BLOW-UP BEHAVIOR FOR A PARABOLIC EQUATION WITH SPATIALLY DEPENDENT COEFFICIENT

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ABSTRACT. We study the initial boundary value problem and Cauchy problem for the semilinear heat equation with power nonlinearity and spatially dependent coefficient. First, for the initial boundary value problem, we establish several conditions that ensure the origin is not a blow-up point. Then the Cauchy problem for a special case is also studied. Finally, we derive the blow-up rate when the origin is not a blow-up point.

Keywords: spatially dependent coefficient, blow-up point, blow-up rate.

1. INTRODUCTION

In this paper, we consider the following initial boundary value problem:

$$\begin{cases} u_t = \Delta u + |x|^\sigma |u|^{p-1}u, & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u = u_0, & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where $p > 1, \sigma > 0$, u_0 is a bounded smooth function with $u_0 > 0$ in Ω and $u_0 = 0$ on $\partial\Omega$, and Ω is a bounded smooth domain in $\mathbb{R}^N$ with the boundary $\partial\Omega$. Here $\bar{\Omega}$ is the closure of Ω. Note that a unique solution u of (1.1) exists locally (in time), say for $t \in [0, \tau]$ for some constant $\tau > 0$, such that $u > 0$ in $\Omega \times [0, \tau]$, by the standard theory of parabolic equations. Moreover, by the Hopf boundary point lemma, $\partial u/\partial\nu < 0$ on $\partial\Omega \times (0, \tau]$, where ν is the unit outward normal on $\partial\Omega$. By re-defining the initial time, we may assume without loss of generality that $\partial u_0/\partial\nu < 0$ on $\partial\Omega$.

We say that the solution u of (1.1) *blows up* if there is some $T = T(u_0) < \infty$ such that $\limsup_{t\nearrow T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$. Here $T(u_0)$ is called the *blow-up time* of the solution u with the initial value u_0. We say that the solution is *global* if the solution u exists for all $t \in [0, \infty)$ as a classical solution. For a given solution u that blows up at $t = T < \infty$, we define its *blow-up set* by

$$B(u_0) = \Big\{a \in \bar{\Omega} \ \Big| \ \exists \{(x_n, t_n)\} \text{ such that } x_n \to a,\ t_n \uparrow T,\ |u(x_n, t_n)| \to \infty \text{ as } n \to \infty\Big\}.$$

Each element of $B(u_0)$ is called a *blow-up point* of u. We note that there is no reaction for the problem (1.1) at $x = 0$. One of the main purposes of this paper is to see whether $x = 0$ is a blow-up point or not.

The phenomena of blow-up have attracted a lot of attention in past years. Most research papers dealing with blow-up are concerned with equations without spatially dependent coefficient. Interesting questions, for example, are about criteria of blow-up,

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blow-up points, blow-up rate, spatial blow-up profile and continuation after blow-up. For example, for the spatially homogeneous equation, we refer the reader to [1, 2, 4, 7, 8, 9, 11, 12, 13, 22, 24, 25, 27, 28] and so on. The authors of [35, 16, 30, 6] considered the Cauchy problem for the spatially inhomogeneous equation in (1.1). They studied the existence and nonexistence of global nonnegative solutions. In particular, the existence of self-similar solutions that blow up at the origin for the Cauchy problem was obtained in [6]. As far as we know, this is the only work which demonstrates blow-up at the origin. In this paper, we shall give some sufficient conditions under which blow-up does not occur at $x = 0$ and prove that such solution always satisfies

$$\limsup_{t \nearrow T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty, \tag{1.2}$$

as far as radially symmetric solutions are concerned.

The blow-up rate estimate is well studied for the problem with spatially homogeneous nonlinearity. It is called that the blow-up is of Type I if u satisfies (1.2). Otherwise, it is called Type II. Indeed, for the spatially homogeneous equation

$$u_t = \Delta u + u^p,$$

it is known that blow-up is of Type I, if $1 < p < (N+2)/(N-2)_+$ (see, e.g., [11, 12, 13, 28]). Furthermore, blow-up is of Type I for any $p > 1$, if $u_t \geq 0$ (cf. [7]). If we restrict ourselves to the radially symmetric solutions, then blow-up is also of Type I even for $p > p_s := (N+2)/(N-2)_+$ under some additional conditions. More precisely, [24, 25] showed that blow-up is of Type I, if $p_s < p < p_{JL}$, where

$$p_{JL} := \begin{cases} 1 + \frac{4}{N-4-2\sqrt{N-1}}, & N \geq 11, \\ \infty, & N \leq 10. \end{cases}$$

The results of [26, 24, 25] imply that blow up is of Type I, if $x = 0$ is not a blow-up point. For Type II blow-up, see [20, 29] for example. They constructed solutions with Type II blow-up in the range $N \geq 11$ and $p > p_{JL}$. See also [25] for Type II blow-up.

Now, let us state our results more precisely. In this paper, we mostly deal with radially symmetric solutions on $\Omega = B_R$, where B_R denotes the ball with center 0 and radius $R > 0$. If u is a radially symmetric solution of (1.1) on $\Omega = B_R$, then $u(x,t) = u(|x|,t)$ satisfies

$$u_t = u_{rr} + \frac{N-1}{r} u_r + r^\sigma |u|^{p-1} u, \quad 0 < r := |x| < R.$$

First, we give some sufficient conditions such that blow-up does not occur at $x = 0$. Let u be a radially symmetric blow-up solution of (1.1) on $\Omega = B_R$. Then $x = 0$ is not a blow-up point, if one of the following holds:

(1) $N = 3$, $\sigma \geq p - 1$,
(2) $N \geq 4$, $\sigma > (p-1)(N-1)/2$.

For the special case when $N = 3$ and $\sigma = p - 1$, we also study the corresponding Cauchy problem. We provide some conditions on the initial data so that the solutions do not blow up at the origin. Also, a criterion for which blow-up occurs only at space infinity is given. We note that blow-up only occurs at space infinity was first considered by Lacey [21] for a one-dimensional problem on the half-line about the semilinear equation $u_t = \Delta u + f(u)$. The similar problem in $\mathbb{R}^N$ was discussed in [14, 15, 33, 9] for the same semilinear equation and in [9, 31, 32] for the porous media equation.

Next, we assume that blow-up does not occur at $x = 0$. We want to prove that the solution u always satisfies (1.2). Moreover, if $r = r_b \in (0, R)$ is a blow-up point, then

$$\lim_{t\to T}(T-t)^{1/(p-1)}u(r_b+(T-t)^{1/2}\xi,t) = \pm\, r_b^{-\sigma/(p-1)}(p-1)^{-1/(p-1)} \tag{1.3}$$

uniformly on $|\xi| \le C$ for any $C > 0$.

This paper is organized as follows. In section 2, we give some conditions such that blow-up does not occur at the origin for boundary value problem. Also, the shape of global blow-up profile is given. Then we study the corresponding Cauchy problem when $N = 3$ and $\sigma = p - 1$ in section 3. Finally, in section 4, we derive the blow-up rate under the assumption that $x = 0$ is not a blow-up point.

2. Blow-up does not occur at the origin

This section is mainly devoted to finding some criteria that blow-up does not occur at the origin. In this section, we shall focus on radially symmetric solutions of (1.1) on $\Omega = B_R \subset \mathbb{R}^N$ with $N \ge 3$. By using the transformation $z(r,t) := r^{\frac{N-1}{2}}u(r,t)$, the problem (1.1) becomes

$$\begin{cases} z_t = z_{rr} + \dfrac{z^p}{r^a} + \dfrac{bz}{r^2}, & r \in (0,R),\ t > 0, \\ z(R,t) = z(0,t) = 0, & t > 0, \\ z(r,0) = z_0(r) := r^{\frac{N-1}{2}}u_0(r), & r \in [0,R], \end{cases} \tag{2.1}$$

where

$$a := \frac{(N-1)(p-1)}{2} - \sigma, \quad b := \frac{(N-1)(3-N)}{4}.$$

Let $T(z_0)$ be the blow-up time of the solution z. In general, we have the inequality $T(z_0) \ge T(u_0)$, since the following inequality holds:

$$z(r,t) \le R^{\frac{N-1}{2}}u(r,t), \quad r \in [0,R],\ t \in [0, \min\{T(u_0), T(z_0)\}).$$

If $r = 0$ is not a blow-up point of u, by using the fact that blow-up set is closed, there exists a small $\varepsilon > 0$ such that B_ε does not include any blow-up point of u. From

$$\varepsilon^{\frac{N-1}{2}}u(r,t) \le z(r,t) \le R^{\frac{N-1}{2}}u(r,t), \quad r \in [\varepsilon,R],\ t \in [0,\min\{T(u_0),T(z_0)\}),$$

it follows that $T(z_0) = T(u_0)$. Moreover, from the identity

$$z_0'(r) = r^{(N-3)/2}\Big[\frac{N-1}{2}u_0(r) + ru_0'(r)\Big],$$

it follows that $z_0'(r) > 0$ near the origin. Also, $z_0'(R) < 0$, since $u_0(R) = 0$ and $u_0'(R) < 0$.

The following lemma is useful in the subsequent argument.

Lemma 2.1. *Let z be the solution of* (2.1) *that blows up in finite time $T = T(z_0)$.*

(i) *If $a \ge 0, b = 0$, then z is bounded in a neighborhood of $r = R$ for all $t \in (0,T)$.*
(ii) *If $a, b \le 0$, then z is bounded in a neighborhood of $r = 0$ for all $t \in (0,T)$.*

Proof. The assertions follow from the combination of the following two steps.

Step 1. This step is based on the reflection argument as the proof of Theorem 5.2 in [10]. Let $\lambda \in (R/2, R)$ and define $I_\lambda := [\lambda, R]$. Also, let z^λ be the reflection of z with respect to $r = \lambda$. If $a, b \ge 0$, then $1/r^a$ and b/r^2 is monotone decreasing in r. Using this monotonicity and the initial-boundary conditions $z(\lambda,t) = z^\lambda(\lambda,t)$, $0 = z(R,t) \le z^\lambda(R,t)$ for $t \in (0, T(z_0))$, and

$$z(\cdot,0) \le z^\lambda(\cdot,0) \quad \text{on } I_\lambda,$$

provided that $\lambda \geq r_0$ for some $r_0 \in (0, R)$ (using $z_0'(R) < 0$), we obtain $z(\cdot, t) \leq z^\lambda(\cdot, t)$ on $I_\lambda \times [0, T(z_0))$. Since $z \not\equiv z^\lambda$ from the initial condition, the Hopf boundary point lemma implies that $z_r(\lambda, t) < 0$ for all $t \in [0, T(z_0))$. Hence $z_r < 0$ on $[r_0, R] \times [0, T(z_0))$.

When $a, b \leq 0$, $1/r^a$ and b/r^2 is monotone increasing in r. Arguing as the above and using $z_0'(r) > 0$ near $r = 0$, we can conclude that $z_r > 0$ in a fixed neighborhood of $r = 0$ for all $t \in (0, T(z_0))$.

Step 2. To prove (i), we assume on the contrary that $r = R$ is a blow-up point. Then, by using the monotonicity of z in r on $[r_0, R]$, we can find constants c, d with $r_0 \leq c < d < R$ such that $[c, d] \subset [r_0, R)$ and $z(\cdot, t) \to \infty$ as $t \to T(z_0)$ uniformly over the interval $[c, d]$.

Following [7], we consider the function $J := z_r + \varepsilon\zeta z^\gamma$ with $\zeta(r) = \sin(\pi(r - c)/(d - c))$, $\gamma \in (1, p)$ and some small positive constant ε to be determined. Then we compute

$$\begin{aligned}
J_t &= z_{rt} + \gamma\varepsilon\zeta z^{\gamma-1} z_t, \\
J_r &= z_{rr} + \gamma\varepsilon\zeta z^{\gamma-1} z_r + \varepsilon\zeta' z^\gamma, \\
J_{rr} &= z_{rrr} + \gamma\varepsilon\zeta z^{\gamma-1} z_{rr} + 2\gamma\varepsilon\zeta' z^{\gamma-1} z_r + \gamma(\gamma-1)\varepsilon\zeta z^{\gamma-2} z_r^2 + \varepsilon\zeta'' z^\gamma.
\end{aligned}$$

Using $\zeta'' = -[\pi/(d - c)]^2\zeta$, $\gamma > 1$ and $a, b \geq 0$, we obtain that

$$J_t - J_{rr} - AJ \leq -\varepsilon\zeta z^\gamma B,$$

where

$$\begin{aligned}
A &:= pr^{-a} z^{p-1} + br^{-2} - 2\gamma\varepsilon\zeta' z^{\gamma-1}, \\
B &:= (p - \gamma) r^{-a} z^{p-1} + br^{-2} - b\gamma r^{-2} - 2\gamma\varepsilon\zeta' z^{\gamma-1} - [\pi/(d - c)]^2.
\end{aligned}$$

Since $1 < \gamma < p$ and $z \to \infty$ as $t \to T(z_0)$ uniformly on $[c, d]$, we can find a $t_0 \in (0, T(z_0))$ such that $B > 0$ on $[c, d] \times [t_0, T(z_0))$. Note that $J < 0$ on $r = c, d$. By choosing $\varepsilon > 0$ sufficiently small, we have $J < 0$ on $[c, d] \times \{t_0\}$. Thus the comparison principle yields $J < 0$ on $[c, d] \times [t_0, T(z_0))$, i.e.,

$$\frac{z_r}{z^\gamma} < -\varepsilon\zeta.$$

Integrating this inequality from c to d for $t \in [t_0, T(z_0))$ and letting $t \to T(z_0)$, we reach a contradiction. Hence (i) is proved.

The proof for (ii) is similar, we omit it here. Hence the lemma is proved. □

Remark 2.1. Lemma 2.1 (i) still holds when $b > 0$, that is, $N = 2$.

We can deduce the following result for the original problem (1.1) from Lemma 2.1.

Proposition 2.1. *Let $N \geq 3$ and let u be a radially symmetric solution of* (1.1) *that blows up in finite time.*

1. *Suppose $N = 3$ and $\sigma \leq p - 1$. Then $r = R$ is not a blow-up point of u.*
2. *Suppose $N \geq 3$, $\sigma \geq (N - 1)(p - 1)/2$. If $r = 0$ is a blow-up point of u, then it is an isolated blow-up point.*
3. *If $N = 3$ and $\sigma \geq p - 1$, then $r = 0$ is not a blow-up point of u.*

Proof. First, part (1) follows from Lemma 2.1(i) and the relation $u = r^{-\frac{N-1}{2}} z$ immediately. To prove part (2), we recall that $T(u_0) \leq T(z_0)$. If $r = 0$ is a blow-up point of u, then it is an isolated blow-up point, by using Lemma 2.1(ii) and the relation $u = r^{-\frac{N-1}{2}} z$.

We now consider part (3). Since $b = 0$ implies $N = 3$, (2.1) becomes

$$\begin{cases}
z_t = z_{rr} + r^{-a} z^p, & r \in (0, R),\ t > 0, \\
z(R, t) = z(0, t) = 0, & t > 0, \\
z(r, 0) = z_0(r) := ru_0, & r \in [0, R].
\end{cases} \tag{2.2}$$

The problem (2.2) can be regarded as a one dimensional problem and we can use the regularity estimate to the solution $z(\cdot,t)$ around $r=0$ using Lemma 2.1. Thus we get an estimate $ru(r,t)=z(r,t)\le Cr$ for all sufficiently small r and $t\in[0,T(z_0)]$, where $C>0$ is some finite constant. Combining this with $T(u_0)\le T(z_0)$, we conclude that u cannot blow up at the origin. This proves the proposition. □

Remark 2.2. By Remark 2.1, Proposition 2.1 (1) also holds when $N=2$ and $\sigma\le(p-1)/2$.

Remark 2.3. In the proof of part (3) in Proposition 2.1, we do not use the boundary condition of u. Thus $r=0$ is not a blow-up point even for the problem with nonzero Dirichlet boundary condition or Neumann boundary condition.

When $N=3$, we have the following result.

Theorem 1. *Let $\Omega=B_R$, $N=3$ and let u be a radially symmetric solution of* (1.1) *that blows up in finite time.*

(i) *If $\sigma\ge p-1$, then $r=0$ is not a blow-up point.*
(ii) *If $\sigma=p-1$, then the blow-up set of u consists of finitely many concentric spheres with positive radii.*

Proof. It is easy to see that part (i) follows from Proposition 2.1(3).

For part (ii), we first note that (2.2) becomes

$$\begin{cases} z_t=z_{rr}+z^p, & r\in(0,R),\ t>0,\\ z(0,t)=z(R,t)=0, & t>0,\\ z(r,0)=ru_0(r):=z_0(r), & r\in[0,R], \end{cases}\tag{2.3}$$

when $\sigma=p-1$. By Theorem E of [4] and Theorem 3.3 of [7], the blow-up set of z consists of finitely many points in $(0,R)$. Combining this with $T(u_0)=T(z_0)$ and $u(r,t)=r^{-1}z(r,t)$ yields the finiteness of blow-up radii. The theorem follows. □

Remark 2.4. From Remark 2.3, Theorem 1(ii) holds even for the problem with nonzero Dirichlet boundary condition or Neumann boundary condition.

For $N\ge 4$, we have $b<0$. Then

Theorem 2. *Let $\Omega=B_R$, $N\ge4$ and $\sigma>(p-1)(N-1)/2$. We assume that u is a radially symmetric solution of* (1.1) *that blows up in finite time T. Then $r=0$ is not a blow-up point of u.*

Proof. First, it follows from (ii) of Lemma 2.1 and the relation $u(r,t)=r^{-\frac{N-1}{2}}z(r,t)$ that

$$u(r,t)\le Cr^{-\frac{N-1}{2}},\quad r\in(0,r_1],\ t\in[0,T).\tag{2.4}$$

for some constants $C>0$ and $r_1>0$.

We shall choose $r_0\in(0,r_1/2]$ sufficiently small and define

$$h(x)=\delta\cos^2\left(\frac{\pi|x|}{2r_0}\right),\ B_0:=\{x:|x|\le r_0\},$$

where $\delta<1$ is a positive constant to be determined later. We also define

$$w(x,t):=\frac{A(r_0)}{[h(x)+(T-t)]^{\frac{1}{p-1}}},\quad A=A(r_0):=Cr_0^{-\frac{N-1}{2}}T^{\frac{1}{p-1}}.$$

Since u_0 is bounded and $A(r_0)\to\infty$ as $r_0\to0$, we can easily check that

$$w(x,0)=\frac{A(r_0)}{[h(x)+T]^{\frac{1}{p-1}}}\ge\frac{A(r_0)}{(\delta+T)^{\frac{1}{p-1}}}\ge u_0(x),\quad x\in B_0,$$

if $r_0 > 0$ is sufficiently small. Also, it follows from (2.4) that

$$w(x,t) = \frac{A(r_0)}{(T-t)^{\frac{1}{p-1}}} \geq C r_0^{-\frac{N-1}{2}} \geq u(x,t), \quad \partial B_0 \times (0,T),$$

where $\partial B_0 := \{x : |x| = r_0\}$. The inequality

$$w_t - \Delta w - |x|^\sigma w^p \geq 0 \quad \text{on} \quad B_0 \times (0,T)$$

is equivalent to

$$1 + \Delta h(x) - \frac{p}{p-1} \frac{|\nabla h(x)|^2}{h(x) + (T-t)} \geq (p-1) A^{p-1} |x|^\sigma \quad \text{on} \quad B_0 \times (0,T). \tag{2.5}$$

Note that

$$A^{p-1} r^\sigma \leq A^{p-1} r_0^\sigma = C^{p-1} T r_0^{\sigma - \frac{(N-1)(p-1)}{2}}$$

Hence (2.5) holds if

$$1 + \Delta h(x) - \frac{p}{p-1} \frac{|\nabla h(x)|^2}{h(x)} \geq (p-1) C^{p-1} T r_0^{\sigma-(N-1)(p-1)/2} \tag{2.6}$$

for all $x \in B_0$.

On the other hand, it is easy to see that Δh and $|\nabla h|^2/h$ are bounded in $B_0 \times (0,T)$ and linear in δ. We first choose r_0 sufficiently small such that the right-hand side of (2.6) is less than 1/3. For this fixed r_0, we then choose δ sufficiently small such that the above inequality (2.6) holds in B_0. Hence the comparison principle yields

$$w(x,t) = \frac{A}{[h(x) + (T-t)]^{\frac{1}{p-1}}} \geq u(x,t), \qquad x \in B_0,\ t \in (0,T).$$

In particular, $x = 0$ is not a blow-up point of u. The theorem is proved. □

Recall that the blow-up set $B(u_0)$ is a closed set and that $a \notin B(u_0)$ if and only if the solution remains bounded as $t \nearrow T$ in a neighborhood of a. Therefore, by the standard regularity theory of parabolic equations, $u(\cdot,t)$ remains bounded in $C^{2+\alpha}_{loc}(\overline{\Omega} \setminus B(u_0))$ as $t \nearrow T$ for some $0 < \alpha < 1$, which then implies the boundedness of $u_t(\cdot,t)$ in $C^{\alpha}_{loc}(\overline{\Omega} \setminus B(u_0))$. By integrating for t from 0 to T, the pointwise limit $u(x,T) := \lim_{t \nearrow T} u(x,t)$ exists for every $x \in \overline{\Omega} \setminus B(u_0)$ and it belongs to $C^{2+\alpha}_{loc}(\overline{\Omega} \setminus B(u_0))$. We call this limit the *global blow-up profile* of u.

Concerning the spatial blow-up profile, we have the following proposition.

Proposition 2.2. *Let $\Omega = B_R$, $N = 3$ and $\sigma = p-1$. Suppose that u is a radially symmetric solution of* (1.1) *that blows up at $T = T(u_0)$. For any blow-up point $r = r_b \neq 0$, one of the following holds:*

$$\lim_{r \to r_b} \Big(\frac{|r - r_b|^2}{|\ln |r - r_b||} \Big)^{\frac{1}{p-1}} r\, u(r,T) = \Big(\frac{8p}{(p-1)^2} \Big)^{\frac{1}{p-1}},$$

$$\lim_{r \to r_b} |r - r_b|^{\frac{m}{p-1}} r\, u(r,T) = C,$$

where C is some constant and $m \geq 4$ is an even integer.

Proof. The proof is just the combination of the fact that $z = ru$ satisfies (2.3) and Theorem 3 of [34] for the one dimensional blow-up problem with spatially homogeneous nonlinearity. □

Remark 2.5. The similar result to this proposition holds for Cauchy problem when $N = 3$ and $\sigma = p-1$, by using Theorem 3 of this paper and the main theorem of [19] or Theorem 3 of [34]. See also [17, 18, 2, 28] for related problem.

Also, we can construct some solutions with prescribed blow-up set as follows.

Proposition 2.3. *Let $\Omega = B_R$, $N = 3$ and $\sigma = p - 1$. Let u be a radially symmetric solution of* (1.1)*. For any finitely many concentric spheres in B_R, there exists an initial datum $u_0 \geq 0$ such that the corresponding solution u blows up on these spheres.*

Proof. By Theorem 1 of [27], for any given finite points $\{r_1, \dots, r_k\}$ in $(0, R)$, there exists a solution z of the problem (2.3) such that z blows up at these points. Hence the proposition follows. □

3. Cauchy problem

In this section, we shall study the following Cauchy problem related to (1.1) when $N = 3$ and $\sigma = p - 1$:

$$\begin{cases} u_t = \Delta u + |x|^{p-1} u^p, & x \in \mathbb{R}^3,\ t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^3, \end{cases} \tag{3.1}$$

where u_0 is a nonnegative bounded smooth function in $\mathbb{R}^3$. First, we shall show that blow-up does not occur at the origin if the support of u_0 is compact and u_0 is radially symmetric.

Theorem 3. *Suppose that u_0 is radially symmetric with compact support. Let u be a blow-up solution of* (3.1) *with the blow-up time T. Then u does not blow up at $x = 0$. Furthermore, its blow-up set consists of finite number of spheres centered at $x = 0$.*

Proof. From the assumption, the solution $u(r, t)$ satisfies

$$\begin{cases} u_t = u_{rr} + \frac{2}{r} u_r + r^{p-1} u^p, & r \in (0, \infty),\ t \in (0, T), \\ u(r, 0) = u_0(r), & r \in [0, \infty). \end{cases} \tag{3.2}$$

We again consider $z(r, t) := r u(r, t)$. Then, by using (3.2), z satisfies

$$\begin{cases} z_t = z_{rr} + z^p, & r \in (0, \infty),\ t \in (0, T_0), \\ z(r, 0) = z_0(r) := r u_0(r), & r \in [0, \infty), \\ z(0, t) = 0, & t \in (0, T_0), \end{cases} \tag{3.3}$$

where T_0 is the maximum existence time of z.

First, since the support of z_0 is compact, Lemma B1 of [25] or the proof of Proposition 5.8 of [13] yields that the blow-up set of z is also compact. From $z = ru$, it follows that $T_0 \geq T$. Indeed, if $r = 0$ is not a blow-up point of u, then $T_0 = T$. Otherwise, if $r = 0$ is the only blow-up point of u, then $T_0 \geq T$. This is due to the fact that $\{r_n u(r_n, t_n) \mid n \in \mathbb{N}\}$ may be bounded when $u(r_n, t_n) \to \infty$ and $(r_n, t_n) \to (0, T)$ as $n \to \infty$.

Next, note that by the strong maximum principle $z_r(0, t) > 0$ for all $t \in (0, T_0)$. By applying the moving plane argument near the boundary in the proof of Theorem 3.3 in [7], we can prove that there is no blow-up point of z in $[0, \varepsilon]$ for some small positive constant ε. Then by the standard parabolic estimates and the boundary condition, there exists some constant $C > 0$ such that $ru(r, t) = z(r, t) < Cr$ for all $(r, t) \in [0, \varepsilon/2] \times [0, T_0]$. Hence $r = 0$ is not a blow-up point of u and $T_0 = T$. Moreover, by the main theorem of [19] or by following the same argument of Section 6 in [4] to the function z, we can conclude that there are only finitely many blow-up points of z. Thus the blow-up set of u consists of finite number of spheres centered at $x = 0$. This proves the theorem. □

Next, we consider the case when the initial function decays very slowly at space infinity. We shall construct a solution that blows up only at space infinity. Here we say that the solution u of (3.1) blows up *only at space infinity* if the following hold:

(a) u blows up in finite time $T(u_0)$;
(b) $\limsup_{t\nearrow T(u_0)} \|u(\cdot,t)\|_{L^\infty(K)} < \infty$ holds for any compact set $K \subset \mathbb{R}^N$.

We define the following functions

$$\varphi(s) := \kappa s^{-\frac{1}{p-1}}, \quad 0 < s < \infty; \qquad \psi(v) := \frac{v^{-(p-1)}}{p-1}, \quad 0 < v < \infty,$$

where $\kappa := (p-1)^{-\frac{1}{p-1}}$. Then the solution of (3.2) with initial data $\dfrac{M}{r}$ is written as $\dfrac{\varphi\big(\psi(M)-t\big)}{r}$. Note that this solution has singularity at the origin for all $t \in [0, \psi(M)]$ and it blows up on the whole space at time $t = \psi(M)$.

Theorem 4. *Let u be a solution of* (3.1)*. Assume $u_0(x)$ satisfies*

$$0 \le u_0(x) \le \frac{M}{|x|}, \quad u_0(x) \not\equiv \frac{M}{|x|}$$

and there exists $R_0 \in (0,\infty)$ such that $u_0(x) = M/|x|$ for all $|x| \ge R_0$. Then $T(u_0) = \psi(M)$ and blow-up occurs only at space infinity.

Proof. First, we prove this theorem for radially symmetric initial data. It follows from the main result of [21, Section 3] that, under the assumption of this theorem, the solution z of the problem (3.3) blows up only at space infinity. More precisely, for any given $\varepsilon > 0$, there exist positive constants c_1, c_2 such that

$$c_1 \exp\left(\frac{r^2}{4(p-1)\,(T(z_0)+\varepsilon)}\right) \le z(r, T(z_0)) \le c_2 \exp\left(\frac{r^2}{4(p-1)\,T(z_0)}\right)$$

for all $r > 0$. This estimate and the relation $u = r^{-1}z$ imply that $T(u_0) \le T(z_0)$.

Since z blows up only at space infinity, there exists $C > 0$ such that $z(r,t) \le Cr$ near $r = 0$ for all $t \in (0, T(z_0)]$ from the standard parabolic estimates. This and the relation $u = r^{-1}z$ imply that u does not blow up at $r = 0$. Therefore, $T(u_0) = T(z_0) = \psi(M)$ and $u(r,t)$ blows up only at space infinity.

Next, we consider general initial datum $u_0(x)$. For this initial datum, we can find two radially symmetric initial data $u_{0,1}(|x|)$ and $u_{0,2}(|x|)$ satisfying the assumption of this theorem and $u_{0,1}(|x|) \le u_0(x) \le u_{0,2}(|x|)$. We shall denote the corresponding solutions of these initial data by $u_1(|x|,t)$ and $u_2(|x|,t)$. The first step imply $T(u_{0,1}) = T(u_{0,2}) = \psi(M)$ and both $u_1(|x|,t)$ and $u_2(|x|,t)$ blow up only at space infinity. By a simple comparison argument, we obtain the desired result for $u(x,t)$. □

4. Blow-up rate

In this section, we shall study the blow-up rate of the solution whose blow-up does not occur at the origin. We do not need to assume that u is nonnegative and so solutions are allowed to be sign-changing. The first theorem implies that the blow-up of the above solution is of Type I. We also obtain its local limiting property at a blow-up point.

Theorem 5. *Let $\Omega = B_R$ and let u be a radially symmetric solution of* (1.1)*. Let $T \in (0,\infty)$ be the blow-up time of the solution u. If $x = 0$ is not a blow-up point of u, then u satisfies* (1.2)*. Furthermore, if $r = r_b \in (0,R)$ is a blow-up point, then* (1.3) *holds uniformly on $|\xi| \le C$ for any $C > 0$.*

Remark 4.1. Similar results to Theorem 5 were also obtained in [3, 5] for spatially inhomogeneous equations. More precisely, they considered the equation $u_t = \Delta u + V(x)u^p$ with $V(x) \geq c > 0$ and $p > 1$. Note that the potential $|x|^\sigma$ in (1.1) vanishes at the origin.

Since u is radially symmetric, the nature of blow-up outside the origin must be similar to that of one dimensional. In order to see this, we set $\beta := 1/(p-1)$ and introduce the transformation $v(r,t) = r^{\beta\sigma}u(r,t)$. Then u satisfies (1.1) if and only if v satisfies

$$\begin{cases} v_t = v_{rr} + \frac{k-1}{r}v_r + |v|^{p-1}v - \frac{l}{r^2}v, & r \in (0,R),\ t > 0, \\ v(R,t) = v(0,t) = 0,\ t > 0, \\ v(r,0) = v_0(r) := r^{\beta\sigma}u_0(r), & r \in [0,R]. \end{cases} \tag{4.1}$$

where

$$k := N - 2\beta\sigma, \quad l := \beta\sigma(N - 2 - \beta\sigma).$$

Let $T(v_0)$ be the blow-up time of the solution v. In general, we have an inequality $T(v_0) \geq T(u_0)$, since the following inequality holds:

$$|v(r,t)| \leq R^{\beta\sigma}|u(r,t)|, \quad r \in [0,R],\, t \in [0, \min\{T(u_0), T(v_0)\}).$$

If $x = 0$ is not a blow-up point, by using the fact that blow-up set is closed, there exists a small $\varepsilon > 0$ such that B_ε does not include any blow-up point. From

$$\varepsilon^{\beta\sigma}|u(r,t)| \leq |v(r,t)| \leq R^{\beta\sigma}|u(r,t)|, \quad r \in [\varepsilon,R],\, t \in [0, \min\{T(u_0), T(v_0)\}),$$

it follows that $T(v_0) = T(u_0)$.

Therefore, Theorem 5 follows immediately from the following proposition.

Proposition 4.1. *Let $R \in (0,\infty)$ and let v be a solution of* (4.1). *Let $T \in (0,\infty)$ be the blow-up time of $v(r,t)$. If $r = 0$ is not a blow-up point of v, then*

$$\limsup_{t \nearrow T}(T-t)^{\frac{1}{p-1}}\|v(\cdot,t)\|_{L^\infty(0,R)} < \infty. \tag{4.2}$$

Furthermore, if $r = r_b \in (0,R)$ is a blow-up point of v, then

$$\lim_{t \to T}(T-t)^{\frac{1}{p-1}}v(r_b + (T-t)^{1/2}\xi, t) = \pm(p-1)^{-1/(p-1)}$$

uniformly on $|\xi| \leq C$ for any $C > 0$.

To prove Proposition 4.1, we need some preparations. First, since u satisfies the inequality

$$|u_t - \Delta u| \leq R^\sigma |u|^p,$$

the following lemma can be easily deduced from Theorem 2.1 of [13].

Lemma 4.1. *Let $B_\delta(a) \subset \Omega$ be an open ball with radius $\delta > 0$ and center $a \in \Omega$. Let u be a solution of* (1.1) *on $Q_\delta(a) := B_\delta(a) \times (T - \delta^2, T)$. Suppose that u satisfies* (1.2) *and*

$$\lim_{t \nearrow T}(T-t)^{\frac{1}{p-1}}u(a + (T-t)^{1/2}y, t) = 0$$

uniformly on any compact set $|y| \leq C$ for any $C > 0$. Then $x = a$ is not a blow-up point of u.

In order to analyze the limiting behavior of blow-up solution v, we introduce a rescaling of coordinates. Suppose that $r = 0$ is not a blow-up point of v. Then there exist

$r_0 \in (0, R/3)$ and $\varepsilon \in (0, r_0)$ such that $B_{r_0+2\varepsilon}$ does not contain any blow-up point. Set $r_* := r_0 - \varepsilon/2$. For any $\eta \in [r_0, R)$ and $T > 0$, we define $w := w_{\eta,T}$ by

$$w_{\eta,T}(\xi, s) = (T-t)^{1/(p-1)} v(r,t), \quad \xi = \frac{r-\eta}{\sqrt{T-t}}, \quad s = -\ln(T-t)$$

for $\xi \in (\xi_1(s), \xi_2(s))$ with $\xi_1(s) = (r_* - \eta)e^{s/2}$, $\xi_2(s) = (R-\eta)e^{s/2}$ and $s > s_0 := -\ln T$. Then w satisfies

$$w_s = w_{\xi\xi} - \frac{\xi}{2} w_\xi + \frac{k-1}{\xi + \eta e^{s/2}} w_\xi - \frac{w}{p-1} + |w|^{p-1}w - \frac{lw}{|\xi + \eta e^{s/2}|^2}. \tag{4.3}$$

We also have initial data $w(\xi, s_0) = T^{1/(p-1)} v_0(\eta + \xi\sqrt{T})$ and boundary conditions $w(-\eta e^{s/2}, s) = w(\xi_2(s), s) = 0$. We consider (4.3) on the space-time domain

$$\mathcal{W}_{\eta,r_*} = \{(\xi, s) :\ s > s_0,\ \xi_1(s) < \xi < \xi_2(s)\}.$$

The equation (4.3) is equivalent to

$$w_s = \frac{1}{\rho(\xi)}(\rho(\xi) w_\xi)_\xi + \frac{k-1}{\xi + \eta e^{s/2}} w_\xi - \frac{w}{p-1} + |w|^{p-1}w - \frac{lw}{|\xi + \eta e^{s/2}|^2},$$

where $\rho(\xi) := e^{-\xi^2/4}$.

Let us choose $\eta_0 \in (r_0+\varepsilon, r_0+2\varepsilon)$ arbitrary. Since $B_{r_0+2\varepsilon}$ does not contain any blow-up point of v, we can see that $\bar{w}(\xi, s) := w_{0,T}(\xi, s)$ converges to zero as $s \to \infty$ uniformly on $\xi \le e^{s/2}\eta_0$ and satisfies

$$\bar{w}_s = \bar{w}_{\xi\xi} - \frac{\xi}{2}\bar{w}_\xi + \frac{k-1}{\xi}\bar{w}_\xi - \frac{\bar{w}}{p-1} + |\bar{w}|^{p-1}\bar{w} - \frac{l}{\xi^2}\bar{w}, \quad 0 < \xi < Re^{s/2},\ s > s_0. \tag{4.4}$$

Applying the regularity estimates to (4.4) yields the L^∞-bound for spatial derivatives $\bar{w}_\xi$ and $\bar{w}_{\xi\xi}$ on $(\delta, \eta_0 e^{s/2} - \delta) \times (s_0 + \delta, \infty)$ for any $\delta > 0$. Combining the relation

$$w_{\eta,T}(\xi, s) = w_{0,T}(\xi + \eta e^{s/2}, s)$$

with these bounds, we obtain

$$|w(\xi, s)| + |w_\xi(\xi, s)| + |w_{\xi\xi}(\xi, s)| \le M, \quad |w_s(\xi, s)| \le M(1 + |\xi|) \tag{4.5}$$

for $s \ge \bar{s}$, $r_* \le \eta + e^{-s/2}\xi < r_0 + \varepsilon$, where $\bar{s} := -\ln(T - \bar{t})$ with a fixed $\bar{t} \in (0, T)$ and the constant $M = M(r_*, p, \sigma, N, \bar{s})$.

Following [26], we introduce the following energy functional:

$$E[w](s) = \int_{\xi_1(s)}^{\xi_2(s)} \Big(\frac{1}{2}|w_\xi|^2 + \frac{|w|^2}{2(p-1)} - \frac{|w|^{p+1}}{p+1}\Big)\rho \, d\xi$$

and we shall next establish some estimates for this energy functional.

Proposition 4.2. *Let $\eta \ge r_0$. For any $\delta \in (0, p-1)$, there exists $s_\delta > 0$ such that*

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}\int_{\xi_1(s)}^{\xi_2(s)} |w|^2 \rho \, d\xi &\ge \int_{\xi_1(s)}^{\xi_2(s)} w w_s \rho \, d\xi \\ &\ge -(2+\delta)E[w](s) \\ &\quad + \frac{p-1-\delta}{p+1}\int_{\xi_1(s)}^{\xi_2(s)} |w|^{p+1}\rho \, d\xi - M^2 \rho(\xi_1(s)) \end{aligned} \tag{4.6}$$

for all $s \geq s_\delta$, where s_δ depends on r_*, p, σ and δ. In addition, there exist positive constants $C_1, C_2 > 0$ such that

$$\frac{dE[w](s)}{ds} \leq -\frac{1}{2}\int_{\xi_1(s)}^{\xi_2(s)} |w_s|^2 \rho\, d\xi + C_1 e^{-s}(E[w](s) + C_2) \tag{4.7}$$

for $s \geq s_*$, where $s_*, C_1, C_2 > 0$ depend on r_*, N, p, σ, M, where $M > 0$ is the constant given in (4.5)*.*

Proof. Since $\xi_1(s) \leq -\varepsilon e^{s/2}/2 < 0$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\int_{\xi_1(s)}^{\xi_2(s)} |w|^2\rho\, d\xi &= \int_{\xi_1(s)}^{\xi_2(s)} ww_s\rho\, d\xi - \frac{1}{4}|w(\xi_1(s), s)|^2 \rho(\xi_1(s))\xi_1(s)\\ &\geq \int_{\xi_1(s)}^{\xi_2(s)} ww_s\rho\, d\xi\end{aligned}$$

for all $s \geq s_0$. This proves the first inequality in (4.6).

Next, we show the second inequality in (4.6). Multiplying (4.3) by $w\rho$, by an integration, for each $s \geq \bar{s}$ we have

$$\begin{aligned}\int_{\xi_1(s)}^{\xi_2(s)} ww_s\rho\, d\xi = -2E[w](s) + \frac{p-1}{p+1}\int_{\xi_1(s)}^{\xi_2(s)} |w|^{p+1}\rho\, d\xi + \int_{\xi_1(s)}^{\xi_2(s)} \frac{(k-1)w_\xi w}{\xi + \eta e^{s/2}}\rho\, d\xi\\ - w(\xi_1(s), s)w_\xi(\xi_1(s), s)\rho(\xi_1(s)) - \int_{\xi_1(s)}^{\xi_2(s)} \frac{lw^2}{|\xi + \eta e^{s/2}|^2}\rho\, d\xi.\end{aligned}$$

This and (4.5) yield

$$\begin{aligned}\int_{\xi_1(s)}^{\xi_2(s)} ww_s\rho\, d\xi \geq -2E[w](s) + \frac{p-1}{p+1}\int_{\xi_1(s)}^{\xi_2(s)} |w|^{p+1}\rho\, d\xi - M^2\rho(\xi_1(s))\\ + \int_{\xi_1(s)}^{\xi_2(s)} \frac{(k-1)w_\xi w}{\xi + \eta e^{s/2}}\rho\, d\xi - \int_{\xi_1(s)}^{\xi_2(s)} \frac{lw^2}{|\xi + \eta e^{s/2}|^2}\rho\, d\xi.\end{aligned} \tag{4.8}$$

Now, we estimate the integral terms in (4.8). For each $\varepsilon_1 > 0$, there exists a positive constant $C(\varepsilon_1)$ such that

$$\begin{aligned}\left|\int_{\xi_1(s)}^{\xi_2(s)} \frac{(k-1)w_\xi w}{\xi + \eta e^{s/2}}\rho\, d\xi\right| &\leq \varepsilon_1 \int_{\xi_1(s)}^{\xi_2(s)} |w|^2\rho\, d\xi + C(\varepsilon_1)e^{-s}\int_{\xi_1(s)}^{\xi_2(s)} \left|\frac{w_\xi}{\eta + e^{-s/2}\xi}\right|^2 \rho\, d\xi\\ &\leq \varepsilon_1 \int_{\xi_1(s)}^{\xi_2(s)} |w|^2\rho\, d\xi + \frac{C(\varepsilon_1)e^{-s}}{r_*^2}\int_{\xi_1(s)}^{\xi_2(s)} |w_\xi|^2\rho\, d\xi.\end{aligned}$$

Hence by a similar calculation,

$$\begin{aligned}&\left|\int_{\xi_1(s)}^{\xi_2(s)} \frac{(k-1)w_\xi w}{\xi + \eta e^{s/2}}\rho\, d\xi\right| + \left|\int_{\xi_1(s)}^{\xi_2(s)} \frac{lw^2}{|\xi + \eta e^{s/2}|^2}\rho\, d\xi\right|\\ \leq\ &\left(\varepsilon_1 + \frac{C(p,\sigma)e^{-s}}{r_*^2}\right)\int_{\xi_1(s)}^{\xi_2(s)} |w|^2\rho\, d\xi + \frac{C(\varepsilon_1)e^{-s}}{r_*^2}\int_{\xi_1(s)}^{\xi_2(s)} |w_\xi|^2\rho\, d\xi\end{aligned}$$

for some positive constant $C(p, \sigma)$. By choosing $\bar{t}$ sufficiently close to T, we can assume that

$$\varepsilon_1 + \frac{C(p,\sigma)e^{-s_\delta}}{r_*^2} \leq \frac{\delta}{2(p-1)}, \qquad \frac{C(\varepsilon_1)e^{-s_\delta}}{r_*^2} < \frac{\delta}{2}$$

for some $s_\delta > \bar{s}$. Note that ε_1, s_δ only depend on δ, r_*, p and σ. Using this relation, we obtain

$$\begin{aligned} &\Big| \int_{\xi_1(s)}^{\xi_2(s)} \frac{(k-1)w_\xi w}{\xi + \eta e^{s/2}} \rho\, d\xi \Big| + \Big| \int_{\xi_1(s)}^{\xi_2(s)} \frac{lw^2}{|\xi+\eta e^{s/2}|^2} \rho\, d\xi \Big| \\ \le\ & \delta\Big(E[w](s) + \int_{\xi_1(s)}^{\xi_2(s)} \frac{1}{p+1} |w|^{p+1} \rho\, d\xi \Big). \end{aligned}$$

Hence, from (4.8), the second inequality in (4.6) follows.

Before starting the proof of the inequality (4.7), we prepare one L^{p+1}-bound. Using (4.6) and the inequality

$$|ww_s| \le \varepsilon_2(|w|^{p+1} + |w_s|^2) + c \quad \text{with} \quad \varepsilon_2 < \frac{p-1-\delta}{2(p+1)},$$

where $c = c(\varepsilon_2)$ is a positive constant depending on the small constant $\varepsilon_2 > 0$, we obtain the following L^{p+1}-bound

$$\begin{aligned} &\frac{1}{2} \int_{\xi_1(s)}^{\xi_2(s)} |w|^{p+1} \rho\, d\xi \\ \le\ & \frac{1}{2} \int_{\xi_1(s)}^{\xi_2(s)} |w_s|^2 \rho\, d\xi + \frac{(2+\delta)(p+1)}{p-1-\delta} E[w](s) + \frac{p+1}{p-1-\delta} [M^2 \rho(\xi_1(s)) + c] \end{aligned} \tag{4.9}$$

for all $s \ge s_\delta$. From now on, we fix $\delta = (p-1)/2$.

The definition of $E[w]$, (4.3), and integrating by part give us

$$\begin{aligned} \frac{dE[w]}{ds}(s) = &- \int_{\xi_1(s)}^{\xi_2(s)} |w_s|^2 \rho\, d\xi + \Big[\Big\{ \Big(\frac{|w_\xi|^2}{2} + \frac{|w|^2}{2(p-1)} - \frac{|w|^{p+1}}{p+1} \Big) \frac{\xi}{2} + w_\xi w_s \Big\} \rho(\xi) \Big] \Big|_{\xi_1(s)}^{\xi_2(s)} \\ &+ \int_{\xi_1(s)}^{\xi_2(s)} \frac{(k-1)w_\xi w_s}{\xi+\eta e^{s/2}} \rho\, d\xi - \int_{\xi_1(s)}^{\xi_2(s)} \frac{lww_s}{|\xi+\eta e^{s/2}|^2} \rho\, d\xi \end{aligned}$$

for $s \ge s_0$. Note that the boundary conditions $v_t(R,t) = v(R,t) = 0$ give us

$$w_s(\xi_2(s), s) + \frac{\xi_2(s)}{2} w_\xi(\xi_2(s), s) = 0, \qquad w(\xi_2(s), s) = 0,$$

where the following identity is used

$$v_t = e^{\frac{ps}{p-1}} \Big\{ w_s + \frac{\xi}{2} w_\xi + \frac{w}{p-1} \Big\}.$$

Therefore, we conclude

$$\begin{aligned} \frac{dE[w]}{ds}(s) \ \le\ & - \int_{\xi_1(s)}^{\xi_2(s)} |w_s|^2 \rho\, d\xi + C_1 |\xi_1(s)| \rho(\xi_1(s)) \\ &+ \int_{\xi_1(s)}^{\xi_2(s)} \frac{(k-1)w_\xi w_s}{\xi+\eta e^{s/2}} \rho\, d\xi - \int_{\xi_1(s)}^{\xi_2(s)} \frac{lww_s}{|\xi+\eta e^{s/2}|^2} \rho\, d\xi \end{aligned}$$

for $s \ge \bar{s}$, where the constant $C_1 = C_1(r_*, N, p, M) > 0$. By a similar calculation as above, for each ε_3, $\varepsilon_4 > 0$, there exist $C(\varepsilon_3)$, $C(\varepsilon_4) > 0$ such that

$$\Big| \int_{\xi_1(s)}^{\xi_2(s)} \frac{(k-1)w_\xi w_s}{\xi+\eta e^{s/2}} \rho\, d\xi \Big| \le \varepsilon_3 \int_{\xi_1(s)}^{\xi_2(s)} |w_s|^2 \rho\, d\xi + \frac{C(\varepsilon_3) e^{-s}}{r_*^2} \int_{\xi_1(s)}^{\xi_2(s)} |w_\xi|^2 \rho\, d\xi,$$

$$\Big| \int_{\xi_1(s)}^{\xi_2(s)} \frac{lww_s}{|\xi+\eta e^{s/2}|^2} \rho\, d\xi \Big| \le \varepsilon_4 \int_{\xi_1(s)}^{\xi_2(s)} |w_s|^2 \rho\, d\xi + \frac{C(\varepsilon_4) e^{-2s}}{r_*^4} \int_{\xi_1(s)}^{\xi_2(s)} |w|^2 \rho\, d\xi.$$

By choosing $s_* > s_\delta$ sufficiently large such that $\frac{C(\varepsilon_4)e^{-2s}}{r_*^4} \le \frac{C(\varepsilon_3)e^{-s}}{(p-1)r_*^2}$ for all $s \ge s_*$, we get

$$\Big|\int_{\xi_1(s)}^{\xi_2(s)} \frac{(k-1)w_\xi w_s}{\xi+\eta e^{s/2}}\rho\, d\xi - \int_{\xi_1(s)}^{\xi_2(s)} \frac{lww_s}{|\xi+\eta e^{s/2}|^2}\rho\, d\xi\Big|$$
$$\le (\varepsilon_3+\varepsilon_4)\int_{\xi_1(s)}^{\xi_2(s)} |w_s|^2\rho\, d\xi + \frac{C(\varepsilon_3)e^{-s}}{r_*^2}\Big(2E[w](s) + \frac{2}{p+1}\int_{\xi_1(s)}^{\xi_2(s)} |w|^{p+1}\rho\, d\xi\Big).$$

Combining the above estimates, we obtain

$$\begin{aligned}\frac{d}{ds}E[w](s) \ &\le\ -\frac{3}{4}\int_{\xi_1(s)}^{\xi_2(s)} w_s^2\rho\, d\xi + C_1|\xi_1(s)|\rho(\xi_1(s))\\ &\quad + \frac{C(\varepsilon_3)e^{-s}}{r_*^2}\Big(2E[w](s) + \frac{2}{p+1}\int_{\xi_1(s)}^{\xi_2(s)} |w|^{p+1}\rho\, d\xi\Big)\end{aligned}$$

by choosing $\varepsilon_3, \varepsilon_4 > 0$ sufficiently small. Applying the L^{p+1}-bound (4.9), we get the desired inequality (4.7) for $s \ge s_*$. This completes the proof of the proposition. □

By using Proposition 4.2, we can get the following lemma.

Lemma 4.2. *There exists $K > 0$ such that $-C_2 \le E[w](s) \le K$ and*

$$\int_{s_*}^{\infty}\int_{\xi_1(s)}^{\xi_2(s)} |w_s|^2\rho\, d\xi ds \le K, \tag{4.10}$$

$$\int_{\xi_1(s)}^{\xi_2(s)} |w|^2\rho\, d\xi \le K, \tag{4.11}$$

$$\int_{s}^{s+1}\Big(\int_{\xi_1(s)}^{\xi_2(s)} |w|^{p+1}\rho\, d\xi\Big)^2 ds \le K \tag{4.12}$$

for all $s \ge s_*$, where s_* and C_2 are constants given in Proposition 4.2 and K depends only on r_*, N, p, σ, M and $E[w](s_*)$ and M is the constant given in (4.5).

Proof. In this proof, we extend the function $w = w_{\eta,T}(\cdot, s)$ to the whole space $\mathbb{R}$, by defining $w = 0$ on $(-\infty, \xi_1(s)) \cup (\xi_2(s), \infty)$.

First we prove the energy bound. Assume on the contrary that there exists $s_1 \ge s_*$ such that $E[w](s_1) < -C_2$. Then (4.7) implies that $E[w](s)$ is monotone decreasing for $s \ge s_1$, in particular, $E[w](s) \le -C_2$ for $s \ge s_1$. This, (4.6) and Jensen's inequality yield

$$\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}} |w|^2\rho\, d\xi \ge \frac{p-1-\delta}{p+1}\Big(\int_{\mathbb{R}} |w|^2\rho\, d\xi\Big)^{\frac{p+1}{2}} + (2+\delta)C_2 - M^2\rho(\xi_1(s)).$$

Then, for s sufficiently large, we have

$$\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}} |w|^2\rho\, d\xi \ge \frac{p-1-\delta}{p+1}\Big(\int_{\mathbb{R}} |w|^2\rho\, d\xi\Big)^{\frac{p+1}{2}}$$

and so w blows up in finite time in L^2-sense. This contradicts the global existence of w. On the other hand, (4.7) also implies that

$$\frac{dE[w]}{ds}(s) \le C_1 e^{-s}(E[w](s) + C_2).$$

By integrating this inequality, we get the following upper bound estimate

$$E[w](s) \le K_1 := -C_2 + (E[w](s_*) + C_2)e^{C_1 e^{-s_*}}.$$

Moreover, (4.10) immediately follows by integrating (4.7) and using the bound for $E[w](s)$.

Next, we prove (4.11). Upper bound of $E[w]$, (4.6) and Jensen's inequality yield

$$\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}}|w|^2\rho\,d\xi \geq \frac{p-1-\delta}{p+1}\Big(\int_{\mathbb{R}}|w|^2\rho\,d\xi\Big)^{\frac{p+1}{2}} - M^2\rho(\xi_1(s)) - (2+\delta)K_1. \tag{4.13}$$

In order that w exists globally, the right-hand side of (4.13) cannot be positive at any $s \geq s_*$. Hence

$$\frac{p-1-\delta}{p+1}\Big(\int_{\mathbb{R}}|w|^2\rho\,d\xi\Big)^{\frac{p+1}{2}} \leq M^2\rho(\xi_1(s)) + (2+\delta)K_1.$$

must hold for all $s \geq s_*$. This proves (4.11).

Finally, we prove the estimate (4.12). Again by (4.6), (4.7) and upper bound of $E[w]$,

$$\begin{aligned}\frac{p-1-\delta}{p+1}\Big(\int_{\mathbb{R}}|w|^{p+1}\rho\,d\xi\Big) &\leq \int_{\mathbb{R}} ww_s\rho\,d\xi + (2+\delta)K_1 + M^2\rho(\xi_1(s))\\ &\leq \Big(\int_{\mathbb{R}}|w|^2\rho\,d\xi\Big)^{1/2}\Big(\int_{\mathbb{R}}|w_s|^2\rho\,d\xi\Big)^{1/2} + K_2\\ &\leq K_3\Big(2\frac{dE[w]}{ds} + 2C_1e^{-s}(E[w](s)+C_2)\Big)^{1/2} + K_2\end{aligned}$$

for some positive constants K_2, K_3. By taking a square and integrating it for time, we obtain the desired inequality (4.12). The proof of the lemma is thus completed. ☐

Proof of Proposition 4.1. Recall $w = w_{\eta,T}$. Note that (4.3) can be rewritten as

$$\begin{aligned} w_s &= w_{\xi\xi} + \Big(\frac{k-1}{\xi+\eta e^{s/2}} - \frac{\xi}{2}\Big)w_\xi + L(\xi,s)w,\\ L(\xi,s) &:= |w|^{p-1} - \frac{1}{p-1} - \frac{l}{|\xi+\eta e^{s/2}|^2}.\end{aligned}$$

We can easily derive that

$$\int_s^{s+1}\Big(\int_{|\xi|\leq 1}|L(\xi,s)|^{(p+1)/(p-1)}\rho\,d\xi\Big)^2 ds \leq K_5, \quad s \geq s_*.$$

This estimate, (4.11) and Theorem 8.1 of [23] give us an L^∞-bound

$$|w_{\eta,T}(\xi,s)| \leq K \quad \text{for} \quad |\xi| \leq \frac{1}{2},\ s \geq s_* + \delta \tag{4.14}$$

for some constants $\delta > 0$ and $K > 0$ so that all the above estimates in Proposition 4.2 and Lemma 4.2 hold. Also, the constants δ and K can be chosen independent of the rescaling point $r = \eta$ for $\eta \in [r_0, R]$. Hence (4.2) follows.

We observe that (4.14) yields

$$\|\nabla u(\cdot,t)\|_{L^\infty(B_R)} \leq M_1(T-t)^{-\frac{1}{p-1}-\frac{1}{2}}, \quad T/2 \leq t < T$$

for some constant $M_1 = M_1(K,p,\sigma,R,T) > 0$. This gradient estimate and $w_\xi(\xi,s) = (T-t)^{\frac{1}{p-1}+\frac{1}{2}}u_r(r,t)$ give us the boundedness of w_ξ. From these and (4.10) we conclude that the ω-limit set of $w_{\eta,T}$ is a compact connected subset included in the set of bounded solutions of the problem

$$U_{\xi\xi} - \frac{\xi}{2}U_\xi - \frac{U}{p-1} + |U|^{p-1}U = 0, \quad \xi \geq 0, \quad U_\xi(0) = 0.$$

It is known from Theorem 1 of [11] that the only bounded solution of this one dimensional elliptic problem is constant, i.e., it is either $\pm\kappa$ or 0. Thus the ω-limit set of $w_{\eta,T}$ is contained in the set $\{\kappa, -\kappa, 0\}$. Furthermore, for a blow-up point $\eta = r_b$, 0 is excluded in the ω-limit set by using Lemma 4.1. This completes the proof. $\square$

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