# FORCED WAVES FOR DIFFUSIVE COMPETITION SYSTEMS IN SHIFTING ENVIRONMENTS

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ABSTRACT. In this paper, we derive the existence of forced waves for diffusive competition systems in shifting environments. First, we derive two different classes of forced waves for a 3-species competition system. Then we obtain forced waves for 2-species competition systems with at least one weak competitor. In all cases, the minimal environmental shifting speeds are determined under the equal diffusivities condition.

## 1. INTRODUCTION

In this paper, we study the following diffusive Lotka-Volterra competition system in a shifting environment

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u[\alpha_1(x-st) - u - a_2 v - a_3 w],\ x \in \mathbb{R},\ t > 0, \\ v_t = d_2 v_{xx} + r_2 v[\alpha_2(x-st) - b_1 u - v - b_3 w],\ x \in \mathbb{R},\ t > 0, \\ w_t = d_3 w_{xx} + r_3 w[\alpha_3(x-st) - c_1 u - c_2 v - w],\ x \in \mathbb{R},\ t > 0, \end{cases} \tag{1.1}$$

where $u, v, w$, functions of $(x,t)$, are three competing species and all parameters $d_i, r_i, a_i, b_j, c_k$ in (1.1) are positive constants in which, for each $i = 1,2,3$, $d_i$ stands for the diffusion coefficient and $r_i \alpha_i$ is the intrinsic growth rate which is spatially and temporally dependent. Parameters $a_i, b_j, c_k$ are inter-specific competition coefficients. The functions $\{\alpha_1, \alpha_2, \alpha_3\}$ model the shifting of their habitat with the same speed $s > 0$. We refer the reader to [2] and the references cited therein for the biological modeling view point of system (1.1) due to climate change.

Throughout this paper, for each $i$ we assume $\alpha_i$ is continuous in $\mathbb{R}$ such that

(a1) the limits $\alpha_i(\pm\infty)$ exist such that $\alpha_i(-\infty) < 0$, $\alpha_i(+\infty) = 1$ and $\alpha_i(z) \le \alpha_i(+\infty)$ for all $z \in \mathbb{R}$;

(a2) there exist $C_i > 0$ and $\rho_i > 0$ such that

$$\alpha_i(+\infty) - \alpha_i(z) \le C_i e^{-\rho_i z} \quad \text{for all large } z. \tag{1.2}$$

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Note that the constant $\rho_i$ in condition (1.2) provides the decay rate of $\alpha_i(+\infty)-\alpha_i(\cdot)$. In particular, (1.2) holds with $\rho_i$ replaced by any exponent $\hat\rho \le \rho_i$.

We are concerned with the existence of forced wave, namely, a traveling wave solution $(u,v,w)$ of (1.1) in the form

$$(u,v,w)(x,t)=(\phi_1,\phi_2,\phi_3)(z),\ z:=x-st,$$

with wave speed the same as the environmental shifting speed $s$ and wave profiles $\{\phi_1,\phi_2,\phi_3\}$. Hence we are looking for unknown $\{\phi_1,\phi_2,\phi_3\}$ that satisfies

$$\begin{cases} d_1\phi_1''+s\phi_1'+r_1\phi_1(\alpha_1-\phi_1-a_2\phi_2-a_3\phi_3)=0,\ z\in\mathbb{R},\\ d_2\phi_2''+s\phi_2'+r_2\phi_2(\alpha_2-b_1\phi_1-\phi_2-b_3\phi_3)=0,\ z\in\mathbb{R},\\ d_3\phi_3''+s\phi_3'+r_3\phi_3(\alpha_3-c_1\phi_1-c_2\phi_2-\phi_3)=0,\ z\in\mathbb{R}. \end{cases} \tag{1.3}$$

For the study of forced waves, we refer the reader to [4, 2, 5, 3, 20] for scalar equations, [1, 19, 11] for 2-species competition systems, [21] for a cooperative model and [9, 10] for predator-prey systems.

Due to the assumption $\alpha_i(-\infty)<0$ in (a1), any positive solution $(\phi_1,\phi_2,\phi_3)$ of (1.3) must satisfy the condition

$$(\phi_1,\phi_2,\phi_3)(-\infty)=(0,0,0). \tag{1.4}$$

This can be verified in the same manner as that of [10, Proposition 2.2] and we omit its proof here. Hereafter $(\phi_1,\phi_2,\phi_3)$ is a *positive* solution of (1.3) means $\phi_i>0$ in $\mathbb{R}$ for $i=1,2,3$. Note that, by the strong maximum principle, $\phi_i>0$ in $\mathbb{R}$ if $(\phi_1,\phi_2,\phi_3)$ is bounded, $\phi_i\ge 0$ and $\phi_i\not\equiv 0$ in $\mathbb{R}$.

It is easy to see that the following homogeneous system

$$\begin{cases} u_t=d_1u_{xx}+r_1u(1-u-a_2v-a_3w),\ x\in\mathbb{R},\ t>0,\\ v_t=d_2v_{xx}+r_2v(1-b_1u-v-b_3w),\ x\in\mathbb{R},\ t>0,\\ w_t=d_3w_{xx}+r_3w(1-c_1u-c_2v-w),\ x\in\mathbb{R},\ t>0, \end{cases} \tag{1.5}$$

always has the constant equilibria $\{(0,0,0),(0,0,1),(0,1,0),(1,0,0)\}$. When $b_3,c_2<1$, system (1.5) has the semi-co-existence state $E_c:=(0,v_c,w_c)$, where

$$v_c:=\frac{1-b_3}{1-b_3c_2}\in(0,1),\ w_c:=\frac{1-c_2}{1-b_3c_2}\in(0,1).$$

The state $(0,0,0)$ is always unstable for the (diffusion-free) ODE system of (1.5). Moreover, it is easy to check that $(0,0,1)$ is unstable, if either $a_3<1$ or $b_3<1$; $(0,1,0)$ is unstable, if either $a_2<1$ or $c_2<1$; while $(1,0,0)$ is unstable, if either $b_1<1$ or $c_1<1$. When $b_3,c_2<1$, the condition

$$\beta:=1-a_2v_c-a_3w_c>0 \tag{1.6}$$

implies that the state $E_c$ is unstable for the ODE system of (1.5). We are interested in forced waves with the limit at $z = \infty$ being one of the above (unstable) nontrivial constant equilibria of system (1.5).

In particular, in the absence of species $v$, system (1.1) is reduced to

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u[\alpha_1(x - st) - u - a_3 w], \ x \in \mathbb{R}, \ t > 0, \\ w_t = d_3 w_{xx} + r_3 w[\alpha_3(x - st) - c_1 u - w], \ x \in \mathbb{R}, \ t > 0. \end{cases} \tag{1.7}$$

The existence and non-existence of forced waves of system (1.7) was studied in [11] for two different cases, namely, one superior with one inferior case and two weak competitors case. In [11], the competition system is transformed to a cooperative system. However, this idea is only applicable for 2-species case, but not for 3-species competition system. Also, forced waves with critical speed was not addressed in [11]. In fact, with the application of Schauder's fixed point theorem and replacing the super-sub solutions by the so-called generalized upper-lower solutions, the idea of [11] can be extended to non-cooperative systems. We refer the reader to, e.g., [18, 16, 15, 17, 23, 22, 8, 12, 6, 7, 9, 10] for the application of this method to derive the existence of traveling waves in various ecological systems.

Motivated by [11, 10], the aim of this paper is twofold, namely, to derive the forced waves with critical speed for two species competition system and to extend the two species case to three species. It is surprising that the forms of generalized upper-lower solutions constructed in [10] for predator-prey systems are in some sense universal. It works well here for competition systems, once the parameters in the generalized upper-lower solutions can be chosen appropriately. On the other hand, after a carefully checking, the monotonicity condition on the shifting function(s) is not needed in [10]. Therefore, we do not impose the monotonicity condition on $\alpha_i$ for any $i$ in this paper. Moreover, we allow each species has its own different shifting function, but with the same shifting speed.

Throughout this paper we let $\kappa_+ := \max\{0, \kappa\}$ for a real number $\kappa$. Now, we describe our main results of this paper as follows.

First, for forced waves connecting $(0, 0, 1)$ for system (1.1) with $a_3 < 1$, we set

$$Q_1(\rho) := \begin{cases} d_1\rho + r_1(1 - a_3)/\rho, & \text{if } \rho \in (0, \lambda_*), \\ s_1^* := 2\sqrt{d_1 r_1(1 - a_3)}, & \text{if } \rho \geq \lambda_* := \sqrt{r_1(1 - a_3)/d_1}. \end{cases}$$

Then we have

**Theorem 1.1.** *Suppose that $a_3 < 1$, $b_3 < 1$, $Q_1(\rho_i) \leq s$ for $\rho_i$ in (1.2), $i = 1, 2, 3$, and*

$$d_2 = d_1 \geq d_3, \quad r_2(1 - b_3) = r_1(1 - a_3) > r_3(c_1 + c_2 - 1)_+. \tag{1.8}$$

*Then there exists a positive solution $(\phi_1, \phi_2, \phi_3)$ of (1.3) satisfying (1.4) and*

$$(\phi_1, \phi_2, \phi_3)(\infty) = (0, 0, 1), \tag{1.9}$$

*for* $s > s_1^*$*; and for* $s = s_1^*$*, if we further impose that* $d_3 = d_2 = d_1$.

It is worth to note that $\phi_i > 0$ in $\mathbb{R}$ for $i = 1, 2, 3$ for all solutions $(\phi_1, \phi_2, \phi_3)$ obtained in Theorem 1.1. Moreover, these waves are of mixed front-pulse types in the sense that $\phi_i$ is of pulse type vanishing at both tails for $i = 1, 2$ and $\phi_3$ is of front type connecting two different constants at $\pm\infty$.

Biologically, condition $d_2 = d_1 \geq d_3$ in (1.8) is natural, since both species $u$ and $v$ must keep pace with species $w$ in order to survive in the favorable habitat. A similar theorem on forced waves with $(\phi_1, \phi_2, \phi_3)(\infty) = (0, 1, 0)$ to Theorem 1.1 can be proved by exchanging the roles of species $v$ and $w$. Same for the case $(1, 0, 0)$. We omit the details.

Secondly, for forced waves connecting $(0, v_c, w_c)$ for system (1.1) with $\beta > 0$, we set

$$Q_2(\rho) := \begin{cases} d_1\rho + r_1\beta/\rho, & \text{if } \rho \in (0, \lambda^*), \\ s_1^{**} := 2\sqrt{d_1 r_1 \beta}, & \text{if } \rho \geq \lambda^* := \sqrt{r_1\beta/d_1}. \end{cases}$$

Then we have

**Theorem 1.2.** *Suppose* $b_3 < 1$*,* $c_2 < 1$*, (1.6) holds,* $Q_2(\rho_i) \leq s$ *for* $\rho_i$ *in (1.2),* $i = 1, 2, 3$*, and*

$$d_1 \geq \max\{d_2, d_3\},\ r_1\beta > \max\{r_2(b_1 + b_3 c_2 v_c), r_3[c_1 + c_2(1 - v_c)]\}. \tag{1.10}$$

*Then there exists a positive solution* $(\phi_1, \phi_2, \phi_3)$ *of (1.3) satisfying (1.4) and*

$$(\phi_1, \phi_2, \phi_3)(\infty) = (0, v_c, w_c), \tag{1.11}$$

*for* $s > s_1^{**}$*; and for* $s = s_1^{**}$*, if we further impose that* $d_3 = d_2 = d_1$.

Again, all wave profiles $(\phi_1, \phi_2, \phi_3)$ obtained in Theorem 1.2 satisfy $\phi_i > 0$ in $\mathbb{R}$ for $i = 1, 2, 3$. They are also of mixed front-pulse types. Also, condition $d_1 \geq \max\{d_2, d_3\}$ in (1.10) is natural, biologically it means that in order to survive species $u$ must keep pace with both species $v$ and $w$. Since the non-existence of forced waves can be proved by exactly the same manner as that of [10, Proposition 4.3], Theorems 1.1 and 1.2 also determine the minimal environmental shifting speeds for both classes of forced waves under the equal diffusivities condition.

Recently, various traveling waves of the limiting system (1.5) of (1.1) with $d_1 = d_2 = d_3 = 1$ are derived in [13]. We make some comments on the differences between traveling waves of (1.5) and those forced waves of (1.1) obtained in Theorems 1.1 and 1.2 as follow. We only consider the equal diffusivities case. First, one should note that at $-\infty$ we always have condition (1.4) for forced waves, but we have a stable constant equilibrium of (1.5) for traveling waves of (1.5). The stable constant equilibrium of (1.5) can be either $(0, 0, 1)$ or the positive co-existence state $(u_*, v_*, w_*)$, under appropriate conditions. Secondly, there is a positive constant (the minimal wave speed) $s^* = s_1^*$ ($s_* = s_1^{**}$, rep.) such that a traveling wave $\{s,(\phi_1,\phi_2,\phi_3)\}$ of (1.5) satisfying (1.9) ((1.11), rep.) exists if and only if its wave speed $s \geq s^*$ ($s \geq s_*$, rep.). However, the wave speed $s$ of forced waves of (1.1) is prescribed a priori by the environmental shifting speed. Moreover, the condition $s \geq s_1^*$ (or, $s \geq s_1^{**}$) gives the admissible environmental shifting speeds for which forced waves do exist. Thirdly, the shapes of waves for (1.1) and (1.5) are different, due to the different conditions of wave tails at $-\infty$. For example, as mentioned earlier forced waves are all of mixed front-pulse types, but traveling waves with wave tail $(u_*,v_*,w_*)$ at $-\infty$ are purely of front types.

Lastly, for forced wave for two species competition system (1.7) with $a_3<1$, as a corollary of Theorem 1.1 we obtain

**Corollary 1.3.** *Suppose that $a_3<1$, $Q_1(\rho_i)\leq s$ for $\rho_i$ in (1.2), $i=1,3$, and*

$$d_1 \geq d_3,\ r_1(1-a_3) > r_3(c_1-1)_+. \tag{1.12}$$

*Then (1.7) has a forced wave $(u,v)(x,t)=(\phi_1,\phi_2)(x-st)$ satisfying $(\phi_1,\phi_2)(-\infty)=(0,0)$ and $(\phi_1,\phi_2)(\infty)=(0,1)$ for $s>s_1^*$; and for $s=s_1^*$, if we further impose that $d_3=d_1$.*

Corollary 1.3 contains both $c_1>1$ (one superior with one inferior) and $c_1<1$ (two weak competitors) cases. When $c_1<1$, by exchanging the roles of $u$ and $w$ we also obtain the forced waves with critical speed $2\sqrt{d_3r_3(1-c_1)}$ connecting $(0,0)$ and $(1,0)$. Forced waves for $s>s_1^*$ in Corollary 1.3 was already obtained in [11]. Note that the decay condition $Q_1(\rho_i)\leq s$, $i=1,3$, is weaker than that in [11]. But, on admissible parameters $\{d_1,d_3,r_1,r_3,a_3,c_1\}$, condition (1.12) is stronger than that in [11]. A detailed discussion is given in §4.

The rest of this paper is organized as follows. First, in §2, some preliminaries are given, including the notion of generalized upper-lower solutions, an existence theorem for solutions of (1.3) and a remark on condition (1.2). Next, we construct various generalized upper-lower solutions for three species competition system (1.1) in §3 to prove Theorems 1.1 and 1.2. The verifications of these generalized upper-lower solutions are quite similar to that in [10] for predator-prey systems. However, conditions imposed on parameters here are different from that in [10], certain modifications are needed here. To support Theorems 1.1 and 1.2, we present some numerical simulations for forced waves of (1.1) at the end of §3. Finally, we study forced waves for two species competition systems and give a proof of Corollary 1.3 in §4.

## 2. Preliminaries

We first introduce the definition of generalized upper-lower solutions as follows.

**Definition 2.1.** *Continuous functions $(\overline{\phi}_1,\overline{\phi}_2,\overline{\phi}_3)$ and $(\underline{\phi}_1,\underline{\phi}_2,\underline{\phi}_3)$ are called a pair of generalized upper-lower solutions of (1.3) if $\overline{\phi}_i''$, $\underline{\phi}_i''$, $\overline{\phi}_i'$, $\underline{\phi}_i'$, $i=1,2,3$, are bounded in $\mathbb{R}$ and* *the following inequalities*

$$\mathcal{U}_1(z) := d_1\overline{\phi}_1''(z) + s\overline{\phi}_1'(z) + r_1\overline{\phi}_1(z)[\alpha_1(z) - \overline{\phi}_1(z) - a_2\underline{\phi}_2(z) - a_3\underline{\phi}_3(z)] \le 0, \tag{2.1}$$

$$\mathcal{U}_2(z) := d_2\overline{\phi}_2''(z) + s\overline{\phi}_2'(z) + r_2\overline{\phi}_2(z)[\alpha_2(z) - b_1\underline{\phi}_1(z) - \overline{\phi}_2(z) - b_3\underline{\phi}_3(z)] \le 0, \tag{2.2}$$

$$\mathcal{U}_3(z) := d_3\overline{\phi}_3''(z) + s\overline{\phi}_3'(z) + r_3\overline{\phi}_3(z)[\alpha_3(z) - c_1\underline{\phi}_1(z) - c_2\underline{\phi}_2(z) - \overline{\phi}_3(z)] \le 0, \tag{2.3}$$

$$\mathcal{L}_1(z) := d_1\underline{\phi}_1''(z) + s\underline{\phi}_1'(z) + r_1\underline{\phi}_1(z)[\alpha_1(z) - \underline{\phi}_1(z) - a_2\overline{\phi}_2(z) - a_3\overline{\phi}_3(z)] \ge 0, \tag{2.4}$$

$$\mathcal{L}_2(z) := d_2\underline{\phi}_2''(z) + s\underline{\phi}_2'(z) + r_2\underline{\phi}_2(z)[\alpha_2(z) - b_1\overline{\phi}_1(z) - \underline{\phi}_2(z) - b_3\overline{\phi}_3(z)] \ge 0, \tag{2.5}$$

$$\mathcal{L}_3(z) := d_3\underline{\phi}_3''(z) + s\underline{\phi}_3'(z) + r_3\underline{\phi}_3(z)[\alpha_3(z) - c_1\overline{\phi}_1(z) - c_2\overline{\phi}_2(z) - \underline{\phi}_3(z)] \ge 0, \tag{2.6}$$

*hold for* $z \in \mathbb{R}\backslash\{z_j \mid j = 1, \cdots, m\}$ *with some finite positive integer* $m$.

Then we have the following existence theorem for system (1.3), by a standard argument from, e.g., [18, 15].

**Proposition 2.2.** *Given* $s > 0$. *If* (1.3) *has a pair of generalized upper-lower solutions* $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ *and* $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ *such that*

$$\underline{\phi}_i \le \overline{\phi}_i,\ i = 1, 2, 3, \tag{2.7}$$

$$\lim_{z\to z_j^+} \overline{\phi}_i'(z) \le \lim_{z\to z_j^-} \overline{\phi}_i'(z),\ \lim_{z\to z_j^-} \underline{\phi}_i'(z) \le \lim_{z\to z_j^+} \underline{\phi}_i'(z),\ \forall\, j = 1, \cdots, m,\ i = 1, 2, 3, \tag{2.8}$$

*then* (1.3) *has a solution* $(\phi_1, \phi_2, \phi_3)$ *such that* $\underline{\phi}_i \le \phi_i \le \overline{\phi}_i$, $i = 1, 2, 3$.

Recall (1.2). Note that, by a suitable translation of system (1.3) and using $\alpha_i(\infty) = 1$, condition (1.2) can be rephrased as

$$\alpha_i(z) \ge 1 - \varepsilon e^{-\hat{\rho} z} \text{ for all } z > 0,\ i = 1, 2, 3, \tag{2.9}$$

for any given positive constant $\varepsilon$ as we need and for any positive constant $\hat{\rho} \le \rho_i$ for all $i = 1, 2, 3$. Indeed, given a positive constant $\hat{\rho} \le \min\{\rho_i \mid i = 1, 2, 3\}$ and a constant $\varepsilon > 0$. By (1.2), there is a constant $K \gg 1$ such that

$$1 - \alpha_i(z + K) \le C_i e^{-\rho_i(z+K)} \le C_i e^{-\rho_i K} e^{-\hat{\rho} z} \le \varepsilon e^{-\hat{\rho} z},\ \forall\, z > 0,\ i = 1, 2, 3,$$

if we make $K$ larger so that $C_i e^{-\rho_i K} \le \varepsilon$ for all $i$. Then a forced wave $(\phi_1, \phi_2, \phi_3)(z)$ of the translated system:

$$\begin{cases} d_1\phi_1''(z) + s\phi_1'(z) + r_1\phi_1(z)[\alpha_1(z + K) - \phi_1(z) - a_2\phi_2(z) - a_3\phi_3(z)] = 0,\ z \in \mathbb{R}, \\ d_2\phi_2''(z) + s\phi_2'(z) + r_2\phi_2(z)[\alpha_2(z + K) - b_1\phi_1(z) - \phi_2(z) - b_3\phi_3(z)] = 0,\ z \in \mathbb{R}, \\ d_3\phi_3''(z) + s\phi_3'(z) + r_3\phi_3(z)[\alpha_3(z + K) - c_1\phi_1(z) - c_2\phi_2(z) - \phi_3(z)] = 0,\ z \in \mathbb{R}, \end{cases}$$

renders a forced wave $(\phi_1, \phi_2, \phi_3)(z - K)$ of the original system (1.3). Note that this idea is already used in [10].

## 3. Forced waves for system (1.1)

### 3.1. Case $(0,0,1)$ with supercritical speed.

Let $s > s_1^*$. Recall from (1.8) that

$$A_1(\lambda) := d_1\lambda^2 - s\lambda + r_1(1-a_3) = d_2\lambda^2 - s\lambda + r_2(1-b_3).$$

Since $s > s_1^*$, there exist $\lambda_1$ and $\lambda_2$ with $0 < \lambda_1 < \lambda_2 < \infty$ such that $A_1(\lambda_i) = 0$, $i = 1, 2$, and $A_1(\lambda) < 0$ for all $\lambda \in (\lambda_1, \lambda_2)$.

Recall (1.8). We may choose $\varepsilon > 0$ such that

$$r_3\varepsilon < r_1(1-a_3) - r_3(c_1+c_2-1)_+ \tag{3.1}$$

and (2.9) holds for this $\varepsilon$ with $\hat{\rho} = \lambda_1$, using $\rho_i \geq \lambda_1$ by the assumption $Q_1(\rho_i) \leq s$ for all $i$. Let

$$\mu_1 \in (\lambda_1, \min\{\lambda_2, 2\lambda_1\}).$$

Define

$$\begin{cases} \overline{\phi}_1(z) := \min\{e^{-\lambda_1 z}, 1\}, \quad \underline{\phi}_1(z) := \max\{e^{-\lambda_1 z} - p_1 e^{-\mu_1 z}, 0\}, \\ \overline{\phi}_2(z) := \min\{e^{-\lambda_1 z}, 1\}, \quad \underline{\phi}_2(z) := \max\{e^{-\lambda_1 z} - p_2 e^{-\mu_1 z}, 0\}, \\ \overline{\phi}_3(z) := 1, \quad \underline{\phi}_3(z) := \max\{1 - e^{-\lambda_1 z}, 0\}, \end{cases} \tag{3.2}$$

where $p_1, p_2$ are constants satisfying

$$p_1 > \max\left\{1, \frac{r_1(1+\varepsilon+a_2)}{-A_1(\mu_1)}\right\}, \tag{3.3}$$

$$p_2 > \max\left\{1, \frac{r_2(1+\varepsilon+b_1)}{-A_1(\mu_1)}\right\}. \tag{3.4}$$

Then we have

**Lemma 3.1.** *Under condition* (1.8)*, the functions defined in* (3.2) *is a pair of upper-lower-solutions of* (1.3) *for a given* $s > s_1^*$ *such that* (2.7) *and* (2.8) *hold.*

*Proof.* Note that it suffices to check (2.1)-(2.6) for non-constant parts.

For $z > 0$, we compute

$$\begin{aligned} \mathcal{U}_1(z) &\leq e^{-\lambda_1 z}(d_1\lambda_1^2 - s\lambda_1) + r_1 e^{-\lambda_1 z}\{1 - e^{-\lambda_1 z} - a_3 + a_3 e^{-\lambda_1 z}\} \\ &= -r_1 e^{-\lambda_1 z}(1-a_3)e^{-\lambda_1 z} \leq 0, \end{aligned}$$

using $\underline{\phi}_2 \geq 0$, $\alpha_1 \leq 1$, $A_1(\lambda_1) = 0$ and $a_3 < 1$. Hence (2.1) holds for all $z \neq 0$.

Similarly, we can show that (2.2) holds for all $z \neq 0$, using $\underline{\phi}_1 \geq 0$, $\alpha_2 \leq 1$, $A_1(\lambda_1) = 0$ and $b_3 < 1$.

It is trivial that $\mathcal{U}_3(z) \leq 0$ for all $z \in \mathbb{R}$.

For $\mathcal{L}_1$, first note that $\underline{\phi}_1(z) = e^{-\lambda_1 z} - p_1 e^{-\mu_1 z}$ for $z > z_1$ and $\underline{\phi}_1(z) = 0$ for $z \le z_1$ for some $z_1 > 0$, using $p_1 > 1$ and $\mu_1 > \lambda_1$. Then, for $z > z_1$, we compute

$$\begin{aligned}\mathcal{L}_1(z) &\ge e^{-\lambda_1 z}(d_1\lambda_1^2 - s\lambda_1) - p_1 e^{-\mu_1 z}(d_1\mu_1^2 - s\mu_1) \\ &\quad + r_1\underline{\phi}_1(z)\{1 - \varepsilon e^{-\lambda_1 z} - \underline{\phi}_1(z) - a_2 e^{-\lambda_1 z} - a_3\} \\ &= -p_1 e^{-\mu_1 z} A_1(\mu_1) - r_1 e^{-\lambda_1 z}(\varepsilon e^{-\lambda_1 z} + e^{-\lambda_1 z} + a_2 e^{-\lambda_1 z}) \\ &\ge e^{-\mu_1 z}\{p_1[-A_1(\mu_1)] - r_1(\varepsilon + 1 + a_2)\} \ge 0,\end{aligned}$$

using $A_1(\lambda_1) = 0$, (2.9) with $\hat{\rho} = \lambda_1$, the choice of $\mu_1$ and (3.3). Hence (2.4) holds for all $z \ne z_1$,

Similarly, there exists $z_2 > 0$ such that (2.5) holds for all $z \ne z_2$, by using $A_1(\lambda_1) = 0$, (2.9) with $\hat{\rho} = \lambda_1$, the choice of $\mu_1$ and (3.4).

Finally, for $z > 0$, we compute

$$\mathcal{L}_3(z) \ge -e^{-\lambda_1 z}(d_1\lambda_1^2 - s\lambda_1) + r_3(1 - e^{-\lambda_1 z})\{-\varepsilon e^{-\lambda_1 z} - (c_1 + c_2 - 1)e^{-\lambda_1 z}\},$$

using (2.9) and $d_3 \le d_1$. When $c_1 + c_2 \le 1$, we obtain

$$\mathcal{L}_3(z) \ge e^{-\lambda_1 z}\{r_1(1 - a_3) - r_3\varepsilon\} \ge 0,\ z > 0,$$

due to (3.1). When $c_1 + c_2 > 1$, we get

$$\mathcal{L}_3(z) \ge e^{-\lambda_1 z}[r_1(1 - a_3) - r_3(c_1 + c_2 - 1) - r_3\varepsilon] \ge 0,\ z > 0,$$

by (3.1) again. Hence (2.6) holds for all $z \ne 0$. The proof is complete. □

### 3.2. **Case** $(0, 0, 1)$ **with critical speed.**

For $s = s_1^*$, $A_1(\lambda) = 0$ has a double root $\lambda = \lambda_*$.

Choose $\varepsilon > 0$ such that (due to (1.8))

$$r_3\varepsilon < e[r_1(1 - a_3) - r_3(c_1 + c_2 - 1)_+] \tag{3.5}$$

and (2.9) holds for this $\varepsilon$ with $\hat{\rho} = \lambda_*$, using $s \ge Q_1(\rho_i)$. Set $B_* := \lambda_* e$, $z_* := 1/\lambda_*$ and define

$$\begin{cases} \overline{\phi}_1(z) := \begin{cases} B_* z e^{-\lambda_* z},\ z > z_*, \\ 1,\ z \le z_* \end{cases} & \underline{\phi}_1(z) := \begin{cases} B_* z e^{-\lambda_* z} - p_3\sqrt{z} e^{-\lambda_* z},\ z > z_3, \\ 0,\ z \le z_3, \end{cases} \\ \overline{\phi}_2(z) := \begin{cases} B_* z e^{-\lambda_* z},\ z > z_*, \\ 1,\ z \le z_* \end{cases} & \underline{\phi}_2(z) := \begin{cases} B_* z e^{-\lambda_* z} - p_4\sqrt{z} e^{-\lambda_* z},\ z > z_4, \\ 0,\ z \le z_4, \end{cases} \\ \overline{\phi}_3(z) := 1, \quad \underline{\phi}_3(z) := \begin{cases} 1 - B_* z e^{-\lambda_* z},\ z > z_*, \\ 0,\ z \le z_*, \end{cases} \end{cases} \tag{3.6}$$

where $z_3 := (p_3/B_*)^2$, $z_4 := (p_4/B_*)^2$, $p_3 > e\sqrt{\lambda_*}$, $p_4 > e\sqrt{\lambda_*}$ and

$$p_3 > \frac{4}{d} r_1 B_* \left\{ B_*(1+a_2) \left( \frac{7}{2\lambda_* e} \right)^{7/2} + \varepsilon \left( \frac{5}{2\lambda_* e} \right)^{5/2} \right\}, \tag{3.7}$$

$$p_4 > \frac{4}{d} r_2 B_* \left\{ B_*(1+b_1) \left( \frac{7}{2\lambda_* e} \right)^{7/2} + \varepsilon \left( \frac{5}{2\lambda_* e} \right)^{5/2} \right\}. \tag{3.8}$$

Then we have

**Lemma 3.2.** *In addition to* (1.8)*, we assume that* $d_1 = d_2 = d_3$*. Then the functions defined in* (3.6) *is a pair of upper-lower-solutions of* (1.3) *for* $s = s_1^*$ *such that* (2.7) *and* (2.8) *hold.*

*Proof.* As before, it suffices to check (2.1)-(2.6) for non-constant parts.

Let $d_1 = d_2 = d_3 = d$. Note that, using $r_2(1-b_3) = r_1(1-a_3)$, $s = 2d\lambda_*$ and $A_1(\lambda_*) = 0$,

$$d(B_* z e^{-\lambda_* z})'' + s(B_* z e^{-\lambda_* z})' = -r_1(1-a_3)B_* z e^{-\lambda_* z} = -r_2(1-b_3)B_* z e^{-\lambda_* z},\ z \in \mathbb{R},$$
$$d(\sqrt{z} e^{-\lambda_* z})'' + s(\sqrt{z} e^{-\lambda_* z})' = -\frac{d}{4} z^{-3/2} e^{-\lambda_* z} - r_1(1-a_3)\sqrt{z} e^{-\lambda_* z},\ z > 0.$$

For $z > z_* = 1/\lambda_*$, we compute

$$\mathcal{U}_1(z) \ \leq\ -r_1(1-a_3)B_* z e^{-\lambda_* z} + r_1 B_* z e^{-\lambda_* z}\{1 - B_* z e^{-\lambda_* z} - a_3 + a_3 B_* z e^{-\lambda_* z}\} \leq 0,$$

using $\alpha_1 \leq 1$, $\underline{\phi}_2 \geq 0$ and $a_3 < 1$.

Similarly, we have $\mathcal{U}_2(z) \leq 0$ for $z > z_*$, using $\alpha_2 \leq 1$, $\underline{\phi}_1 \geq 0$ and $b_3 < 1$.

It is clearly that $\mathcal{U}_3(z) \leq 0$ for all $z \in \mathbb{R}$.

Next, for $z > z_3$, we have

$$\begin{aligned}
\mathcal{L}_1(z) \ &\geq\ \frac{1}{4} d p_3 z^{-3/2} e^{-\lambda_* z} - r_1(B_* z - p_3 z^{1/2})\{\varepsilon + B_* z + a_2 B_* z\} e^{-2\lambda_* z} \\
&\geq\ \frac{1}{4} d p_3 z^{-3/2} e^{-\lambda_* z} - r_1 B_* z e^{-2\lambda_* z}\{\varepsilon + (1+a_2)B_* z\} \\
&=\ \frac{1}{4} z^{-3/2} e^{-\lambda_* z} \left\{ d p_3 - 4 r_1 B_* [\varepsilon z^{5/2} e^{-\lambda_* z} + B_*(1+a_2) z^{7/2} e^{-\lambda_* z}] \right\} \\
&\geq\ \frac{1}{4} z^{-3/2} e^{-\lambda_* z} \left\{ d p_3 - 4 r_1 B_* \left[ \varepsilon \left( \frac{5}{2\lambda_* e} \right)^{5/2} + B_*(1+a_2) \left( \frac{7}{2\lambda_* e} \right)^{7/2} \right] \right\} \geq 0,
\end{aligned}$$

using (2.9) with $\hat{\rho} = \lambda_*$, the fact

$$\max_{z>0} \left\{ z^{\gamma} e^{-\lambda_* z} \right\} \leq \left( \frac{\gamma}{\lambda_* e} \right)^{\gamma},\ \gamma > 0, \tag{3.9}$$

and (3.7).

A similar calculation also leads $\mathcal{L}_2(z) \geq 0$ for all $z > z_4$, using (3.8) instead of (3.7).

Finally, for $z > z_*$, by the same argument as that in the proof of Lemma 3.1 we obtain

$$\mathcal{L}_3(z) \geq r_1(1-a_3)B_* z e^{\lambda_* z} + r_3(1 - B_* z e^{\lambda_* z})\{-\varepsilon e^{-\lambda_* z} - (c_1 + c_2 - 1)B_* z e^{\lambda_* z}\} \geq 0,$$

using (2.9) with $\hat{\rho} = \lambda_*$, $B_* z > e$ for all $z > z_*$, (1.8) and (3.5). The lemma is thus proved. $\square$

It is clear that the functions defined in both (3.2) and (3.6) satisfy

$$(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)(+\infty) = (\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)(+\infty) = (0, 0, 1).$$

Then Theorem 1.1 is proved by combining Lemmas 3.1 and 3.2 with Proposition 2.2.

### 3.3. **Case** $(0, v_c, w_c)$ **with supercritical speed.**

Given $s > s_1^{**}$. Let $\lambda_i$, $i = 3, 4$, be the two positive solutions of

$$A_2(\lambda) := d_1\lambda^2 - s\lambda + r_1\beta = 0$$

such that $\lambda_3 < \lambda_4$. Note that $A_2(\lambda) < 0$ for $\lambda \in (\lambda_3, \lambda_4)$. Due to (1.10), we may choose $\varepsilon > 0$ such that

$$\varepsilon < \min\{r_1\beta/r_2 - (b_1 + b_3c_2v_c), r_1\beta/r_3 - [c_1 + c_2(1 - v_c)]\} \tag{3.10}$$

and (2.9) holds for this $\varepsilon$ with $\hat{\rho} = \lambda_3$, using (1.10) and $s \geq Q_2(\rho_i)$ for all $i$. We then define

$$\begin{cases} \overline{\phi}_1(z) := \min\{1, e^{-\lambda_3 z}\},\ \underline{\phi}_1(z) := \max\{0, e^{-\lambda_3 z} - pe^{-\mu_2 z}\}, \\ \overline{\phi}_2(z) := \min\{1, v_c + (1 - v_c)e^{-\lambda_3 z}\},\ \underline{\phi}_2(z) := \max\{0, v_c(1 - e^{-\lambda_3 z})\}, \\ \overline{\phi}_3(z) := \min\{1, w_c + c_2v_ce^{-\lambda_3 z}\},\ \underline{\phi}_3(z) := \max\{0, w_c(1 - e^{-\lambda_3 z})\}, \end{cases} \tag{3.11}$$

where $\mu_2 \in (\lambda_3, \min\{2\lambda_3, \lambda_4\})$ (so that $A_2(\mu_2) < 0$) and $p$ satisfies

$$p > \max\{1, r_1[\varepsilon + 1 + a_2(1 - v_c) + a_3c_2v_c]/[-A_2(\mu_2)]\}. \tag{3.12}$$

Then we have

**Lemma 3.3.** *Under condition* (1.10)*, the functions defined in* (3.11) *are a pair of generalized upper-lower solutions of* (1.3) *for a given* $s > s_1^{**}$.

*Proof.* We only need to check (2.1)-(2.6) for non-constant parts.

For $z > 0$, we compute

$$\begin{aligned} \mathcal{U}_1(z) &\leq e^{-\lambda_3 z}(d_1\lambda_3^2 - s\lambda_3) + r_1e^{-\lambda_3 z}\{1 - e^{-\lambda_3 z} - a_2v_c(1 - e^{-\lambda_3 z}) - a_3w_c(1 - e^{-\lambda_3 z})\} \\ &= -r_1\beta e^{-2\lambda_3 z} \leq 0, \end{aligned}$$

using $\alpha_1 \leq 1$, $\beta = 1 - a_2v_c - a_3w_c$ and $A_2(\lambda_3) = 0$. Hence (2.1) holds for all $z \neq 0$.

For $z > 0$, since $\underline{\phi}_1 \geq 0$, we have

$$\begin{aligned} \mathcal{U}_2(z) &\leq (1 - v_c)e^{-\lambda_3 z}(d_2\lambda_3^2 - s\lambda_3) + r_2\overline{\phi}_2(z)\{1 - v_c - (1 - v_c)e^{-\lambda_3 z} - b_3w_c + b_3w_ce^{-\lambda_3 z}\} \\ &\leq (1 - v_c)e^{-\lambda_3 z}(d_1\lambda_3^2 - s\lambda_3) = -r_1\beta(1 - v_c)e^{-\lambda_3 z} \leq 0, \end{aligned}$$

using $\alpha_2 \leq 1$, $1 - v_c - b_3w_c = 0$, $d_2 \leq d_1$ and $A_2(\lambda_3) = 0$. Hence (2.2) holds for $z \neq 0$.

For $z > 0$, we have

$$\begin{aligned}\mathcal{U}_3(z) &\leq c_2 v_c e^{-\lambda_3 z}(d_3\lambda_3^2 - s\lambda_3) + r_3\overline{\phi}_3(z)\{1 - c_2 v_c(1-e^{-\lambda_3 z}) - w_c - c_2 v_c e^{-\lambda_3 z}\} \\ &\leq c_2 v_c e^{-\lambda_3 z}(d_1\lambda_3^2 - s\lambda_3) = -r_1\beta c_2 v_c e^{-\lambda_3 z} \leq 0,\end{aligned}$$

using $\alpha_3 \leq 1$, $1 - c_2 v_c - w_c = 0$, $d_3 \leq d_1$ and $A_2(\lambda_3) = 0$. Hence (2.3) holds for $z \neq 0$.

Now, for $\underline{\phi}_1$, using $p > 1$, there is $z_3 > 0$ such that $\underline{\phi}_1(z) = e^{-\lambda_3 z} - pe^{-\mu z}$ for $z > z_3$ and $\underline{\phi}_1(z) = 0$ for $z \leq z_3$. For $z > z_3$, we compute

$$\begin{aligned}\mathcal{L}_1(z) &\geq e^{-\lambda_3 z}(d_1\lambda_3^2 - s\lambda_3) - pe^{-\mu_2 z}(d_1\mu_2^2 - s\mu_2) \\ &\quad + r_1\underline{\phi}_1(z)\{1 - \varepsilon e^{-\lambda_3 z} - e^{-\lambda_3 z} - a_2 v_c - a_2(1-v_c)e^{-\lambda_3 z} - a_3 w_c - a_3 c_2 v_c e^{-\lambda_3 z}\} \\ &= -pe^{-\mu_2 z}A_2(\mu_2) + r_1\underline{\phi}_1(z)\{-\varepsilon - 1 - a_2(1-v_c) - a_3 c_2 v_c\}e^{-\lambda_3 z} \\ &\geq e^{-\mu_2 z}\{-pA_2(\mu_2) - r_1[\varepsilon + 1 + a_2(1-v_c) + a_3 c_2 v_c]e^{(\mu_2 - 2\lambda_3)z}\} \\ &\geq -A_2(\mu_2)e^{-\mu_2 z}\{p - r_1[\varepsilon + 1 + a_2(1-v_c) + a_3 c_2 v_c]/[-A_2(\mu_2)]\} \geq 0,\end{aligned}$$

using (2.9) with $\hat{\rho} = \lambda_3$, the choice of $\mu_2$ and (3.12). Hence (2.4) holds for all $z \neq z_3$.

For $z > 0$, we calculate, using $1 - v_c - b_3 w_c = 0$,

$$\begin{aligned}\mathcal{L}_2(z) &\geq -v_c e^{-\lambda_3 z}(d_2\lambda_3^2 - s\lambda_3) + r_2\underline{\phi}_2(z)\{-\varepsilon e^{-\lambda_3 z} - b_1 e^{-\lambda_3 z} + v_c e^{-\lambda_3 z} - b_3 c_2 v_c e^{-\lambda_3 z}\} \\ &\geq r_1\beta v_c e^{-\lambda_3 z} - r_2 v_c(\varepsilon + b_1 + b_3 c_2 v_c)e^{-\lambda_3 z} \geq 0,\end{aligned}$$

due to $d_2 \leq d_1$, $A_2(\lambda_3) = 0$, (1.10) and (3.10). Hence (2.5) holds for all $z \neq 0$.

Finally, for $z > 0$, we compute, using $A_2(\lambda_3) = 0$ and $1 - c_2 v_c - w_c = 0$,

$$\mathcal{L}_3(z) \geq r_1\beta w_c e^{-\lambda_3 z} - r_3 w_c[\varepsilon + c_1 + c_2(1-v_c)]e^{-\lambda_3 z} \geq 0,$$

using again (1.10) and (3.10). Hence (2.6) holds for all $z \neq 0$. This completes the proof of the lemma. □

3.4. **Case** $(0, v_c, w_c)$ **with critical speed.**

For $s = s_1^{**}$, $A_2(\lambda) = 0$ has a double root $\lambda^* > 0$. Note that $s_1^{**} = 2d_1\lambda^*$.

Choose $\varepsilon > 0$ such that

$$\varepsilon < e(\min\{r_1\beta/r_2 - (b_1 + b_3 c_2 v_c), r_1\beta/r_3 - [c_1 + c_2(1-v_c)]\}) \tag{3.13}$$

and (2.9) holds for this $\varepsilon$ with $\hat{\rho} = \lambda^*$, due to (1.10) and $s \geq Q_2(\rho_i)$ for all $i$.

Set $B := \lambda^* e$ and $z^* := 1/\lambda^*$. We define

$$
\begin{cases}
\overline{\phi}_1(z) := \begin{cases} 1, \ z \le z^*, \\ Bze^{-\lambda_* z}, \ z > z^*, \end{cases} & \underline{\phi}_1(z) := \begin{cases} 0, \ z \le \hat{z}, \\ Bze^{-\lambda_* z} - q\sqrt{z}e^{-\lambda_* z}, \ z > \hat{z}, \end{cases} \\
\overline{\phi}_2(z) := \begin{cases} 1, \ z \le z^*, \\ v_c + (1-v_c)Bze^{-\lambda_* z}, \ z > z^*, \end{cases} & \\
\underline{\phi}_2(z) := \begin{cases} 0, \ z \le z^*, \\ v_c(1 - Bze^{-\lambda_* z}), \ z > z^*, \end{cases} & \\
\overline{\phi}_3(z) := \begin{cases} 1, \ z \le z^*, \\ w_c + c_2 v_c Bze^{-\lambda_* z}, \ z > z^*, \end{cases} & \underline{\phi}_3(z) := \begin{cases} 0, \ z \le z^*, \\ w_c(1 - Bze^{-\lambda_* z}), \ z > z^*, \end{cases}
\end{cases} \tag{3.14}
$$

where $q > B/\sqrt{\lambda^*}$ so that $\hat{z} := (q/B)^2 > z^*$ and $q$ satisfies

$$
q > \frac{4}{d_1} r_1 B \left\{ B[1 + a_2(1-v_c) + a_3 c_2 v_c] \left( \frac{7}{2\lambda^* e} \right)^{7/2} + \varepsilon \left( \frac{5}{2\lambda^* e} \right)^{5/2} \right\}. \tag{3.15}
$$

Then we have

**Lemma 3.4.** *In addition to* (1.10)*, we assume that $d_1 = d_2 = d_3$. Then the functions defined in* (3.14) *are a pair of generalized upper-lower solutions of* (1.3) *for $s = s_1^{**}$.*

*Proof.* As before, it suffices to check (2.1)-(2.6) for non-constant parts.

Let $d_1 = d_2 = d_3 = d$. The following identity shall be used in our computations:

$$
d(Bze^{-\lambda_* z})'' + s(Bze^{-\lambda_* z})' = -r_1 \beta Bze^{-\lambda_* z}, \tag{3.16}
$$

using $s = s_1^{**} = 2d\lambda^*$ and $A_2(\lambda^*) = 0$.

For $z > z^*$, we compute

$$
\begin{aligned}
\mathcal{U}_1(z) &\le -r_1 \beta Bze^{-\lambda_* z} + r_1 Bze^{-\lambda_* z}\{(1 - a_2 v_c - a_3 w_c) - Bze^{-\lambda_* z}(1 - a_2 v_c - a_3 w_c)\} \\
&= -r_1 Bze^{-\lambda_* z}\{\beta Bze^{-\lambda_* z}\} \le 0,
\end{aligned}
$$

using $\alpha_1 \le 1$ and $1 - a_2 v_c - a_3 w_c = \beta$. Hence (2.1) holds for all $z \ne z^*$.

For $z > z^*$, using $\underline{\phi}_1 \ge 0$, we have

$$
\begin{aligned}
\mathcal{U}_2(z) &\le -(1-v_c)(r_1\beta)Bze^{-\lambda_* z} + r_2 \overline{\phi}_2(z)\{(1 - v_c - b_3 w_c)(1 - Bze^{-\lambda_* z})\} \\
&= -(1-v_c)B(r_1\beta)ze^{-\lambda_* z} \le 0,
\end{aligned}
$$

using $\alpha_2 \le 1$ and $1 - v_c - b_3 w_c = 0$. Hence (2.2) holds for all $z \ne z^*$.

For $z > z^*$, using again $\underline{\phi}_1 \ge 0$, we compute

$$
\mathcal{U}_3(z) \le -c_2 v_c r_1 \beta Bze^{-\lambda_* z} \le 0,
$$

due to $\alpha_3 \le 1$ and $1 - c_2 v_c - w_c = 0$. Hence (2.3) holds for all $z \ne z^*$.

Next, we note that

$$d(\sqrt{z}e^{-\lambda_* z})'' + s(\sqrt{z}e^{-\lambda_* z})' = -\frac{d}{4}z^{-3/2}e^{-\lambda_* z} - r_1\beta\sqrt{z}e^{-\lambda_* z},\ z > 0,$$

using $s = 2d\lambda^*$. Then, for $z > \hat{z}$, we have

$$\begin{aligned}
\mathcal{L}_1(z) &\geq -r_1\beta Bze^{-\lambda_* z} - q\left[-\frac{d}{4}z^{-3/2}e^{-\lambda_* z} - r_1\beta\sqrt{z}e^{-\lambda_* z}\right] \\
&\quad + r_1\underline{\phi}_1(z)\{(1-a_2v_c-a_3w_c) - \varepsilon e^{-\lambda_* z} - [1+a_2(1-v_c)+a_3c_2v_c]Bze^{-\lambda_* z}\} \\
&= q\frac{d}{4}z^{-3/2}e^{-\lambda_* z} - r_1\underline{\phi}_1(z)\{\varepsilon e^{-\lambda_* z} + [1+a_2(1-v_c)+a_3c_2v_c]Bze^{-\lambda_* z}\} \\
&\geq \frac{d}{4}z^{-3/2}e^{-\lambda_* z}\Big\{q - \frac{4}{d}r_1B\Big(B[1+a_2(1-v_c)+a_3c_2v_c]\left(z^{7/2}e^{-\lambda_* z}\right) + \varepsilon(z^{5/2}e^{-\lambda_* z})\Big)\Big\} \\
&\geq \frac{d}{4}z^{-3/2}e^{-\lambda_* z}\Big\{q - \frac{4}{d}r_1B\Big(B[1+a_2(1-v_c)+a_3c_2v_c]\left(\frac{7}{2\lambda^* e}\right)^{7/2} + \varepsilon\left(\frac{5}{2\lambda^* e}\right)^{5/2}\Big)\Big\} \\
&\geq 0,
\end{aligned}$$

using (3.9) and condition (3.15). Hence (2.4) holds for all $z \neq \hat{z}$.

For $z > z^*$, we compute

$$\begin{aligned}
\mathcal{L}_2(z) &\geq r_1\beta v_cBze^{-\lambda_* z} \\
&\quad + r_2\underline{\phi}_2(z)\{(1-v_c-b_3w_c) - \varepsilon e^{-\lambda_* z} + v_cBze^{-\lambda_* z} - (b_1+b_3c_2v_c)Bze^{-\lambda_* z}\} \\
&\geq \{r_1\beta - r_2[\varepsilon/e + (b_1+b_3c_2v_c)]\}v_cBze^{-\lambda_* z} \geq 0,
\end{aligned}$$

using $1 - v_c - b_3w_c = 0$, (3.13) and (1.10). Hence (2.5) holds for all $z \neq z^*$. Similarly, we have

$$\mathcal{L}_3(z) \geq w_cBze^{-\lambda_* z}\{r_1\beta - r_3\varepsilon/e - r_3[c_1 + c_2(1-v_c)]\} \geq 0,\ z > z^*.$$

Hence (2.6) also holds for all $z \neq z^*$. Thereby, we complete the proof of the lemma. □

It is clear that the functions defined in both (3.11) and (3.14) satisfy

$$(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)(+\infty) = (\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)(+\infty) = (0, v_c, w_c).$$

Then Theorem 1.2 is proved by combining Lemmas 3.3 and 3.4 with Proposition 2.2.

### 3.5. **Some numerical simulations for forced waves of** (1.1).

In this subsection, we provide some numerical simulations for system (1.1) to demonstrate forced waves derived in Theorems 1.1 and 1.2. In our numerical simulations, we take the following shifting functions

$$\alpha_i(z) = \frac{2}{\pi}\arctan(10z),\ z = x - st,\ i = 1, 2, 3,$$

for different admissible shifting speeds $s$.

For Theorem 1.1, we choose the following parameters

$$\begin{cases} d_1 = d_2 = d_3 = 1, \ r_1 = r_2 = 1, r_3 = 0.1, \\ a_2 = 1, a_3 = b_3 = 0.25, c_2 = 2, b_1 = 3, c_1 = 4. \end{cases}$$

Hence $s_1^* = \sqrt{3}$. Then the wave profiles of forced waves satisfying

$$(\phi_1, \phi_2, \phi_3)(-\infty) = (0, 0, 0), \ (\phi_1, \phi_2, \phi_3)(\infty) = (0, 0, 1),$$

for $s = 5 > s_1^*$ and $s = \sqrt{3} = s_1^*$ are shown in Figures 1 and 2, respectively.

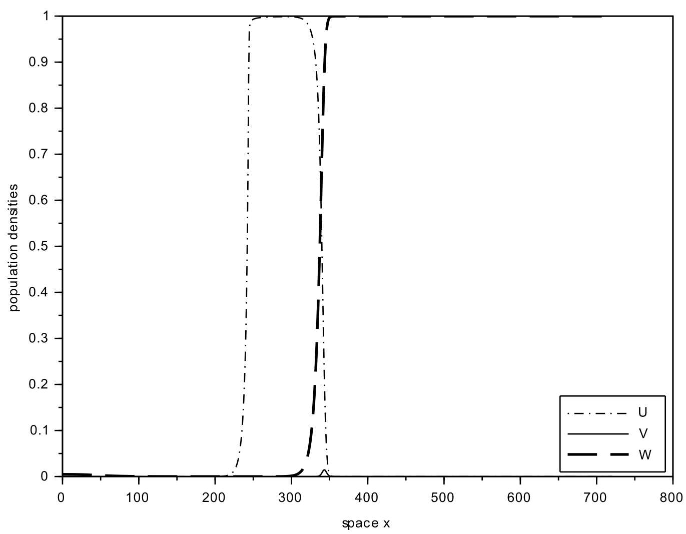


Figure 1 : Wave profile of forced wave with the shifting speed $s = 5$.

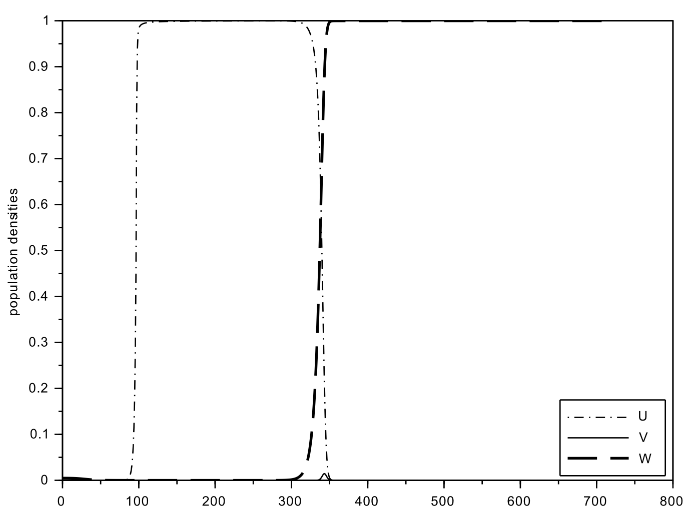


Figure 2 : Wave profile of forced wave with the critical shifting speed $s = \sqrt{3}$.

For Theorem 1.2, we set

$$\begin{cases} d_1 = d_2 = d_3 = 1,\ r_1 = 1, r_2 = 0.1, r_3 = 0.12, \\ b_3 = 0.2, c_2 = 0.25, b_1 = 3, c_1 = 4, a_2 = a_3 = 0.25. \end{cases}$$

Then $s_1^{**} = 2\sqrt{45/76}$ and the wave profiles of forced waves with

$$(\phi_1, \phi_2, \phi_3)(-\infty) = (0, 0, 0),\ (\phi_1, \phi_2, \phi_3)(\infty) = (0, v_c, w_c),$$

for $s = 5 > s_1^{**}$ and $s = 2\sqrt{45/76} = s_1^{**}$ are given in Figures 3 and 4, respectively.

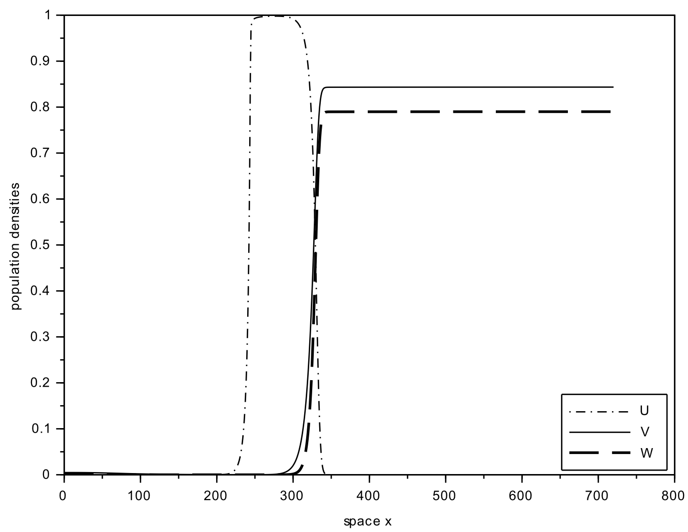


Figure 3 : Wave profile of forced wave with the shifting speed $s = 5$.

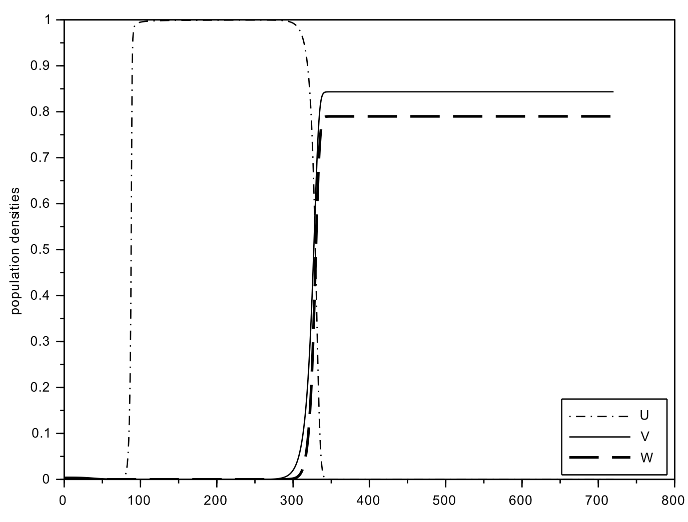


Figure 4 : Wave profile of forced wave with the critical shifting speed $s = 2\sqrt{45/76}$.

## 4. Forced waves for system (1.7)

In this section, we shall study the forced waves of (1.7). We are looking for positive solution $(\phi_1, \phi_3)$ of

$$\begin{cases} d_1\phi_1'' + s\phi_1' + r_1\phi_1(\alpha_1 - \phi_1 - a_3\phi_3) = 0, \ z \in \mathbb{R}, \\ d_3\phi_3'' + s\phi_3' + r_3\phi_3(\alpha_3 - c_1\phi_1 - \phi_3) = 0, \ z \in \mathbb{R}, \end{cases} \tag{4.1}$$

with $s \geq s_1^*$ such that

$$(\phi_1, \phi_3)(-\infty) = (0, 1), \quad (\phi_1, \phi_3)(\infty) = (0, 0).$$

For system (1.7), the definition of generalized upper-lower solutions is the same as that in Definition 2.1 by putting $\overline{\phi}_2 = \underline{\phi}_2 = a_2 = c_2 = 0$ everywhere. Therefore, Proposition 2.2 also holds for system (1.7). It is interesting to remark that $\{(\overline{\phi}, 1 - \underline{\psi}), (\underline{\phi}, 1 - \overline{\psi})\}$ is a pair of generalized upper-lower solutions for a pair of super-sub solutions $\{(\overline{\phi}, \overline{\psi}), (\underline{\phi}, \underline{\psi})\}$ constructed in [11] for system (1.7).

*Proof of Corollary 1.3.* As remarked above, recall from (3.2) and (3.6) that

$$\begin{cases} \overline{\phi}_1(z) := \min\{e^{-\lambda_1 z}, 1\}, \quad \underline{\phi}_1(z) := \max\{e^{-\lambda_1 z} - p_1 e^{-\mu_1 z}, 0\}, \\ \overline{\phi}_3(z) := 1, \quad \underline{\phi}_3(z) := \max\{1 - e^{-\lambda_1 z}, 0\}, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \overline{\phi}_1(z) := \begin{cases} B_* z e^{-\lambda_* z}, \ z > z_*, \\ 1, \ z \leq z_* \end{cases} & \underline{\phi}_1(z) := \begin{cases} B_* z e^{-\lambda_* z} - p_3\sqrt{z} e^{-\lambda_* z}, \ z > z_3, \\ 0, \ z \leq z_3, \end{cases} \\ \overline{\phi}_3(z) := 1, \quad \underline{\phi}_3(z) := \begin{cases} 1 - B_* z e^{-\lambda_* z}, \ z > z_*, \\ 0, \ z \leq z_*, \end{cases} \end{cases} \tag{4.3}$$

are generalized upper-lower solutions of (4.1) for $s > s_1^*$ and $s = s_1^*$, respectively, such that

$$(\overline{\phi}_1, \overline{\phi}_3)(+\infty) = (\underline{\phi}_1, \underline{\phi}_3)(+\infty) = (0, 1).$$

Hence Corollary 1.3 follows from the same proof as that of Theorem 1.1. $\square$

Finally, we make some comments on Corollary 1.3 as follows.

In [11], the following system was considered:

$$\begin{cases} U_t = d_1 U_{xx} + U\{R_1(x - st) - U - \gamma_1 W\}, \\ W_t = d_3 W_{xx} + W\{R_3(x - st) - \gamma_3 U - W\}. \end{cases} \tag{4.4}$$

In fact, by setting

$$u = \frac{U}{R_1(\infty)}, \quad w = \frac{W}{R_3(\infty)},$$

system (4.4) is equivalent to system (1.7) with

$$r_1 = R_1(\infty),\ \alpha_1(\cdot) = \frac{R_1(\cdot)}{R_1(\infty)},\ a_3 = \frac{\gamma_1 R_3(\infty)}{R_1(\infty)},$$
$$r_3 = R_3(\infty),\ \alpha_3(\cdot) = \frac{R_3(\cdot)}{R_3(\infty)},\ c_1 = \frac{\gamma_3 R_1(\infty)}{R_3(\infty)}.$$

Hence the constant $\overline{\mu}_2(\infty)$ defined in [11] is given by

$$\overline{\mu}_2(\infty) = \sqrt{r_1(1-a_3)/d_1} = \lambda_*.$$

Since $Q_1(\rho_i) \le s$ implies that $\rho_i \ge \lambda_1 \ge \lambda_*$, the decay rates of $R_1$ and $R_3$ imposed in [11] are stronger than that in Corollary 1.3.

On the other hand, condition **(LD)** in [11] for system (1.7) reads

$$\min\left\{1, 2 - \frac{d_3}{d_1}\right\} \ge \frac{r_3(a_3c_1 - 1)_+}{r_1(1-a_3)}. \tag{4.5}$$

Fixing $d_1 > 0$, the optimal range for admissible $d_3$ is $(0, 2d_1]$, when $a_3c_1 \le 1$. Hence condition **(LD)** is weaker than our condition (1.12) in Corollary 1.3.

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