compact support.

Under the assumption

$$\beta > \mu + \sigma, \tag{1.5}$$

there is a unique stable positive endemic equilibrium (S^*, I^*), where

$$S^* = \frac{\mu + \sigma + \alpha\mu}{\alpha\mu + \beta}, \quad I^* = \frac{\mu(\beta - \mu - \sigma)}{(\mu + \sigma)(\alpha\mu + \beta)},$$

which corresponds to the coexistence state of (S, I). Hereafter we set $\gamma := \mu + \sigma$ and define

$$c^* := \inf_{0<\lambda<\hat{\lambda}_2} \frac{d_2\left[\int_{\mathbb{R}} J_2(y)e^{\lambda y}dy - 1\right] + \beta - \gamma}{\lambda}. \tag{1.6}$$

Note that the constant c^* is well-defined and $c^* > 0$, since $\beta - \gamma > 0$ due to (1.5).

We now state the main theorem of this paper as follows.

Theorem 1.1. *Let $\alpha \geq 0$ and $J_i \in \mathcal{P}(\hat{\lambda}_i)$ for some $\hat{\lambda}_i \in (0, \infty]$, $i = 1, 2$. Assume* (1.5). *In the case $\alpha = 0$, we further assume that $d_1 = d_2$ and $J_1 = J_2$. Let (S, I) be a solution of* (1.1), (1.2) *and* (1.4) *with a nonnegative nontrivial compactly supported continuous initial data I_0. Then*

$$\lim_{t\to\infty} \sup_{|x|\leq ct} \{|S(x,t) - S^*| + |I(x,t) - I^*|\} = 0,\ \forall\, c \in (0, c^*),$$

where c^* is defined by (1.6).

To prove Theorem 1.1, we present in this paper a Liouville type theorem (Theorem 2.1 below) to characterize entire solutions for more general reaction-diffusion systems including system (1.1)-(1.2) as a special case. Hereafter, a solution is called an entire solution if it is defined for all $t \in \mathbb{R}$. For the characterization of entire solutions in the study of the asymptotic behavior of the associated reaction-diffusion systems, we refer the reader to, e.g., the references cited in [12] for the case of classical diffusion and [13] for the fractional diffusion. In fact, the proof of Theorem 2.1 is quite similar to the one given in [13]. However, extending a Lyapunov function for ODE to a Lyapunov functional for PDE in an unbounded spatial domain relies on a suitable choice of the weight function. We are able to find such a weight function to overcome this difficulty. Consequently, Theorem 2.1 can be applied to a large class of systems in ecology and epidemiology such as those reaction-diffusion systems studied in [12, 13] with diffusions replaced by nonlocal dispersals.

The rest of this paper is organized as follows. We present a Liouville type theorem along with its proof in §2. Then we give the detailed proof of Theorem 1.1 for the precise asymptotic spreading behavior of system (1.1)-(1.2) in §3.

2. A Liouville type theorem

In this section, we consider the following general reaction-diffusion system

$$\frac{\partial u_i}{\partial t} = d_i \mathcal{N}_i[u_i(\cdot,t)](x) + f_i(u_1,\dots,u_m),\ x \in \mathbb{R}^n,\ t \in \mathbb{R},\ i = 1,\dots,m, \tag{2.1}$$

where m, n are positive integers, $d_i > 0$ and $f_i : \mathbb{R}^m \to \mathbb{R}$ is a C^1 function for each $i = 1,\dots,m$. Moreover, $J_i \in \mathcal{P}(\hat{\lambda}_i)$ for some constant $\hat{\lambda}_i \in (0,\infty]$ for $i = 1,\dots,m$. Note that conditions (J1)-(J3) are well-defined in $\mathbb{R}^n$ for $n \geq 1$.

We assume that (2.1) has a unique positive constant equilibrium $u^* := (u_1^*,\dots,u_m^*)$ such that $u_i^* \in (0,\infty)$ for each i. Set $g(\theta) := \theta - 1 - \ln\theta$, $\theta > 0$. Note that g is a strictly convex smooth function on $(0,\infty)$ such that $g(1) = 0$ and $g(\theta) > 0$ for all $\theta \neq 1$. Then we have the following Liouville type theorem for (2.1).

Theorem 2.1. *Let $u = (u_1,\dots,u_m)$ be an entire solution of (2.1) such that $0 < a_i \leq u_i \leq A_i < \infty$ for $i = 1,\dots,m$ for some positive constants $\{a_i, A_i\}$ with $a_i \leq u_i^* \leq A_i$. Suppose that the corresponding diffusion-free system of (2.1) admits a nonnegative bounded Lyapunov function in the form*

$$F(u) = \sum_{i=1}^{m} F_i(u_i),\ u = (u_1,\dots,u_m) \in \mathbb{R}_+^m,$$

where $F_i(u_i) = c_i g(u_i/u_i^*)$ for some positive constant c_i for $i = 1,\dots,m$ such that

$$\sum_{i=1}^{m} F_i'(u_i) f_i(u) \leq -\kappa F(u) \ \text{ for } u_i \in [a_i, A_i],\ 1 \leq i \leq m, \tag{2.2}$$

for some positive constant κ. Then $u = u^*$.

To prove Theorem 2.1, we first prepare the following lemma.

Lemma 2.2. *Let $J \in \mathcal{P}(\hat{\lambda})$ for some $\hat{\lambda} \in (0,\infty]$. Then for any $\varepsilon > 0$ there exists $R > 0$ sufficiently large such that*

$$\mathcal{N}[\rho_R](x) := (J * \rho_R)(x) - \rho_R(x) \le \varepsilon \rho_R(x), \quad \forall\, x \in \mathbb{R}^n,$$

where

$$\rho_R(x) := e^{-|x|/R}.$$

Proof. First, $J * e^{-|x|/R}$ is well-defined because of (J2).

Next, writing

$$\mathcal{N}[\rho_R](x) = \int_{\mathbb{R}^n} J(y)\left[e^{-|x-y|/R} - e^{-|x|/R}\right] dy = \rho_R(x) \int_{\mathbb{R}^n} J(y)\left[e^{|x|/R - |x-y|/R} - 1\right] dy$$

and using $|x| - |x-y| \le |y|$ for any $x, y \in \mathbb{R}^n$, we obtain

$$\mathcal{N}[\rho_R](x) \le \rho_R(x) \int_{\mathbb{R}^n} J(y)\{e^{|y|/R} - 1\}\, dy. \tag{2.3}$$

Now, let $\varepsilon > 0$ be given and let $R_0 > 1$ be sufficiently large such that $1/R_0 < \hat{\lambda}$. Then, by (J2) and (J3), there exists $r > 0$ sufficiently large such that

$$0 < \int_{|y|\ge r} J(y)\{e^{|y|/R} - 1\}\, dy \le \int_{|y|\ge r} J(y)\{e^{|y|/R_0} - 1\}\, dy < \varepsilon/2,\ \forall\, R \ge R_0. \tag{2.4}$$

Moreover, since the sequence $J(y)\{e^{|y|/R} - 1\}$ converges to 0 as $R \to \infty$ uniformly over $\{|y| \le r\}$, we may choose a large enough $R \ge R_0$ such that

$$0 < \int_{|y|\le r} J(y)\{e^{|y|/R} - 1\}\, dy < \varepsilon/2.$$

Then the lemma follows from this estimate together with (2.3) and (2.4). □

With the weight ρ_R, we introduce the functional

$$\mathcal{F}_R(t) := \int_{\mathbb{R}^n} F(u(x,t))\rho_R(x) dx$$

for an entire solution u of (2.1) satisfying

$$0 < a_i \le u_i \le A_i < \infty,\ i = 1, \dots, m.$$

Note that $F(u(x,t))$ is uniformly bounded over $\mathbb{R}^n \times \mathbb{R}$ and ρ_R is integrable over $\mathbb{R}^n$. Hence $\mathcal{F}_R(t)$ is well-defined and uniformly bounded for $t \in \mathbb{R}$. Then Theorem 2.1 can be proved in the same manner as that for [13, Theorem 1.1]. To be self-contained and for the reader's convenience, we provide some details here.

First, we compute

$$\frac{d}{dt}\mathcal{F}_R(t) = \sum_{i=1}^m \int_{\mathbb{R}^n} F_i'(u_i) f_i(u) \rho_R dx + \sum_{i=1}^m d_i \int_{\mathbb{R}^n} F_i'(u_i) \mathcal{N}_i[u_i] \rho_R\, dx.$$

It follows from (2.2) and $g'(\theta) = 1 - 1/\theta$ that

$$\frac{d}{dt}\mathcal{F}_R(t) \le -\kappa \mathcal{F}_R(t) + \sum_{i=1}^m d_i c_i \frac{1}{u_i^*} \int_{\mathbb{R}^n} \left(1 - \frac{u_i^*}{u_i}\right) \mathcal{N}_i[u_i] \rho_R\, dx. \tag{2.5}$$

Next, for a fixed i set

$$I_i(t) := \int_{\mathbb{R}^n} \Big(1 - \frac{u_i^*}{u_i(x,t)}\Big)\mathcal{N}_i[u_i](x,t)\rho_R(x)\,dx.$$

Then we obtain from

$$\mathcal{N}_i[u_i](x,t) = \int_{\mathbb{R}^n} J_i(x-y)\{u_i(y,t) - u_i(x,t)\}dy,$$

that

$$I_i(t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)u_i^*\Big[1 - \frac{u_i(x,t)}{u_i^*} + \frac{u_i(y,t)}{u_i^*} - \frac{u_i(y,t)}{u_i(x,t)}\Big]\rho_R(x)\,dydx. \tag{2.6}$$

By changing the order of integration, we also have

$$I_i(t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)u_i^*\left[1 - \frac{u_i(x,t)}{u_i^*} + \frac{u_i(y,t)}{u_i^*} - \frac{u_i(y,t)}{u_i(x,t)}\right]\rho_R(x)\,dxdy. \tag{2.7}$$

On the other hand, by exchanging the roles of x and y and using $J(x-y) = J(y-x)$, we get from (2.6) that

$$I_i(t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)u_i^*\left[1 - \frac{u_i(y,t)}{u_i^*} + \frac{u_i(x,t)}{u_i^*} - \frac{u_i(x,t)}{u_i(y,t)}\right]\rho_R(y)\,dxdy. \tag{2.8}$$

Summing over (2.7) and (2.8), we obtain $2I_i(t) = I_{i1}(t) + I_{i2}(t)$, where

$$\begin{aligned} I_{i1}(t) &:= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)u_i^*\left[\rho_R(x) + \rho_R(y) - \frac{u_i(x,t)\rho_R(y)}{u_i(y,t)} - \frac{u_i(y,t)\rho_R(x)}{u_i(x,t)}\right]dxdy \\ I_{i2}(t) &:= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)\Big\{[u_i(y,t) - u_i(x,t)]\rho_R(x) + [u_i(x,t) - u_i(y,t)]\rho_R(y)\Big\}dxdy. \end{aligned}$$

Then, by exchanging x and y and using $J(x-y) = J(y-x)$, we obtain

$$\begin{aligned} I_{i1}(t) &= 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)u_i^*\left[1 - \frac{u_i(y,t)}{u_i(x,t)}\right]\rho_R(x)\,dxdy, \\ I_{i2}(t) &= 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)\Big[u_i(y,t) - u_i(x,t)\Big]\rho_R(x)\,dxdy. \end{aligned}$$

It follows that

$$\begin{aligned} I_i(t) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)\left[u_i(y,t) + u_i^* - u_i(x,t) - u_i^*\frac{u_i(y,t)}{u_i(x,t)}\right]\rho_R(x)\,dxdy \\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)\left[u_i(y,t) - u_i(x,t) + u_i^*\ln\Big(\frac{u_i(x,t)}{u_i(y,t)}\Big)\right]\rho_R(x)\,dxdy, \end{aligned}$$

using $1 - X \le \ln(1/X)$ for all $X > 0$. Thus we get

$$I_i(t) \le \frac{u_i^*}{c_i}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)\Big[F_i(u_i(y,t)) - F_i(u_i(x,t))\Big]\rho_R(x)\,dxdy. \tag{2.9}$$

Moreover, using

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)F_i(u_i(y,t))\rho_R(x)\,dxdy &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(y-x)F_i(u_i(x,t))\rho_R(y)\,dydx\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)F_i(u_i(x,t))\rho_R(y)\,dxdy,\end{aligned}$$

it follows from (2.9) that

$$I_i(t) \le \frac{u_i^*}{c_i}\int_{\mathbb{R}^n} F_i(u_i(x,t))\,\mathcal{N}_i[\rho_R](x)\,dx.$$

According to Lemma 2.2, for each J_i, for any $\varepsilon \in (0,\kappa/2)$, there exists $R_i > 0$ such that

$$\mathcal{N}_i[\rho_{R_i}] \le \frac{\varepsilon}{md_i}\rho_{R_i}.$$

If we choose $R \ge \max_{1\le i\le m} R_i$, then

$$I_i(t) \le \frac{\varepsilon u_i^*}{md_ic_i}\int_{\mathbb{R}^n} F_i(u_i(x,t))\,\rho_R(x)\,dx. \tag{2.10}$$

Finally, from (2.5) and (2.10) it follows that

$$\frac{d}{dt}\mathcal{F}_R(t) \le -(\kappa-\varepsilon)\mathcal{F}_R(t) \le -\frac{\kappa}{2}\mathcal{F}_R(t), \quad \forall\, t\in\mathbb{R}.$$

By integrating in time from $-\infty$ to t, we deduce that $\mathcal{F}_R(t) = 0$ for all $t \in \mathbb{R}$. Hence $F(u(x,t)) \equiv 0$ and so $u(x,t) \equiv u^*$ for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. Theorem 2.1 is thereby proved. ☐

3. Proof of Theorem 1.1

First, we recall the following proposition from [11].

Proposition 3.1. *Let $\alpha > 0$ and let (S,I) be a solution of* (1.1), (1.2) *and* (1.4) *with a nonnegative nontrivial compactly supported continuous initial data I_0. Assume the condition* (1.5) *is enforced. Then the constant c^* defined in* (1.6) *is the (asymptotic) spreading speed of I in the sense*

$$\lim_{t\to\infty}\sup_{|x|>ct} I(x,t) = 0,\ \forall\, c > c^*; \quad \liminf_{t\to\infty}\inf_{|x|<ct} I(x,t) > 0,\ \forall\, c\in(0,c^*). \tag{3.1}$$

Let $\alpha \ge 0$. Since 0 is a sub-solution of (1.2) with $I(x,0) \ge 0$ for any $S(x,t) \in \mathbb{R}$, by comparison we obtain $I \ge 0$ in $\mathbb{R}\times[0,\infty)$. Similarly, 0 is a sub-solution and 1 is a super-solution of (1.1) with $S(\cdot,0) \equiv 1$. Hence, by comparison, we have $0 \le S \le 1$ in $\mathbb{R}\times[0,\infty)$. With this information, one can check that the proof of [11, Theorem 1.1] works well for $\alpha = 0$. Hence we obtain

Corollary 3.2. *Let $\alpha = 0$ and let (S,I) be a solution of* (1.1), (1.2) *and* (1.4) *with a nontrivial nonnegative compactly supported continuous initial data I_0. Assume the condition* (1.5) *is enforced. Then* (3.1) *holds with the constant c^* defined in* (1.6).

Note that, for a given $\alpha > 0$, the uniform persistence of S follows from that $S \geq \alpha\mu/(\beta+\alpha\mu)$ in $\mathbb{R} \times [0, \infty)$, since $\alpha\mu/(\beta + \alpha\mu)$ is a sub-solution of (1.1) with $S(\cdot, 0) \equiv 1$. Recall also from [11, (2.2)] that $I \leq \max\{\|I_0\|_\infty, (\beta - \gamma)/(\alpha\gamma)\}$.

The case for $\alpha = 0$ is more delicate. We only consider the case when $d_1 = d_2 := d$ and $J_1 = J_2 := J$. Then equations (1.1)-(1.2) are reduced to

$$\begin{cases} S_t(x,t) = d\mathcal{N}[S(\cdot,t)](x) + \mu - \mu S(x,t) - \beta S(x,t)I(x,t), \ t > 0, \ x \in \mathbb{R}, \\ I_t(x,t) = d\mathcal{N}[I(\cdot,t)](x) + \beta S(x,t)I(x,t) - (\mu+\sigma)I(x,t), \ t > 0, \ x \in \mathbb{R}, \end{cases} \tag{3.2}$$

where

$$\mathcal{N}[\varphi](x) := \int_{\mathbb{R}} J(x-y)\varphi(y)dy - \varphi(x), \ x \in \mathbb{R}.$$

Set $W := 1 - (S + I)$. Then W satisfies

$$W_t = d\mathcal{N}[W] - \mu W + \sigma I \geq d\mathcal{N}[W] - \mu W, \ x \in \mathbb{R}, \, t > 0.$$

It follows that

$$(e^{\mu t}W)_t(x,t) \geq d\mathcal{N}[e^{\mu t}W(\cdot,t)](x), \ x \in \mathbb{R}, \, t > 0.$$

Since $W(x, 0) = -I_0(x) \geq -\|I_0\|_\infty$, by comparison, we obtain the estimate

$$S(x,t) + I(x,t) \leq 1 + e^{-\mu t}\|I_0\|_\infty \leq 1 + \|I_0\|_\infty := \theta, \ x \in \mathbb{R}, \, t > 0. \tag{3.3}$$

Using $S \geq 0$ and $I \geq 0$, we conclude that

$$I \text{ is uniformly bounded in } \mathbb{R} \times [0, \infty). \tag{3.4}$$

Moreover, since the constant $\mu/(\mu + \beta\theta)$ is a sub-solution of S-equation in (3.2), we obtain

$$S \geq \mu/(\mu + \beta\theta) > 0 \text{ in } \mathbb{R} \times [0, \infty), \tag{3.5}$$

by comparison.

Next, with the help of Theorem 2.1, Proposition 3.1, Corollary 3.2, and a uniform persistent result on (S, I) in the zone $\{(x, t) \mid |x| \leq ct, \, t \gg 1\}$ for $c \in (0, c^*)$, the proof of Theorem 1.1 can be done by a similar argument as that of [12, Theorem 1.4] with some modifications due to the regularity of solutions. We provide a proof as follows.

Proof of Theorem 1.1. First, we recall from [18, 20] that both $S(\cdot, t)$ and $I(\cdot, t)$ are uniformly continuous on $\mathbb{R}$ for each $t \geq 0$. Moreover, the uniform boundedness of (S, I) and (1.1)-(1.2) implies that both S_t and I_t are uniformly bounded. This implies that both S and I are uniformly continuous in $\mathbb{R} \times [0, \infty)$. Furthermore, it follows from (1.1)-(1.2) that both S_t and I_t are uniformly continuous in $\mathbb{R} \times [0, \infty)$.

Next, we let

$$k_0 \leq S(x,t) \leq 1, \ 0 \leq I(x,t) \leq k_1 < \infty, \ x \in \mathbb{R}, \, t \geq 0, \tag{3.6}$$

for some positive constants k_0, k_1. Following [12], we assume for contradiction that there is a positive constant δ such that

$$|S(x_j,t_j) - S^*| + |I(x_j,t_j) - I^*| \geq \delta, \ \forall \, j \geq 1, \tag{3.7}$$

for some sequence $\{(x_j, t_j)\}$ with $t_j \to \infty$ as $j \to \infty$ and $|x_j| \le c_0 t_j$ for all $j \ge 1$ for some constant $c_0 \in (0, c^*)$. Set

$$(S_j, I_j)(x,t) := (S, I)(x + x_j, t + t_j),\ (x,t) \in \mathbb{R}^2,\ j \ge 1.$$

It follows from the above regularity result that $\{(S_j, I_j)\}$ and $\{((S_j)_t, (I_j)_t)\}$ are uniformly bounded and equi-continuous sequences on $\mathbb{R}^2$. Hence, by Arzelá-Ascoli theorem with the help of a diagonal process, the limit

$$(S_\infty, I_\infty)(x,t) := \lim_{j\to\infty} (S_j, I_j)(x,t),\ (x,t) \in \mathbb{R}^2,$$

exists (up to a subsequence) such that (S_∞, I_∞) is an entire solution of system (1.1)-(1.2).

Finally, note that (3.6) holds for (S_∞, I_∞) in $\mathbb{R}^2$. Also, by (3.1) with $c \in (c_0, c^*)$, there is a positive constant k_3 such that $I_\infty \ge k_3$ in $\mathbb{R}^2$. Hence $(S_\infty, I_\infty) = (S^*, I^*)$ by Theorem 2.1 and a Lyapunov function given in [19, 12]. However, $|S_\infty(0,0) - S^*| + |I_\infty(0,0) - I^*| \ge \delta > 0$ by (3.7), a contradiction. This completes the proof of Theorem 1.1. □

References

[1] M. Alfaro, J. Coville, *Rapid traveling waves in the nonlocal Fisher equation connect two unstable states*, Appl. Math. Letters, 25 (2012), 2095–2099.

[2] F. Andreu-Vaillo, J. Mazón, J. Rossi, J. Toledo-Melero, *Nonlocal diffusion problems*, Mathematical Surveys and Monographs, 165, American Mathematical Society, Providence, RI, 2010.

[3] N. Apreutesei, N. Bessonov, V. Volpert, V. Vougalter, *Spatial structures and generalized travelling waves for an integro-differential equation*, Disc. Cont. Dyn. Syst. Ser. B, 13 (2010), 537–557.

[4] H. Berestycki, G. Nadin, B. Perthame, L. Ryzhik, *The non-local Fisher-KPP equation: traveling waves and steady states*, Nonlinearity, 22 (2009), 2813–2844.

[5] V. Capasso, G. Serio, *A generalization of the Kermack-Mckendrick deterministic epidemic model*, Math. Biosci., 42 (1978), 41–61.

[6] J. Coville, J. Dávila, S. Martinez, *Existence and uniqueness of solutions to a nonlocal equation with monostable nonlinearity*, SIAM J. Math. Anal., 39 (2008), 1693–1709.

[7] J. Coville, J. Dávila, S. Martinez, *Pulsating fronts for nonlocal dispersion and KPP nonlinearity*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 30 (2013), 179–223.

[8] J. Fang, X.-Q. Zhao, *Monotone wavefronts of the nonlocal Fisher-KPP equation*, Nonlinearity, 24 (2011), 3043–3054.

[9] G. Faye, M. Holzer, *Modulated traveling fronts for a nonlocal Fisher-KPP equation: a dynamical systems approach*, J. Diff. Equations, 258 (2015), 2257–2289.

[10] S. Genieys, V. Volpert, P. Auger, *Pattern and waves for a model in population dynamics with nonlocal consumption of resources*, Math. Model. Nat. Phenom., 1 (2006), 63–80.

[11] J.-S. Guo, A. A-L. Poh, M. Shimojo, *The spreading speed of an SIR epidemic model with nonlocal dispersal*, Asymp. Anal. 120 (2020), 163-174.

[12] J.-S. Guo, M. Shimojo, *Stabilization to a positive equilibrium for some reaction-diffusion systems*, Nonlinear Analysis: Real World Applications, 62 (2021), Art. 103378, 12pp.

[13] J.-S. Guo, M. Shimojo, *A Liouville theorem for a class of reaction-diffusion systems with fractional diffusion*, Applied Mathematics Letters, 133 (2022), Art. 108254.

[14] V. Hutson, M. Grinfeld, *Non-local dispersal and bistability*, Euro. J. Appl. Math., 17 (2006), 221–232.

[15] V. Hutson, S. Martinez, K. Mischailow, G. T. Vickers, *The evolution of dispersal*, J. Math. Biol., 47 (2003), 483–517.

[16] V. Hutson, W. Shen, G. T. Vickers, *Spectral theory for nonlocal dispersal with periodic or almost-periodic time dependence*, Rocky Mountain J. Math., 38 (2008), 1147–1175.

[17] L.I. Ignat, J.D. Rossi, *A nonlocal convection-diffusion equation*, J. Funct. Anal., 251 (2007), 399–437.

[18] Y. Jin, X.-Q. Zhao, *Spatial dynamics of a periodic population model with dispersal*, Nonlinearity, 22 (2009), 1167-1189.

[19] A. Korobeinikov, *Lyapunov functions and global stability for SIR and SIRS epidemiological models with non-Linear transmission*, Bull. Math. Biol., 68 (2006), 615-626.

[20] W.-T. Li, J.-B. Wang, X.-Q. Zhao, *Spatial dynamics of a nonlocal dispersal population model in a shifting environment*, J. Nonlinear Sci., 28 (2018), 1189-1219.

[21] Y. Li, W.-T. Li, G. Lin, *Traveling waves of a delayed diffusive SIR epidemic model*, Comm. Pure Appl. Anal., 14 (2015), 1001–1022.

[22] F. Lutscher, E. Pachepsky, M. A. Lewis, *The effect of dispersal patterns on stream populations*, SIAM J. Appl. Math., 65 (2005), 1305–1327.

[23] C. McCluskey, *Global stability for an SIR epidemic model with delay and nonlinear incidence*, Nonlinear Anal. Real Word Appl., 11 (2010), 3106–3109.

[24] G. Nadin, L. Rossi, L. Ryzhik, B. Perthame, *Wave-like solutions for nonlocal reaction-diffusion equations: a toy model*, Math. Model. Nat. Phenom., 8 (2013), 33–41.

[25] W. Shen, A. Zhang, *Spreading speeds for monostable equations with nonlocal dispersal in space periodic habitats*, J. Differential Equations, 15 (2010), 747–795.

[26] F.-Y. Yang, Y. Li, W.-T. Li, Z.-C. Wang, *Traveling waves in a nonlocal dispersal Kermack-McKendrick epidemic model*, Disc. Cont. Dyn. Syst., Ser. B, 18 (2013), 1969–1993.

[27] G.-B. Zhang, W.-T. Li, Z.-C. Wang, *Spreading speeds and traveling waves for nonlocal dispersal equations with degenerate monostable nonlinearity*, J. Differential Equations, 252 (2012), 5096–5124.

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