

FORCED WAVES OF PULSE TYPE FOR 3-SPECIES COMPETITION SYSTEMS IN SHIFTING ENVIRONMENTS

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ABSTRACT. We investigate the existence and non-existence of forced waves of pulse type for three-species diffusive competition systems in shifting environments. With a careful construction of generalized upper-lower solutions, we are able to construct infinitely many forced waves of pulse type for each admissible wave speed. This extends the work by Fang, Lou and Wu (SIAM J. Appl. Math. 76 (2016), 1633-1657) from scalar equations to non-cooperative systems. Moreover, the construction of forced waves at the critical speed is also provided.

1. INTRODUCTION

In this paper, we study the following 3-species diffusive Lotka-Volterra competition system in a shifting environment

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u[\alpha(x - st) - u - kv - aw], & x \in \mathbb{R}, t > 0, \\ v_t = d_2 v_{xx} + r_2 v[\alpha(x - st) - hu - v - bw], & x \in \mathbb{R}, t > 0, \\ w_t = d_3 w_{xx} + r_3 w[\alpha(x - st) - c_1 u - c_2 v - w], & x \in \mathbb{R}, t > 0, \end{cases} \quad (1.1)$$

where u, v, w , functions of (x, t) , are three competing species with inter-specific competition coefficients h, k, a, b, c_1, c_2 , diffusion coefficients d_i and maximal intrinsic growth rates r_i , $i = 1, 2, 3$. All parameters are assumed to be positive. Also, we have normalized the intra-specific competition coefficients to 1. The function α models the shifting environment with the same shifting speed $s > 0$.

Climate change, such as global warming, is one of the reasons for environmental changes in ecology. It can affect the survival and extinction of species in ecological systems. It may cause changes in the environment, such as desertification and drastic fluctuation of precipitation. Due to global warming, this planet has suffered from many natural disasters in recent years. Starting with the pioneering work [2] on modeling climate change, it has attracted significant attention to the study of ecological systems in a shifting environment. We refer the reader to, e.g. [1, 9, 6, 10] for works on persistence of species, and [4, 5, 12, 3, 16, 17, 8, 15, 7, 11] for the existence of forced waves. Here, a forced wave is a traveling wave with a speed the same as the environmental shifting speed.

Throughout this paper, we assume that α is continuous in $\mathbb{R}$ such that

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- (a1) the limits $\alpha(\pm\infty)$ exist with $\alpha(-\infty) < 0$ and $\alpha(z) \leq 1 = \alpha(+\infty)$ for all $z \in \mathbb{R}$;
 (a2) there are constants $C > 0$, $K > 0$ and $\rho > 0$ such that

$$1 - \alpha(z) \leq Ce^{-\rho z}, \quad \forall z > K. \quad (1.2)$$

We are concerned with the existence of a forced wave, namely, a traveling wave solution (u, v, w) of (1.1) in the form

$$(u, v, w)(x, t) = (\phi_1, \phi_2, \phi_3)(z), \quad z := x - st,$$

with wave speed the same as the environmental shifting speed s and wave profiles $\{\phi_1, \phi_2, \phi_3\}$. Hence we are looking for unknown $\{\phi_1, \phi_2, \phi_3\}$ that satisfies

$$\begin{cases} d_1\phi_1'' + s\phi_1' + r_1\phi_1(\alpha - \phi_1 - k\phi_2 - a\phi_3) = 0, & z \in \mathbb{R}, \\ d_2\phi_2'' + s\phi_2' + r_2\phi_2(\alpha - h\phi_1 - \phi_2 - b\phi_3) = 0, & z \in \mathbb{R}, \\ d_3\phi_3'' + s\phi_3' + r_3\phi_3(\alpha - c_1\phi_1 - c_2\phi_2 - \phi_3) = 0, & z \in \mathbb{R}. \end{cases} \quad (1.3)$$

Due to the assumption $\alpha(-\infty) < 0$ in (a1), any positive solution (ϕ_1, ϕ_2, ϕ_3) of (1.3) must satisfy the condition

$$(\phi_1, \phi_2, \phi_3)(-\infty) = (0, 0, 0). \quad (1.4)$$

Hereafter we call (ϕ_1, ϕ_2, ϕ_3) a *positive* solution of (1.3) if $\phi_1, \phi_2, \phi_3 > 0$ in $\mathbb{R}$. Note that, by the strong maximum principle, $\phi_1 > 0$ in $\mathbb{R}$ if $\phi_1 \geq 0$ and $\phi_1 \not\equiv 0$ in $\mathbb{R}$. Similar results hold for ϕ_2 and ϕ_3 .

Remark 1.1. *The limiting condition (1.4) can be verified in the same manner as that of [7, Proposition 2.2]. Indeed, the proof is by a contradiction argument as follows. Assume for contradiction that $\phi_1^+ := \limsup_{z \rightarrow -\infty} \phi_1(z) > 0$. In the case when $\phi_1(z)$ is oscillatory at $z = -\infty$, we can find a sequence of maximal points $\{z_n\}$ of ϕ_1 such that $z_n \rightarrow -\infty$ and $\phi_1(z_n) \rightarrow \phi_1^+$ as $n \rightarrow \infty$. Then, using condition $\alpha(-\infty) < 0$, we derive*

$$\begin{aligned} 0 &\leq -(d_1\phi_1'' + s\phi_1')(z_n) = r_1\phi_1(z_n)(\alpha - \phi_1 - k\phi_2 - a\phi_3)(z_n) \\ &\leq r_1\phi_1(z_n)(\alpha - \phi_1)(z_n) \rightarrow r_1\phi_1^+[\alpha(-\infty) - \phi_1^+] < 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

a contradiction. On the other hand, when $\phi_1(z)$ is monotone ultimately at $z = -\infty$, we can find a positive constant L and a sequence $\{z_n\} \subset (-\infty, -L)$ such that $\alpha(z) \leq 0$, $\phi_1(z) \geq \phi_1^+/2$, $\forall z \leq -L$, $z_n \rightarrow -\infty$ and $\phi_1'(z_n) \rightarrow 0$ as $n \rightarrow \infty$. Integrating ϕ_1 -equation over $(z_n, -L)$, we obtain

$$d_1\phi_1'(-L) - d_1\phi_1'(z_n) + s\phi_1(-L) - s\phi_1(z_n) \geq r_1 \int_{z_n}^{-L} [\phi_1(z)]^2 dz \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

a contradiction, since the left-hand side is uniformly bounded for all n . Hence $\phi_1(-\infty) = 0$. The same argument shows that $\phi_j(-\infty) = 0$ for $j = 2, 3$.

Forced waves of mixed front-pulse types for diffusive competition systems were studied in our previous paper [11]. Forced waves of pulse type were constructed in the work [9] for a scalar equation with KPP-type nonlinearity. Since a scalar equation enjoys a comparison principle, the classical monotone iteration method is applicable as long as a pair of suitable upper-lower-solutions can be constructed. In particular, they were able to construct infinitely many different forced waves of pulse type for each admissible shifting speed s (see [9, Theorem 2.1(iii) & p.1653] for more details), under the condition that $1 - \alpha$ is integrable at $+\infty$. However, the existence of forced waves with critical speed is left open in [9]. The aim of this paper is to extend the work [9] on forced waves of pulse type from scalar equations to non-cooperative systems with a slightly stronger condition (1.2) on α . We also provide the derivation of forced waves of pulse type with critical speed.

Our main theorem reads

Theorem 1.2. *For any $s \geq s^* := 2 \max_{1 \leq j \leq 3} \{\sqrt{d_j r_j}\}$, let λ_j be the minimal positive root of $d_j \lambda_j^2 - s \lambda_j + r_j = 0$, $j = 1, 2, 3$, respectively. Let $\{\gamma_1, \gamma_2, \gamma_3\}$ be any positive numbers. For each $s > s^*$, (1.3) has a positive solution (ϕ_1, ϕ_2, ϕ_3) satisfying $(\phi_1, \phi_2, \phi_3)(+\infty) = (0, 0, 0)$ such that*

$$\lim_{z \rightarrow +\infty} e^{\lambda_j z} \phi_j(z) = \gamma_j, \quad j = 1, 2, 3. \quad (1.5)$$

While for $s = s^*$, (1.3) has a positive solution (ϕ_1, ϕ_2, ϕ_3) satisfying $(\phi_1, \phi_2, \phi_3)(+\infty) = (0, 0, 0)$ such that

$$\lim_{z \rightarrow +\infty} e^{\lambda_j z} \phi_j(z) = \gamma_j, \quad \text{if } s^* > 2\sqrt{d_j r_j}; \quad \lim_{z \rightarrow +\infty} \frac{e^{\lambda_j z}}{z} \phi_j(z) = \gamma_j, \quad \text{if } s^* = 2\sqrt{d_j r_j}. \quad (1.6)$$

Conversely, a positive solution of (1.3) with $(\phi_1, \phi_2, \phi_3)(+\infty) = (0, 0, 0)$ exists only when $s \geq s^*$.

In fact, one can easily check from our derivation (see §3 below) that Theorem 1.2 extends to an n -species competition system, $n \geq 2$, of the form

$$(u_i)_t = d_i(u_i)_{xx} + r_i u_i \left[\alpha(x - st) - u_i - \sum_{k \neq i} a_{i,k} u_k \right], \quad i = 1, \dots, n,$$

with positive competition coefficients $a_{i,k}$. In this case, the critical speed is given by

$$s^* = 2 \max_{1 \leq i \leq n} \sqrt{d_i r_i}.$$

Indeed, the construction of the generalized upper and lower solutions, the fixed-point argument, and the non-existence argument extend in the same manner. We restrict our presentation to the 3-species system to avoid cumbersome notation and to make the main construction more transparent.

Note that the limit γ_j in (1.5) can be any arbitrarily positive constant for each $j = 1, 2, 3$. In comparison with [9] on the KPP-type scalar equation, they only provide infinitely many forced waves with discrete multiples γ . Note that a multiple of a given solution is not necessarily a solution to a nonlinear equation. On the other hand, since the classical monotone iteration method is not applicable to non-cooperative systems, we adopt the method of generalized upper-lower-solutions with the help of Schauder's fixed-point theorem, which is a very powerful method in deriving traveling waves of systems without a monotonicity property, such as competition systems, predator-prey systems, and epidemiology models.

The rest of this paper is organized as follows. In the next section, we shall introduce the notion of generalized upper-lower solutions and provide an existence theorem for forced waves of (1.3). Finally, in §3, a detailed proof of Theorem 1.2 is given.

2. PRELIMINARIES

We first introduce the definition of generalized upper-lower solutions of (1.3) as follows.

Definition 2.1. *Continuous functions $(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ are called a pair of generalized upper-lower solutions of (1.3) if they have bounded first and second derivatives in $\mathbb{R}$ such that the following inequalities*

$$\mathcal{U}_1 := d_1 \bar{\phi}_1'' + s \bar{\phi}_1' + r_1 \bar{\phi}_1 (\alpha - \bar{\phi}_1 - k \underline{\phi}_2 - a \underline{\phi}_3) \leq 0, \quad (2.1)$$

$$\mathcal{U}_2 := d_2 \bar{\phi}_2'' + s \bar{\phi}_2' + r_2 \bar{\phi}_2 (\alpha - h \underline{\phi}_1 - \bar{\phi}_2 - b \underline{\phi}_3) \leq 0, \quad (2.2)$$

$$\mathcal{U}_3 := d_3 \bar{\phi}_3'' + s \bar{\phi}_3' + r_3 \bar{\phi}_3 (\alpha - c_1 \underline{\phi}_1 - c_2 \underline{\phi}_2 - \bar{\phi}_3) \leq 0, \quad (2.3)$$

$$\mathcal{L}_1 := d_1 \underline{\phi}_1'' + s \underline{\phi}_1' + r_1 \underline{\phi}_1 (\alpha - \underline{\phi}_1 - k \bar{\phi}_2 - a \bar{\phi}_3) \geq 0, \quad (2.4)$$

$$\mathcal{L}_2 := d_2 \underline{\phi}_2'' + s \underline{\phi}_2' + r_2 \underline{\phi}_2 (\alpha - h \bar{\phi}_1 - \underline{\phi}_2 - b \bar{\phi}_3) \geq 0, \quad (2.5)$$

$$\mathcal{L}_3 := d_3 \underline{\phi}_3'' + s \underline{\phi}_3' + r_3 \underline{\phi}_3 (\alpha - c_1 \bar{\phi}_1 - c_2 \bar{\phi}_2 - \underline{\phi}_3) \geq 0, \quad (2.6)$$

hold in $\mathbb{R} \setminus \{z_j \mid j = 1, \dots, m\}$ with some finite positive integer m .

In fact, Theorem 1.2 holds for system (1.1) with different α for different species, namely, replacing α by α_1 in u -equation, α_2 in v -equation and α_3 in w -equation, respectively, in which α_j , $j = 1, 2, 3$, are different such that (1.2) holds with different constants C_j, K_j, ρ_j . Then, by choosing

$$C = \max\{C_1, C_2, C_3\}, \quad K = \max\{K_1, K_2, K_3\}, \quad \rho = \min\{\rho_1, \rho_2, \rho_3\},$$

condition (1.2) holds for all α_j , $j = 1, 2, 3$. Moreover, by a suitable translation of system (1.3) and using $\alpha(+\infty) = 1$, condition (1.2) can be rephrased as

$$\alpha(z) \geq 1 - \varepsilon e^{-\rho z}, \quad \forall z > 0, \quad (2.7)$$

1 for any specified positive constant ε . Indeed, given any constant $\varepsilon > 0$, by (1.2), we have

$$1 - \alpha(z + K) \leq Ce^{-\rho(z+K)} \leq Ce^{-\rho K} e^{-\rho z} \leq \varepsilon e^{-\rho z}, \quad \forall z > 0,$$

2 if we make K larger so that $Ce^{-\rho K} \leq \varepsilon$. Then a forced wave $(\phi_1, \phi_2, \phi_3)(z)$ of the translated
3 system:

$$\begin{cases} d_1 \phi_1''(z) + s \phi_1'(z) + r_1 \phi_1(z) [\alpha(z + K) - \phi_1(z) - k \phi_2(z) - a \phi_3(z)] = 0, & z \in \mathbb{R}, \\ d_2 \phi_2''(z) + s \phi_2'(z) + r_2 \phi_2(z) [\alpha(z + K) - h \phi_1(z) - \phi_2(z) - b \phi_3(z)] = 0, & z \in \mathbb{R}, \\ d_3 \phi_3''(z) + s \phi_3'(z) + r_3 \phi_3(z) [\alpha(z + K) - c_1 \phi_1(z) - c_2 \phi_2(z) - \phi_3(z)] = 0, & z \in \mathbb{R}, \end{cases}$$

4 gives a forced wave $(\phi_1, \phi_2, \phi_3)(z - K)$ of the original system (1.3). Therefore, from now on
5 we shall assume (2.7) with a suitable choice of ε instead of (1.2).

6 With Definition 2.1, the following existence theorem for system (1.3) can be proved as
7 that in, e.g. [14, 13]. We omit its proof here.

8 **Proposition 2.2.** *Given $s > 0$. If (1.3) has a pair of generalized upper-lower solutions*
9 *$(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ such that*

$$\underline{\phi}_1 \leq \bar{\phi}_1, \quad \underline{\phi}_2 \leq \bar{\phi}_2, \quad \underline{\phi}_3 \leq \bar{\phi}_3 \quad \text{in } \mathbb{R}, \quad (2.8)$$

$$\bar{\phi}_1'(z_j^+) \leq \bar{\phi}_1'(z_j^-), \quad \underline{\phi}_1'(z_j^-) \leq \underline{\phi}_1'(z_j^+), \quad \forall j = 1, \dots, m, \quad (2.9)$$

$$\bar{\phi}_2'(z_j^+) \leq \bar{\phi}_2'(z_j^-), \quad \underline{\phi}_2'(z_j^-) \leq \underline{\phi}_2'(z_j^+), \quad \forall j = 1, \dots, m, \quad (2.10)$$

$$\bar{\phi}_3'(z_j^+) \leq \bar{\phi}_3'(z_j^-), \quad \underline{\phi}_3'(z_j^-) \leq \underline{\phi}_3'(z_j^+), \quad \forall j = 1, \dots, m, \quad (2.11)$$

10 then (1.3) has a solution (ϕ_1, ϕ_2, ϕ_3) such that $\underline{\phi}_1 \leq \phi_1 \leq \bar{\phi}_1$, $\underline{\phi}_2 \leq \phi_2 \leq \bar{\phi}_2$ and $\underline{\phi}_3 \leq \phi_3 \leq \bar{\phi}_3$.
11 Here, $\phi(z^-)$ and $\phi(z^+)$ denote the left-hand and right-hand limits of ϕ at z , respectively.

12 For the reader's convenience, we provide an outline of the proof as follows. For simplicity,
13 we give the argument under the assumption that $0 \leq \underline{\phi}_j \leq \bar{\phi}_j \leq 1$ ($j = 1, 2, 3$) holds, but the
14 argument can proceed without this assumption provided that the generalized upper-lower
15 solutions are bounded.

16 First, we take a sufficiently large constant β such that $F_j(\phi_1, \phi_2, \phi_3)$ is non-decreasing in
17 ϕ_j and decreasing in ϕ_i for $i \neq j$, where

$$\begin{cases} F_1(\phi_1, \phi_2, \phi_3) := \beta \phi_1 + r_1 \phi_1 (\alpha - \phi_1 - k \phi_2 - a \phi_3), \\ F_2(\phi_1, \phi_2, \phi_3) := \beta \phi_2 + r_2 \phi_2 (\alpha - h \phi_1 - \phi_2 - b \phi_3), \\ F_3(\phi_1, \phi_2, \phi_3) := \beta \phi_3 + r_3 \phi_3 (\alpha - c_1 \phi_1 - c_2 \phi_2 - \phi_3) \end{cases}$$

18 for $(\phi_1, \phi_2, \phi_3) \in X := [0, 1]^3$ for $j = 1, 2, 3$. Then (1.3) is transformed to

$$d_i \phi_i'' + s \phi_i' - \beta \phi_i + F_i(\phi_1, \phi_2, \phi_3) = 0, \quad z \in \mathbb{R}, \quad i = 1, 2, 3.$$

19 Next, for a given $s > 0$, we introduce the integral operator

$$P_i(\phi_1, \phi_2, \phi_3)(z) := \frac{1}{d_i(\lambda_i^+ - \lambda_i^-)} \left[\int_{-\infty}^z e^{\lambda_i^-(z-y)} + \int_z^{\infty} e^{\lambda_i^+(z-y)} \right] F_i(\phi_1, \phi_2, \phi_3)(y) dy$$

where $\lambda_i^+ > 0 > \lambda_i^-$ are the two roots of $d_i\lambda^2 + s\lambda - \beta = 0$, $i = 1, 2, 3$. Also, for a given pair of upper-lower solutions $(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$, we choose the set

$$\Sigma := \{(\phi_1, \phi_2, \phi_3) \in [0, 1]^3 \mid \underline{\phi}_j \leq \phi_j \leq \bar{\phi}_j \text{ in } \mathbb{R}, j = 1, 2, 3\}.$$

Due to the choice of β , (2.9)-(2.11), and the definition of upper-lower solutions, one can check that $P := (P_1, P_2, P_3)$ maps Σ into itself.

Finally, by a similar argument as in [14], one shows that $P : \Sigma \rightarrow \Sigma$ is completely continuous in the suitable weighted sup-norm. The proof is then completed by Schauder's fixed-point theorem.

3. PROOF OF THEOREM 1.2

This section is devoted to the proof of Theorem 1.2. We assume that

$$s \geq s^* = \max\{2\sqrt{d_1 r_1}, 2\sqrt{d_2 r_2}, 2\sqrt{d_3 r_3}\}$$

so that there exists the minimal positive root λ_j of

$$d_j \lambda_j^2 - s \lambda_j + r_j = 0 \tag{3.1}$$

for each $j \in \{1, 2, 3\}$. Also, in the sequel we set $\sigma := \min\{\lambda_1, \lambda_2, \lambda_3, \rho\}$.

3.1. Existence: super-critical speeds.

Fix $\varepsilon \in (0, 1)$ in (2.7). Let $s > s^*$. We fix constants $\eta_j \in (\lambda_j, \lambda_j + \sigma)$ satisfying

$$d_j \eta_j^2 - s \eta_j + r_j < 0, j = 1, 2, 3.$$

Let $\gamma_1, \gamma_2, \gamma_3$ be any positive numbers. Set $\gamma_j = e^{\lambda_j z_j}$ for some $z_j \in \mathbb{R}$, $j = 1, 2, 3$. Then we introduce

$$\begin{aligned} \bar{\phi}_1(z) &= \min\{1, e^{-\lambda_1(z-z_1)}\}, & \underline{\phi}_1(z) &= \max\{0, e^{-\lambda_1(z-z_1)} - C_1 e^{-\eta_1(z-z_1)}\}, \\ \bar{\phi}_2(z) &= \min\{1, e^{-\lambda_2(z-z_2)}\}, & \underline{\phi}_2(z) &= \max\{0, e^{-\lambda_2(z-z_2)} - C_2 e^{-\eta_2(z-z_2)}\}, \\ \bar{\phi}_3(z) &= \min\{1, e^{-\lambda_3(z-z_3)}\}, & \underline{\phi}_3(z) &= \max\{0, e^{-\lambda_3(z-z_3)} - C_3 e^{-\eta_3(z-z_3)}\}, \end{aligned}$$

where C_1, C_2, C_3 are sufficiently large constants to be fixed later.

For convenience, we prepare the following lemma which can be easily checked.

Lemma 3.1. *Let $A > 0$ and $a_2 > a_1 > 0$. Then $e^{-a_1 z} - A e^{-a_2 z} > 0$ if and only if*

$$z > \frac{\ln A}{a_2 - a_1}.$$

Based on Lemma 3.1, we set

$$\xi_j := z_j + \frac{\ln C_j}{\eta_j - \lambda_j}, j = 1, 2, 3,$$

1 to obtain

$$\underline{\phi}_j(z) = \begin{cases} 0, & z < \xi_j, \\ e^{-\lambda_j(z-z_j)} - C_j e^{-\eta_j(z-z_j)}, & z > \xi_j, \end{cases}$$

2 for $j = 1, 2, 3$. Also, note that for $j = 1, 2, 3$,

$$\bar{\phi}_j(z) = \begin{cases} 1, & z < z_j, \\ e^{-\lambda_j(z-z_j)}, & z > z_j. \end{cases}$$

3 Then we fix C_1 sufficiently large so that $\xi_1 > \max\{0, z_2, z_3\}$ and

$$C_1 > \frac{r_1\{(\varepsilon + ke^{\lambda_2 z_2} + ae^{\lambda_3 z_3})e^{\lambda_1 z_1} + e^{2\lambda_1 z_1}\}}{-(d_1\eta_1^2 - s\eta_1 + r_1)e^{\eta_1 z_1}}. \quad (3.2)$$

4 Likewise, choose C_2 sufficiently large so that $\xi_2 > \max\{0, z_1, z_3\}$ and

$$C_2 > \frac{r_2\{(\varepsilon + he^{\lambda_1 z_1} + be^{\lambda_3 z_3})e^{\lambda_2 z_2} + e^{2\lambda_2 z_2}\}}{-(d_2\eta_2^2 - s\eta_2 + r_2)e^{\eta_2 z_2}}. \quad (3.3)$$

5 Lastly, fix C_3 sufficiently large so that $\xi_3 > \max\{0, z_1, z_2\}$ and

$$C_3 > \frac{r_3\{(\varepsilon + c_1 e^{\lambda_1 z_1} + c_2 e^{\lambda_2 z_2})e^{\lambda_3 z_3} + e^{2\lambda_3 z_3}\}}{-(d_3\eta_3^2 - s\eta_3 + r_3)e^{\eta_3 z_3}}. \quad (3.4)$$

6 Using $\alpha \leq 1$, for $z < z_1$ we compute

$$\mathcal{U}_1(z) = r_1[\alpha(z) - 1 - k\underline{\phi}_2(z) - a\underline{\phi}_3(z)] \leq 0.$$

7 When $z > z_1$, we obtain

$$\mathcal{U}_1(z) \leq [d_1\lambda_1^2 - s\lambda_1 + r_1\alpha(z)]e^{-\lambda_1(z-z_1)} = r_1(\alpha(z) - 1)e^{-\lambda_1(z-z_1)} \leq 0$$

8 by using (3.1). Hence $\mathcal{U}_1(z) \leq 0$ for all $z \neq z_1$. Similarly, we can verify that $\mathcal{U}_j(z) \leq 0$ for
9 all $z \neq z_j$, $j = 2, 3$.

10 Next, let us show that $\mathcal{L}_1(z) \geq 0$ for any $z \in \mathbb{R} \setminus \{\xi_1\}$. Since $\underline{\phi}_1(z) \equiv 0$ for $z \in (-\infty, \xi_1)$,
11 it follows that $\mathcal{L}_1(z) \equiv 0$ on this interval. For $z > \xi_1$, we have

$$\begin{aligned} \mathcal{L}_1(z) &= \{d_1\lambda_1^2 - s\lambda_1 + r_1[\alpha(z) - k\bar{\phi}_2(z) - a\bar{\phi}_3(z)]\}e^{-\lambda_1(z-z_1)} \\ &\quad - C_1\{d_1\eta_1^2 - s\eta_1 + r_1[\alpha(z) - k\bar{\phi}_2(z) - a\bar{\phi}_3(z)]\}e^{-\eta_1(z-z_1)} - r_1(\underline{\phi}_1(z))^2 \\ &\geq r_1\{-\varepsilon e^{-\rho z} - ke^{-\lambda_2(z-z_2)} - ae^{-\lambda_3(z-z_3)}\}e^{-\lambda_1(z-z_1)} \\ &\quad - C_1(d_1\eta_1^2 - s\eta_1 + r_1)e^{-\eta_1(z-z_1)} - r_1e^{-2\lambda_1(z-z_1)} \\ &\geq e^{-\eta_1 z}\{-C_1(d_1\eta_1^2 - s\eta_1 + r_1)e^{\eta_1 z_1} - r_1[(\varepsilon + ke^{\lambda_2 z_2} + ae^{\lambda_3 z_3})e^{\lambda_1 z_1} + e^{2\lambda_1 z_1}]\} \geq 0, \end{aligned}$$

12 using (2.7), $\lambda_1 < \eta_1 < \lambda_1 + \sigma$ and (3.2). Thus, we obtain $\mathcal{L}_1(z) \geq 0$ for any $z \in \mathbb{R} \setminus \{\xi_1\}$.

13 By similar calculations, one can show that $\mathcal{L}_j(z) \geq 0$ in $\mathbb{R} \setminus \{\xi_j\}$, using (2.7), $\lambda_j < \eta_j <$
14 $\lambda_j + \sigma$, $j = 2, 3$, (3.3) and (3.4). Therefore, we have (2.1)-(2.6) for all $z \notin \{z_1, z_2, z_3, \xi_1, \xi_2, \xi_3\}$.

1 It is obvious that (2.8)-(2.11) hold. Thus the existence of forced waves with (1.5) for super-
 2 critical speeds follows from Proposition 2.2. Moreover, (1.5) holds due to $\underline{\phi}_j \leq \phi_j \leq \bar{\phi}_j$,
 3 $j = 1, 2, 3$, since $\gamma_j = e^{\lambda_j z_j}$. $\square$

4 3.2. Existence: the critical speed.

5

6 For notational simplicity, we may assume that $d_1 r_1 \geq d_2 r_2 \geq d_3 r_3$. Given any positive
 7 numbers $\gamma_1, \gamma_2, \gamma_3$.

8 **Case 1.** $s = s^* = 2\sqrt{d_1 r_1} > 2\sqrt{d_2 r_2}$.

9 Same notation as the super-critical case, except that λ_1 is a double root in (3.1). Also, we
 10 choose $z_1 \in \mathbb{R}$ such that $e^{\lambda_1 z_1} = \gamma_1/(\lambda_1 e)$ and z_j satisfying $e^{\lambda_j z_j} = \gamma_j$, $j = 2, 3$.

11 We introduce the following upper-lower-solutions:

$$\bar{\phi}_1(z) = \begin{cases} 1, & z < z_1 + 1/\lambda_1, \\ A_1(z - z_1)e^{-\lambda_1(z-z_1)}, & z > z_1 + 1/\lambda_1, \end{cases} \quad (3.5)$$

$$\underline{\phi}_1(z) = \begin{cases} 0, & z < z_1 + \xi_1, \\ A_1(z - z_1)e^{-\lambda_1(z-z_1)} - p_1\sqrt{z - z_1}e^{-\lambda_1(z-z_1)}, & z > z_1 + \xi_1, \end{cases} \quad (3.6)$$

12 where $A_1 := \lambda_1 e$, p_1 is a positive constant to be determined, $\xi_1 := (p_1/A_1)^2$ and

$$\bar{\phi}_j(z) = \begin{cases} 1, & z < z_j, \\ e^{-\lambda_j(z-z_j)}, & z > z_j, \end{cases} \quad \underline{\phi}_j(z) = \begin{cases} 0, & z < \xi_j, \\ e^{-\lambda_j(z-z_j)} - C_j e^{-\eta_j(z-z_j)}, & z > \xi_j, \end{cases}$$

13 for $j = 2, 3$.

14 Here, we introduce the following lemma, which can be easily shown.

15 **Lemma 3.2.** *Let $A > 0$ and $a_2 > a_1 > 0$. Then $e^{-a_1 y} - Aye^{-a_2 y} \geq 0$ holds for any $y \in \mathbb{R}$ if*

$$A \leq e(a_2 - a_1).$$

16 One can also check that

$$\max_{y \geq 0} \{y^{7/2} e^{-\sigma y}\} = \left(\frac{7}{2\sigma e}\right)^{7/2} =: B. \quad (3.7)$$

17 Then we fix p_1 sufficiently large so that $\xi_1 \geq \max\{1, -z_1, z_2 - z_1, z_3 - z_1\}$ and

$$p_1 \geq \frac{4r_1 A_1 B}{d_1} (\varepsilon e^{-\rho z_1} + A_1 + k\gamma_2 e^{-\lambda_2 z_1} + a\gamma_3 e^{-\lambda_3 z_1}). \quad (3.8)$$

18 We also choose C_2 and C_3 large enough to ensure that

$$\xi_2 > \max\{0, z_1 + 1/\lambda_1, z_3\}, \quad \xi_3 > \max\{0, z_1 + 1/\lambda_1, z_2\},$$

1 and that

$$C_2 \geq -\frac{r_2\gamma_2(\varepsilon\lambda_1e^{-\rho z_1} + hA_1 + \gamma_2\lambda_1e^{-\lambda_2 z_1} + b\gamma_3\lambda_1e^{-\lambda_3 z_1})}{e^{\eta_2(z_2-z_1)+\lambda_2 z_1+1}(\lambda_2 + \sigma - \eta_2)(d_2\eta_2^2 - s\eta_2 + r_2)}, \quad (3.9)$$

$$C_3 \geq -\frac{r_3\gamma_3(\varepsilon\lambda_1e^{-\rho z_1} + c_1A_1 + c_2\gamma_2\lambda_1e^{-\lambda_2 z_1} + \gamma_3\lambda_1e^{-\lambda_3 z_1})}{e^{\eta_3(z_3-z_1)+\lambda_3 z_1+1}(\lambda_3 + \sigma - \eta_3)(d_3\eta_3^2 - s\eta_3 + r_3)} \quad (3.10)$$

2 hold.

3 Many of the arguments are the same as those in Section 3.1; hence, we focus only on the
4 key points required to verify (2.1)-(2.6). Using the inequality $\alpha \leq 1$, we obtain $\mathcal{U}_1(z) \leq 0$
5 for $z < z_1 + 1/\lambda_1$. When $z > z_1 + 1/\lambda_1$, we have

$$\begin{aligned} \mathcal{U}_1(z) &\leq d_1\bar{\phi}_1''(z) + s\bar{\phi}_1'(z) + r_1\bar{\phi}(z) \\ &= A_1(d_1\lambda_1^2 - s\lambda_1 + r_1)(z - z_1)e^{-\lambda_1(z-z_1)} - A_1(2d_1\lambda_1 - s)e^{-\lambda_1(z-z_1)} = 0, \end{aligned}$$

6 where we have used (3.1) and the fact that λ_1 is the double root of (3.1). Thus, $\mathcal{U}_1(z) \leq 0$
7 holds for any $z \neq z_1 + 1/\lambda_1$. By the same argument as in Section 3.1, we also obtain $\mathcal{U}_j(z) \leq 0$
8 for all $z \neq z_j$ and $j = 2, 3$.

9 Next, for $j = 1, 2, 3$, we have $\mathcal{L}_j(z) \equiv 0$ on the interval where $\phi_j(z) \equiv 0$. Recall $\sigma =$
10 $\min\{\lambda_1, \lambda_2, \lambda_3, \rho\}$. For $z > z_1 + \xi_1 > \max\{0, z_2, z_3\}$, setting $y := z - z_1 > \xi_1 > 1$, we compute

$$\begin{aligned} \mathcal{L}_1(z) &= d_1\phi_1''(z) + s\phi_1'(z) + r_1\phi_1(z) + r_1\phi_1(z)(\alpha(z) - 1 - \phi_1(z) - k\bar{\phi}_2(z) - a\bar{\phi}_3(z)) \\ &\geq \frac{d_1p_1}{4y^{3/2}}e^{-\lambda_1 y} - r_1A_1ye^{-\lambda_1 y}(\varepsilon e^{-\rho z} + A_1ye^{-\lambda_1 y} + k\bar{\phi}_2(z) + a\bar{\phi}_3(z)) \quad (3.11) \\ &= \frac{d_1p_1}{4y^{3/2}}e^{-\lambda_1 y} - r_1A_1ye^{-\lambda_1 y}(\varepsilon e^{-\rho z} + A_1ye^{-\lambda_1 y} + k\gamma_2e^{-\lambda_2 z} + a\gamma_3e^{-\lambda_3 z}) \\ &\geq \frac{e^{-\lambda_1 y}}{y^{3/2}} \left[\frac{d_1p_1}{4} - r_1A_1y^{7/2}e^{-\sigma y}(\varepsilon e^{-\rho z_1} + A_1 + k\gamma_2e^{-\lambda_2 z_1} + a\gamma_3e^{-\lambda_3 z_1}) \right] \geq 0, \end{aligned}$$

11 where the last inequality follows from (3.7) and (3.8). When $z > \xi_2$, we have $y = z - z_1 > 1/\lambda_1$
12 and we deduce that

$$\begin{aligned} \mathcal{L}_2(z) &= d_2\phi_2''(z) + s\phi_2'(z) + r_2\phi_2(z) + r_2\phi_2(z)(\alpha(z) - 1 - h\bar{\phi}_1(z) - \phi_2(z) - b\bar{\phi}_3(z)) \\ &\geq -C_2(d_2\eta_2^2 - s\eta_2 + r_2)e^{-\eta_2(z-z_2)} \\ &\quad - r_2\gamma_2e^{-\lambda_2 z}(\varepsilon e^{-\rho z} + hA_1ye^{-\lambda_1 y} + \gamma_2e^{-\lambda_2 z} + b\gamma_3e^{-\lambda_3 z}) \\ &\geq -C_2(d_2\eta_2^2 - s\eta_2 + r_2)e^{-\eta_2(y+z_1-z_2)} \\ &\quad - r_2\gamma_2ye^{-(\lambda_2+\sigma)y-\lambda_2 z_1}(\varepsilon\lambda_1e^{-\rho z_1} + hA_1 + \gamma_2\lambda_1e^{-\lambda_2 z_1} + b\gamma_3\lambda_1e^{-\lambda_3 z_1}) \\ &\geq 0, \end{aligned}$$

13 where the last inequality follows from Lemma 3.2 and (3.9). By a similar calculation, it
14 follows from Lemma 3.2 and (3.10) that $\mathcal{L}_3(z) \geq 0$ for all $z > \xi_3$.

From the above, we have (2.1)-(2.6) for all $z \notin \{z_1 + 1/\lambda_1, z_2, z_3, z_1 + \xi_1, \xi_2, \xi_3\}$. It is straightforward to verify that (2.8)-(2.11) hold. Therefore, in this case, the existence of forced waves satisfying (1.6) at the critical speed follows from Proposition 2.2. Note that we have

$$\lim_{z \rightarrow +\infty} \frac{e^{\lambda_1 z}}{z} \phi_1(z) = A_1 e^{\lambda_1 z_1} = \gamma_1, \quad \lim_{z \rightarrow +\infty} e^{\lambda_j z} \phi_j(z) = e^{\lambda_j z_j} = \gamma_j, \quad \text{for } j = 2, 3,$$

since $\underline{\phi}_j \leq \phi_j \leq \bar{\phi}_j$, $j = 1, 2, 3$.

Case 2. $s = s^* = 2\sqrt{d_2 r_2} > 2\sqrt{d_3 r_3}$. Note that, by the above assumption, $d_1 r_1 = d_2 r_2$.

In this case, we choose $z_j \in \mathbb{R}$ such that $e^{\lambda_j z_j} = \gamma_j/(\lambda_j e)$, $j = 1, 2$, and z_3 satisfying $e^{\lambda_3 z_3} = \gamma_3$. Let us introduce the following upper-lower-solutions. For $j = 1, 2$, define

$$\begin{aligned} \bar{\phi}_j(z) &= \begin{cases} 1, & z < z_j + 1/\lambda_j, \\ A_j(z - z_j)e^{-\lambda_j(z - z_j)}, & z > z_j + 1/\lambda_j, \end{cases} \\ \underline{\phi}_j(z) &= \begin{cases} 0, & z < z_j + \xi_j, \\ A_j(z - z_j)e^{-\lambda_j(z - z_j)} - p_j \sqrt{z - z_j} e^{-\lambda_j(z - z_j)}, & z > z_j + \xi_j, \end{cases} \end{aligned}$$

where $A_j := \lambda_j e$, p_j is a positive constant to be determined, $\xi_j := (p_j/A_j)^2$ and

$$\bar{\phi}_3(z) = \begin{cases} 1, & z < z_3, \\ e^{-\lambda_3(z - z_3)}, & z > z_3, \end{cases} \quad \underline{\phi}_3(z) = \begin{cases} 0, & z < \xi_3, \\ e^{-\lambda_3(z - z_3)} - C_3 e^{-\eta_3(z - z_3)}, & z > \xi_3. \end{cases}$$

Let p_1 and p_2 be sufficiently large constants such that

$$\xi_1 > \max\{1, -z_1, z_2 - z_1 + 1/\lambda_2, z_3 - z_1\}, \quad \xi_2 > \max\{1, -z_2, z_1 - z_2 + 1/\lambda_1, z_3 - z_2\}$$

and

$$p_1 \geq \frac{4r_1 A_1 B}{d_1} \{\varepsilon e^{-\rho z_1} + A_1 + k\gamma_2(1 + |z_1 - z_2|)e^{-\lambda_2 z_1} + a\gamma_3 e^{-\lambda_3 z_1}\}, \quad (3.12)$$

$$p_2 \geq \frac{4r_2 A_2 B}{d_2} \{\varepsilon e^{-\rho z_2} + h\gamma_1(1 + |z_1 - z_2|)e^{-\lambda_1 z_2} + A_2 + b\gamma_3 e^{-\lambda_3 z_2}\}. \quad (3.13)$$

In addition, we fix C_3 large enough so that $\xi_3 > \max\{1, z_1 + 1/\lambda_1, z_2 + 1/\lambda_2, z_3\}$ and

$$C_3 \geq -\frac{r_3 \gamma_3 \{\varepsilon \lambda_1 e^{-\rho z_1} + c_1 A_1 + c_2 \gamma_2(1 + \lambda_1 |z_1 - z_2|)e^{-\lambda_2 z_1} + \lambda_1 \gamma_3 e^{-\lambda_3 z_1}\}}{e^{\eta_3(z_3 - z_1) + \lambda_3 z_1} (\lambda_3 + \sigma - \eta_3)(d_3 \eta_3^2 - s \eta_3 + r_3)} \quad (3.14)$$

hold.

It follows from the parallel computations as in Case 1 that $\mathcal{U}_j(z) \leq 0$ for $j = 1, 2, 3$ and for all $z \notin \{z_1 + 1/\lambda_1, z_2 + 1/\lambda_2, z_3\}$. Similarly, we obtain $\mathcal{L}_j(z) \equiv 0$ for $z < z_j + \xi_j$, $j = 1, 2$, and $\mathcal{L}_3(z) \equiv 0$ for $z < \xi_3$. When $z > z_1 + \xi_1 > 0$, i.e. $y = z - z_1 > \xi_1 > 1$, $\mathcal{L}_1(z)$ is evaluated

1 as in (3.11), and then estimated as

$$\begin{aligned}\mathcal{L}_1(z) &\geq \frac{d_1 p_1}{4y^{3/2}} e^{-\lambda_1 y} - r_1 A_1 y e^{-\lambda_1 y} \{ \varepsilon e^{-\rho z} + A_1 y e^{-\lambda_1 y} + k \gamma_2 (z - z_2) e^{-\lambda_2 z} + a \gamma_3 e^{-\lambda_3 z} \} \\ &\geq \frac{e^{-\lambda_1 y}}{y^{3/2}} \left[\frac{d_1 p_1}{4} - r_1 A_1 y^{7/2} e^{-\sigma y} \{ \varepsilon e^{-\rho z_1} + A_1 + k \gamma_2 (1 + |z_1 - z_2|) e^{-\lambda_2 z_1} + a \gamma_3 e^{-\lambda_3 z_1} \} \right] \\ &\geq 0\end{aligned}$$

2 where the last inequality follows from (3.7) and (3.12). In the same manner, we have $\mathcal{L}_2(z) \geq$
 3 0 for all $z > z_2 + \xi_2$, by a similar argument applied to $\mathcal{L}_1(z)$ together with (3.7) and (3.13).
 4 Moreover, for $z > \xi_3 > 1$, that is, $y = z - z_1 > 1/\lambda_1$, we deduce that

$$\begin{aligned}\mathcal{L}_3(z) &\geq -C_3(d_3 \eta_3^2 - s \eta_3 + r_3) e^{-\eta_3(z-z_3)} \\ &\quad - r_3 \gamma_3 e^{-\lambda_3 z} \{ \varepsilon e^{-\rho z} + c_1 A_1 y e^{-\lambda_1 y} + c_2 \gamma_2 (z - z_2) e^{-\lambda_2 z} + \gamma_3 e^{-\lambda_3 z} \} \\ &\geq -C_3(d_3 \eta_3^2 - s \eta_3 + r_3) e^{-\eta_3(y+z_1-z_3)} \\ &\quad - r_3 \gamma_3 y e^{-(\lambda_3+\sigma)y-\lambda_3 z_1} \{ \varepsilon \lambda_1 e^{-\rho z_1} + c_1 A_1 + c_2 \gamma_2 (1 + \lambda_1 |z_1 - z_2|) e^{-\lambda_2 z_1} + \lambda_1 \gamma_3 e^{-\lambda_3 z_1} \} \\ &\geq 0.\end{aligned}$$

5 Here, the last inequality follows from Lemma 3.2 and (3.14).

6 By the preceding arguments, we obtain (2.1)-(2.6) for all $z \notin \{z_1 + 1/\lambda_1, z_2 + 1/\lambda_2, z_3, z_1 +$
 7 $\xi_1, z_2 + \xi_2, \xi_3\}$. Furthermore, (2.8)-(2.11) can be verified without difficulty. Consequently,
 8 in this case, the existence of forced waves satisfying (1.6) at the critical speed is ensured by
 9 Proposition 2.2 and the above construction of upper-lower solutions.

10 **Case 3.** $s = s^* = 2\sqrt{d_1 r_1} = 2\sqrt{d_2 r_2} = 2\sqrt{d_3 r_3}$.

11 In this case, we choose $z_j \in \mathbb{R}$ such that $e^{\lambda_j z_j} = \gamma_j/(\lambda_j e)$, $j = 1, 2, 3$. Let us introduce the
 12 following upper-lower-solutions. For $j = 1, 2, 3$, define

$$\begin{aligned}\bar{\phi}_j(z) &= \begin{cases} 1, & z < z_j + 1/\lambda_j, \\ A_j(z - z_j) e^{-\lambda_j(z-z_j)}, & z > z_j + 1/\lambda_j, \end{cases} \\ \underline{\phi}_j(z) &= \begin{cases} 0, & z < z_j + \xi_j, \\ A_j(z - z_j) e^{-\lambda_j(z-z_j)} - p_j \sqrt{z - z_j} e^{-\lambda_j(z-z_j)}, & z > z_j + \xi_j, \end{cases}\end{aligned}$$

13 where $A_j := \lambda_j e$ and $\xi_j := (p_j/A_j)^2$. We take p_1, p_2 , and p_3 sufficiently large so that

$$\begin{cases} \xi_1 > \max\{1, -z_1, z_2 - z_1 + 1/\lambda_2, z_3 - z_1 + 1/\lambda_3\}, \\ \xi_2 > \max\{1, -z_2, z_1 - z_2 + 1/\lambda_1, z_3 - z_2 + 1/\lambda_3\}, \\ \xi_3 > \max\{1, -z_3, z_1 - z_3 + 1/\lambda_1, z_2 - z_3 + 1/\lambda_2\} \end{cases}$$

1 and

$$p_1 \geq \frac{4r_1 A_1 B}{d_1} \{ \varepsilon e^{-\rho z_1} + A_1 + k\gamma_2(1 + |z_1 - z_2|)e^{-\lambda_2 z_1} + a\gamma_3(1 + |z_1 - z_3|)e^{-\lambda_3 z_1} \}, \quad (3.15)$$

$$p_2 \geq \frac{4r_2 A_2 B}{d_2} \{ \varepsilon e^{-\rho z_2} + h\gamma_1(1 + |z_1 - z_2|)e^{-\lambda_1 z_2} + A_2 + b\gamma_3(1 + |z_2 - z_3|)e^{-\lambda_3 z_2} \}, \quad (3.16)$$

$$p_3 \geq \frac{4r_3 A_3 B}{d_3} \{ \varepsilon e^{-\rho z_3} + c_1\gamma_1(1 + |z_1 - z_3|)e^{-\lambda_1 z_3} + c_2\gamma_2(1 + |z_2 - z_3|)e^{-\lambda_2 z_3} + A_3 \}. \quad (3.17)$$

2 From the same argument as in Case 1, we have $\mathcal{U}_j(z) \leq 0$ for $j = 1, 2, 3$ and for all
 3 $z \in \mathbb{R} \setminus \{z_j + 1/\lambda_j \mid j = 1, 2, 3\}$. For $j = 1, 2, 3$, we also obtain $\mathcal{L}_j(z) \equiv 0$ on $(-\infty, z_j + \xi_j)$,
 4 since $\underline{\phi}_j(z) \equiv 0$ in this interval. When $z > z_1 + \xi_1 > 0$, i.e. $y > \xi_1 > 1$, $\mathcal{L}_1(z)$ is evaluated as
 5 in (3.11), and then estimated as

$$\begin{aligned} \mathcal{L}_1(z) &\geq \frac{d_1 p_1}{4y^{3/2}} e^{-\lambda_1 y} - r_1 A_1 y e^{-\lambda_1 y} \{ \varepsilon e^{-\rho z} + A_1 y e^{-\lambda_1 y} \\ &\quad + k\gamma_2(z - z_2)e^{-\lambda_2 z} + a\gamma_3(z - z_3)e^{-\lambda_3 z} \} \\ &\geq \frac{d_1 p_1}{4y^{3/2}} e^{-\lambda_1 y} - r_1 A_1 y^2 e^{-(\lambda_1 + \sigma)y} \{ \varepsilon e^{-\rho z_1} + A_1 \\ &\quad + k\gamma_2(1 + |z_1 - z_2|)e^{-\lambda_2 z_1} + a\gamma_3(1 + |z_1 - z_3|)e^{-\lambda_3 z_1} \} \\ &\geq \frac{d_1 p_1}{4y^{3/2}} e^{-\lambda_1 y} \left\{ 1 - \frac{1}{B} y^{7/2} e^{-\sigma y} \right\} \geq 0, \end{aligned}$$

6 where the last inequalities follow from (3.7) and (3.15). By a similar argument applied to
 7 $\mathcal{L}_j(z)$, together with (3.7), (3.16) and (3.17), it follows that $\mathcal{L}_j(z) \geq 0$ for all $z > z_j + \xi_j$ for
 8 $j = 2, 3$.

9 Combining the above results, we obtain (2.1)-(2.6) hold for all $z \notin \{z_j + 1/\lambda_j, z_j + \xi_j \mid$
 10 $j = 1, 2, 3\}$. It can be readily verified that (2.8)-(2.11) are satisfied. Hence, in this case, the
 11 existence of forced waves satisfying (1.6) also follows from Proposition 2.2 and the above
 12 construction of upper-lower solutions. $\square$

13 **3.3. Non-existence.** For the non-existence, we use an idea from [7, Proposition 4.3]. Sup-
 14 pose that there is a positive solution (ϕ_1, ϕ_2, ϕ_3) of (1.3) satisfying $(\phi_1, \phi_2, \phi_3)(+\infty) = (0, 0, 0)$
 15 for some $s > 0$. Set $\zeta(z) := \phi'_1(z)/\phi_1(z)$. Then ζ satisfies

$$d_1 \zeta'(z) + d_1 \zeta^2(z) + s\zeta(z) + r_1 \Phi(z) = 0, \quad \Phi(z) := \alpha(z) - \phi_1(z) - k\phi_2(z) - a\phi_3(z).$$

16 Note that $\Phi(+\infty) = 1$ and $\phi'_1(z) < 0$ for all $z \gg 1$. Following the proof of [7, Proposition
 17 4.3], we obtain that the limit $-\lambda := \lim_{z \rightarrow +\infty} \zeta(z)$ exists and satisfies

$$d_1 \lambda^2 - s\lambda + r_1 = 0.$$

18 This implies that $s \geq 2\sqrt{d_1 r_1}$. Similar argument also leads that $s \geq 2\sqrt{d_j r_j}$, $j = 2, 3$. The
 19 non-existence part of Theorem 1.2 is thereby proved. $\square$

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