FORCED WAVES OF A THREE SPECIES PREDATOR-PREY SYSTEM IN A SHIFTING ENVIRONMENT

WONHYUNG CHOI AND JONG-SHENQ GUO

Abstract. This paper is concerned with the existence and non-existence of forced waves for a three species predator-prey system in a shifting environment. The speed of these forced waves is the same as the shifting speed of the living environment of these species. We assume that their habitat changes to the hostile environment as time increases. This makes all species go extinction eventually. Under certain conditions on parameters, we obtain two different types (front and mixed front-pulse types) forced waves that connecting different constant states and the extinction state. Moreover, we are able to characterize the minimal shifting speed for each mixed front-pulse type forced waves.

1. Introduction

Global warming is one of the major long-term threats and challenges worldwide. Climate change causes environmental changes such as sea level rise, precipitation change, and desertification. This indeed is one of the major challenges in ecology, because it directly affects the survival and extinction of species. In addition, as many species become extinct due to climate change, species diversity is expected to decrease which would cause an irreversible influence of the ecological system. Recently, mathematical biologists have established and studied various mathematical models for climate change and its effect on species ecology. For a single species model, the following scalar equation is considered:

$$u_t(x,t) = du_{xx}(x,t) + u(x,t)f(x-st,u(x,t))$$

in which t is the time variable and x is the spatial variable. Here, s is the climate change speed, f models a population growth depending on the climate change and d is the diffusion coefficient of the species u.

For the scalar model, Berestycki et al. [1] considered the discontinuous moving habitat patch, and showed that minimum patch size for the persistence of species. For a continuous population growth, authors in [12, 16] studied the existence of forced waves for Fisher's equation with

$$f(x-st,u) = \alpha(x-st) - u,$$

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where α is a monotone bounded function which takes both positive and negative values. Here a *forced wave* is a traveling wave solution with wave speed s (the given climate change speed). A single species model with some general KPP type nonlinearity was investigated in [2]. In [3, 4], authors studied the forced waves in higher spatial dimension with general type of functional response. In addition, propagation of species in time-periodic shifting habitat is studied in [13]. See also [20, 23, 21, 5, 25, 9] for some more related works on scalar equation.

For two interacting species, most works on the climate change were in competition and cooperative models. We refer the reader to [27] for a cooperative model; the existence of forced wave [24, 10] and the persistence and extinction of species [29, 28, 26] for a competition model; and a gap formation in competition model when the species' favorable habitats shift with opposite directions [1]. We also refer the reader to [19, 11, 17, 18] for the study of forced waves in a free boundary formulation.

However, due to the lack of comparison principle, predator-prey models with the climate change effect were not studied too much. Not until recently, Choi et al. [8] investigated the existence of forced waves and the persistence of species for a two-species predator-prey model with either local or nonlocal dispersal. To our knowledge, three or more species predator-prey models with climate change have not been studied so far. For the studies of traveling waves on predator-prey models without the climate change (i.e., in the homogeneous environment), we refer the reader to, e.g., the survey paper [14] and references cited therein.

In this paper, we consider the following diffusive predator-prey model with two preys and one predator:

$$
\begin{cases}
u_t(x,t) = d_1 u_{xx}(x,t) + r_1 u(x,t)[\alpha(x-st) - (u + hv + aw)(x,t)],\ x \in \mathbb{R},\ t > 0,\\
v_t(x,t) = d_2 v_{xx}(x,t) + r_2 v(x,t)[\alpha(x-st) - (ku + v + aw)(x,t)],\ x \in \mathbb{R},\ t > 0,\\
w_t(x,t) = d_3 w_{xx}(x,t) + r_3 w(x,t)(-1 + bu + bv - w)(x,t),\ x \in \mathbb{R},\ t > 0,
\end{cases}
\tag{1.1}
$$

where the unknown functions u, v and w respectively stand for the population densities of two preys and predator species at position x and time t. Parameters $d_1, d_2, d_3, r_1, r_2, r_3, h, k, a, b$ are positive and represent the diffusion coefficients, intrinsic growth rates, competing rates, predation rate and conversion rate, respectively. The given positive constant s denotes the climate change speed.

The function $\alpha(\cdot)$ models the climate change which depends on a shifting variable, and throughout the paper we assume that it satisfies the following properties:

(α1) α is continuous and nondecreasing in $\mathbb{R}$;
(α2) $-\infty < \alpha(-\infty) < 0 < \alpha(\infty) < \infty$;
(α3) There exist $C > 0$ and $\rho > 0$ such that $\alpha(\infty) - \alpha(z) \le Ce^{-\rho z}$ for large z.

This means that the environment is favourable to the prey ahead of the climate change, then gradually deteriorates until it becomes hostile to the species. Here, without loss of generality (up to a rescaling) we choose $\alpha(\infty) = 1$.

In this paper, we assume that two preys compete weakly, i.e. $h, k < 1$. For a given pair (h, k), we impose the following condition on the parameters a and b:

$$
0 < a < \min\left\{\frac{1-h}{2b}, \frac{1-k}{2b}\right\}.
\tag{1.2}
$$

In addition, we assume that $b > 1$, which means the predator can survive in given system when at least one prey exists.

We are mainly interested in the existence of forced waves for (1.1), namely, a solution of (1.1) in the form $(u, v, w)(x, t) = (\phi_1, \phi_2, \phi_3)(z)$, $z := st - x$. Then (ϕ_1, ϕ_2, ϕ_3) satisfies

$$\begin{cases} s\phi_1'(z) = d_1\phi_1''(z) + r_1\phi_1(z)[\alpha(-z) - \phi_1(z) - h\phi_2(z) - a\phi_3(z)],\ z \in \mathbb{R}, \\ s\phi_2'(z) = d_2\phi_2''(z) + r_2\phi_2(z)[\alpha(-z) - k\phi_1(z) - \phi_2(z) - a\phi_3(z)],\ z \in \mathbb{R}, \\ s\phi_3'(z) = d_3\phi_3''(z) + r_3\phi_3(z)[-1 + b\phi_1(z) + b\phi_2(z) - \phi_3(z)],\ z \in \mathbb{R}. \end{cases} \tag{1.3}$$

Due to the hostile environment (by the assumption on α), all species go extinction eventually. This is equivalent to the boundary condition $(\phi_1, \phi_2, \phi_3)(+\infty) = (0, 0, 0)$.

We shall consider the following possible limiting behaviors at $z = -\infty$:

$$E_1 = (1, 0, 0), \quad E_2 = \Big(\frac{1-h}{1-hk}, \frac{1-k}{1-hk}, 0\Big), \quad E_3 = \Big(\frac{1+a}{1+ab}, 0, \frac{b-1}{1+ab}\Big),$$

$$E_4 = \Big(\frac{(1+a)(1-h)}{1-hk+ab(2-h-k)}, \frac{(1+a)(1-k)}{1-hk+ab(2-h-k)}, \frac{b(2-h-k)-1+hk}{1-hk+ab(2-h-k)}\Big).$$

In fact, there are two other possible limits

$$\hat{E}_1 = (0, 1, 0), \quad \hat{E}_3 = \Big(0, \frac{1+a}{1+ab}, \frac{b-1}{1+ab}\Big).$$

Since these two cases can be treated similarly, we omit it here.

Biologically, the state E_1 at $z = -\infty$ can be thought as a saturated aboriginal prey living in the habitat and there are one invading alien predator along with one invading alien prey; while E_2: two competing aboriginal co-existent preys and an invading alien predator; and E_3: a pair of aboriginal co-existent predator-prey and an invading alien prey.

For a scalar wave profile $\phi(x - st)$, it is called a front type if $\phi(-\infty) \neq \phi(+\infty)$; and it is a pulse type if $\phi(\pm\infty) = 0$. We now describe our main results as follows.

First, for the front type (for all components) waves, we have

Theorem 1.1. *Suppose that $b > 1$ and (1.2) hold. If we assume further that*

$$a < \frac{-1 + b(2-h-k)}{2b(2b-1)}, \quad b > \frac{1}{2-h-k}, \tag{1.4}$$

then there exists a positive solution (ϕ_1, ϕ_2, ϕ_3) of (1.3) such that

$$(\phi_1, \phi_2, \phi_3)(-\infty) = E_4,\ (\phi_1, \phi_2, \phi_3)(+\infty) = (0, 0, 0)$$

for any $s > 0$.

Next, we consider the mixed front-pulse type waves. For waves connecting E_1, we let

$$s_2^* := 2\sqrt{d_2 r_2(1-k)},\ s_3^* := 2\sqrt{d_3 r_3(b-1)}.$$

Also, we assume without loss of generality that $s_3^* \geq s_2^*$ and define

$$s_0^*(\rho) := \begin{cases} d_3\rho + r_3(b-1)/\rho, & \text{if } \rho < \rho_{**}, \\ s_3^*, & \text{if } \rho \geq \rho_{**}, \end{cases}$$

where the constant ρ is defined in (α3) and $\rho_{**} := \sqrt{r_3(b-1)/d_3}$.

Theorem 1.2. *Given $\rho > 0$ in $(\alpha 3)$. Suppose $b > 1$, (1.2) and*

$$d_2 = d_3 \geq d_1,\ r_2(1-k) = r_3(b-1) \tag{1.5}$$

hold. Then there exists a positive solution (ϕ_1, ϕ_2, ϕ_3) of (1.3) with

$$(\phi_1, \phi_2, \phi_3)(-\infty) = E_1,\ (\phi_1, \phi_2, \phi_3)(+\infty) = (0, 0, 0), \tag{1.6}$$

provided that $s \geq s_0^*(\rho)$ and $s > s_3^* = s_2^*$. Moreover, in addition to (1.5), if we further assume $d_1 = d_2 = d_3$, then a positive solution with (1.6) exists for $s = s_0^*(\rho) = s_3^* = s_2^*$.

For the waves connecting E_2, we let

$$s_3^{**} := 2\sqrt{d_3 r_3 \beta_2},\ \beta_2 := -1 + b(u_c + v_c),\ (u_c, v_c) := \left(\frac{1-h}{1-hk}, \frac{1-k}{1-hk}\right)$$

and define

$$s_c^*(\rho) := \begin{cases} d_3\rho + r_3\beta_2/\rho, & \text{if } \rho \in (0, \rho_*), \\ s_3^{**}, & \text{if } \rho \geq \rho_*, \end{cases}$$

where $\rho_* := \sqrt{r_3\beta_2/d_3}$. Then we have

Theorem 1.3. *Given $\rho > 0$ in $(\alpha 3)$. Suppose $b > 1$, $d_3 \geq \max\{d_1, d_2\}/2$ and (1.2) hold. Then there exists a positive solution (ϕ_1, ϕ_2, ϕ_3) of (1.3) such that*

$$(\phi_1, \phi_2, \phi_3)(-\infty) = E_2,\ (\phi_1, \phi_2, \phi_3)(+\infty) = (0, 0, 0), \tag{1.7}$$

provided $s \geq s_c^*(\rho)$ and $s > s_3^{**}$. Moreover, if we further assume that $d_3 \leq \min\{d_1, d_2\}$, then a positive solution with (1.7) exists for $s = s_c^*(\rho) = s_3^{**}$.

For E_3, we let

$$s_2^{**} := 2\sqrt{d_2 r_2 \delta_2},\ \delta_2 := 1 - ku_p - aw_p,\ (u_p, w_p) := \left(\frac{1+a}{1+ab}, \frac{b-1}{1+ab}\right)$$

and define

$$s_p^*(\rho) := \begin{cases} d_2\rho + r_2\delta_2/\rho, & \text{if } \rho \in (0, \rho^*), \\ s_2^{**}, & \text{if } \rho \geq \rho^*, \end{cases}$$

where $\rho^* := \sqrt{r_2\delta_2/d_1}$. Then we have

Theorem 1.4. *Given $\rho > 0$ in $(\alpha 3)$. Suppose $b > 1$ and*

$$d_2 \geq \max\{d_1, d_3\}, r_2\delta_2 > \max\{r_1 a(2b-1), r_3\}. \tag{1.8}$$

Then there exists a positive solution (ϕ_1, ϕ_2, ϕ_3) of (1.3) with

$$(\phi_1, \phi_2, \phi_3)(-\infty) = E_3,\ (\phi_1, \phi_2, \phi_3)(+\infty) = (0, 0, 0), \tag{1.9}$$

provided $s \geq s_p^*(\rho)$ and $s > s_2^{**}$. Moreover, if we assume that

$$d_1 = d_2 \in (d_3/2, d_3],\quad r_2\delta_2(2 - d_3/d_2) \geq r_3 b u_p,\quad r_2\delta_2 > r_1 a(2b-1) \tag{1.10}$$

holds, then a positive solution exists for $s = s_p^*(\rho) = s_2^{**}$.

Since our model is a non-monotone system, the classical monotone iteration method to derive the existence of traveling waves cannot be applied. To overcome it, we apply Schauder's fixed point theorem with the help of generalized upper-lower solutions [22] to derive the existence of forced waves. This method has been proved to be very successful in dealing with non-monotone systems. However, due to the climate change involved, some more cares are needed for the mixed type waves. In particular, with the exponential decay condition on α, we introduce a shifted system (see (4.2)) to derive the existence of forced waves for this new system first. Then by shifting back we are able to obtain the existence of forced waves for the original system.

In this paper, we construct the suitable upper-lower solutions pairs not only for the front type waves but also for the mixed front-pulse type waves. With the shifting heterogeneity, the dynamics of this three-species predator-prey system is much more complex than the corresponding homogeneous case. Moreover, we also obtain the minimal shifting speed for each mixed front-pulse type waves under a faster exponential decay condition on α and certain conditions on the parameters of predator-prey model. The characterization of minimal speeds for the existence of mixed type forced waves are stated and proved in Propositions 4.3, 4.6 and 4.9 (see §4 below). A new idea of the proof of non-existence of forced waves (see Proposition 4.3) is introduced.

The remainder of this paper is organized as follows. Some preliminary results are presented in Section 2. In Section 3, we study the existence of front type forced waves that connect $(0, 0, 0)$ to E_4. Then the proofs of Theorems 1.2-1.4, the existence and non-existence of mixed front-pulse type waves for (1.1) connecting $(0, 0, 0)$ to E_1, E_2 and E_3, respectively, are given in Section 4.

2. Preliminaries

We first introduce the following notion of (generalized) upper-lower solutions of (1.3).

Definition 2.1. Continuous functions $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ are called a pair of upper and lower solutions of (1.3) if $\overline{\phi_i} \geq \underline{\phi_i}$, $i = 1, 2, 3$, and the following inequalities

$$s\overline{\phi}_1'(z) \geq d_1\overline{\phi}_1''(z) + r_1\overline{\phi}_1(z)[\alpha(-z) - \overline{\phi}_1(z) - h\underline{\phi}_2(z) - a\underline{\phi}_3(z)], \tag{2.1}$$

$$s\overline{\phi}_2'(z) \geq d_2\overline{\phi}_2''(z) + r_2\overline{\phi}_2(z)[\alpha(-z) - k\underline{\phi}_1(z) - \overline{\phi}_2(z) - a\underline{\phi}_3(z)], \tag{2.2}$$

$$s\overline{\phi}_3'(z) \geq d_3\overline{\phi}_3''(z) + r_3\overline{\phi}_3(z)[-1 + b\overline{\phi}_1(z) + b\overline{\phi}_2(z) - \overline{\phi}_3(z)], \tag{2.3}$$

$$s\underline{\phi}_1'(z) \leq d_1\underline{\phi}_1''(z) + r_1\underline{\phi}_1(z)[\alpha(-z) - \underline{\phi}_1(z) - h\overline{\phi}_2(z) - a\overline{\phi}_3(z)], \tag{2.4}$$

$$s\underline{\phi}_2'(z) \leq d_2\underline{\phi}_2''(z) + r_2\underline{\phi}_2(z)[\alpha(-z) - k\overline{\phi}_1(z) - \underline{\phi}_2(z) - a\overline{\phi}_3(z)], \tag{2.5}$$

$$s\underline{\phi}_3'(z) \leq d_3\underline{\phi}_3''(z) + r_3\underline{\phi}_3(z)[-1 + b\underline{\phi}_1(z) + b\underline{\phi}_2(z) - \underline{\phi}_3(z)] \tag{2.6}$$

hold for all $z \in \mathbb{R} \setminus E$ for some finite subset E of $\mathbb{R}$.

Then we have the following lemma for the existence of wave profiles. Since its proof by now is standard, we omit it and refer the reader to, e.g., [22].

Lemma 2.1. *Let $s > 0$ be given. Let $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ be a pair of upper and lower solutions of* (1.3) *satisfying, for $i = 1, 2, 3$,*

$$\overline{\phi}_i'(z-) \geq \overline{\phi}_i'(z+) \text{ and } \underline{\phi}_i'(z-) \leq \underline{\phi}_i'(z+) \text{ for } z \in E. \tag{2.7}$$

Then (1.3) *admits a solution (ϕ_1, ϕ_2, ϕ_3) such that $\underline{\phi}_i(z) \leq \phi_i(z) \leq \overline{\phi}_i(z)$ for all $z \in \mathbb{R}$ for $i = 1, 2, 3$.*

Next, we provide a proof of the right-hand tail limit as follows.

Proposition 2.2. *It holds $(\phi_1, \phi_2, \phi_3)(\infty) = (0, 0, 0)$ for any nonnegative solution (ϕ_1, ϕ_2, ϕ_3) of* (1.3)*.*

Proof. For contradiction, we assume that $\phi_1^+ := \limsup_{z\to\infty} \phi_1(z) > 0$. When ϕ_1 is oscillatory near $z = +\infty$, there is a maximal sequence $\{z_n\}$ of ϕ_1 such that $z_n \to \infty$ and $\phi_1(z_n) \to \phi_1^+$ as $n \to \infty$. It follows from the ϕ_1-equation of (1.3) and $\alpha(-\infty) < 0$ that

$$\begin{aligned} 0 &= \limsup_{n\to\infty} \{d_1 \phi_1''(z_n) + r_1 \phi_1(z_n)[\alpha(-z_n) - \phi_1(z_n) - h\phi_2(z_n) - a\phi_3(z_n)]\} \\ &\leq r_1 \phi_1^+[\alpha(-\infty) - \phi_1^+ - h \liminf_{n\to\infty} \phi_2(z_n) - a \liminf_{n\to\infty} \phi_3(z_n)] < 0, \end{aligned}$$

a contradiction. The inequality holds because $d_1\phi_1''(z_n) \leq 0$ for maximally chosen z_n.

On the other hand, suppose that ϕ_1 is monotone ultimately at $z = +\infty$. Then $\phi_1(z) \to \phi_1^+$ as $z \to \infty$. Also, we can find a sequence $\{z_n\}$ with $z_n \to \infty$ such that $\phi_1'(z_n) \to 0$ as $n \to \infty$. Integrating the ϕ_1-equation in (1.3) from 0 to z_n, we obtain

$$d[\phi_1'(0) - \phi_1'(z_n)] + s[\phi_1(z_n) - \phi_1(0)] = r_1 \int_0^{z_n} \{\phi_1(y)[\alpha(-y) - (\phi_1 + h\phi_2 + a\phi_3)(y)]\} dy. \tag{2.8}$$

By taking $K \gg 1$ so that $\phi_1(y) \geq \phi_1^+/2$ and $\alpha(-y) \leq 0$ for all $y \geq K$, we get

$$\phi_1(y)[\alpha(-y) - (\phi_1 + h\phi_2 + a\phi_3)(y)] \leq -\phi_1^2(y) \leq -(\phi_1^+)^2/4 < 0, \quad \forall\, y \geq K.$$

Hence the integral

$$\int_0^\infty \{\phi_1(y)[\alpha(-y) - (\phi_1 + h\phi_2 + a\phi_3)(y)]\} dy$$

diverges. However, the left-hand side of (2.8) is uniformly bounded with respect to n, a contradiction. This proves that $\phi_1(\infty) = 0$. Similarly, we obtain $\lim_{z\to\infty} \phi_2(z) = 0$.

Next, we assume for contradiction that $\phi_3^+ := \limsup_{z\to\infty} \phi_3(z) > 0$. When ϕ_3 is oscillatory near $z = +\infty$, we have a maximal sequence $\{z_n\}$ of ϕ_3 such that $z_n \to \infty$ and $\phi_3(z_n) \to \phi_3^+$ as $n \to \infty$. It follows from the ϕ_3-equation of (1.3) that

$$\begin{aligned} 0 &= \limsup_{n\to\infty} \{d_3 \phi_3''(z_n) + r_3 \phi_3(z_n)[-1 + b(\phi_1(z_n) + \phi_2(z_n)) - \phi_3(z_n)]\} \\ &\leq r_3 \phi_3^+(-1 - \phi_3^+) < 0, \end{aligned}$$

a contradiction again. The case when ϕ_3 is monotone ultimately at $z = +\infty$ can be treated as the above argument for ϕ_1. Hence $\phi_3(\infty) = 0$ and so we have proved

$$(\phi_1, \phi_2, \phi_3)(\infty) = (0, 0, 0) \tag{2.9}$$

for any nonnegative solution (ϕ_1, ϕ_2, ϕ_3) of (1.3). $\square$

We remark here, by the strong maximum principle, that any nonnegative nontrivial solution (ϕ_1, ϕ_2, ϕ_3) of (1.3) must be positive (in the sense that $\phi_i > 0$ in $\mathbb{R}$ for all $i = 1, 2, 3$). We shall consider separately two different classes of forced waves in the following sections.

3. Front type forced waves

Recall (1.2) and (1.4). First, it follows from [16, Theorem 1.1] with $z \mapsto -z$ that there is a non-increasing functions $\underline{\phi}_1$ and $\underline{\phi}_2$ such that

$$s\underline{\phi}_1'(z) = d_1\underline{\phi}_1''(z) + r_1\underline{\phi}_1(z)[\alpha(-z) - h - a(2b-1) - \underline{\phi}_1(z)], \quad z \in \mathbb{R}, \tag{3.1}$$
$$s\underline{\phi}_2'(z) = d_2\underline{\phi}_2''(z) + r_2\underline{\phi}_2(z)[\alpha(-z) - k - a(2b-1) - \underline{\phi}_2(z)], \quad z \in \mathbb{R}, \tag{3.2}$$

and

$$\lim_{z\to-\infty} \underline{\phi}_1(z) = 1 - h - a(2b-1) > 0, \quad \lim_{z\to\infty} \underline{\phi}_1(z) = 0,$$
$$\lim_{z\to-\infty} \underline{\phi}_2(z) = 1 - k - a(2b-1) > 0, \quad \lim_{z\to\infty} \underline{\phi}_2(z) = 0.$$

Also, it follows from [16, Theorem 1.1] again that there exists a non-increasing function $\underline{\phi}_3$ such that

$$s\underline{\phi}_3'(z) = d_3\underline{\phi}_3''(z) + r_3\underline{\phi}_3(z)[-1 + b(\underline{\phi}_1(z) + \underline{\phi}_2(z)) - \underline{\phi}_3(z)], \quad z \in \mathbb{R}, \tag{3.3}$$

and

$$\lim_{z\to-\infty} \underline{\phi}_3(z) = -1 + b[2 - h - k - 2a(2b-1)] > 0, \quad \lim_{z\to\infty} \underline{\phi}_3(z) = 0.$$

With the function $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$, we have

Lemma 3.1. *Suppose that* (1.2) *holds. Then there exists a solution* (ϕ_1, ϕ_2, ϕ_3) *of* (1.3) *such that* $\underline{\phi}_1 \le \phi_1 \le 1$, $\underline{\phi}_2 \le \phi_2 \le 1$ *and* $\underline{\phi}_3 \le \phi_3 \le 2b-1$ *in* $\mathbb{R}$.

Proof. Let $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3) = (1, 1, 2b-1)$. Then, by (3.1)-(3.3), (2.4)-(2.6) hold for all $z \in \mathbb{R}$.

It remains to show $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ satisfy (2.1)-(2.3). Since $\alpha(-z) \le 1$ for $z \in \mathbb{R}$,

$$d_1\overline{\phi}_1''(z) + r_1\overline{\phi}_1(z)[\alpha(-z) - \overline{\phi}_1(z) - h\underline{\phi}_2(z) - a\underline{\phi}_3(z)] \le r_1[\alpha(-z) - 1] \le 0 = s\overline{\phi}_1'(z),$$
$$d_2\overline{\phi}_2''(z) + r_2\overline{\phi}_2(z)[\alpha(-z) - k\underline{\phi}_1(z) - \overline{\phi}_2(z) - a\underline{\phi}_3(z)] \le r_2[\alpha(-z) - 1] \le 0 = s\overline{\phi}_2'(z),$$

and so (2.1) and (2.2) hold for all $z \in \mathbb{R}$. Finally, it is easy to check that

$$\begin{aligned} &d_3\overline{\phi}_3''(z) + r_3\overline{\phi}_3(z)[-1 + b\overline{\phi}_1(z) + b\overline{\phi}_2(z) - \overline{\phi}_3(z)] \\ &= r_2(2b-1)[-1 + 2b - (2b-1)] = 0 = s\overline{\phi}_3'(z). \end{aligned}$$

Since $\overline{\phi_i} \ge \underline{\phi_i}$, $i = 1, 2, 3$, $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ are a pair of upper and lower solutions. Clearly, condition (2.7) in Lemma 2.1 holds. Hence, by Lemma 2.1, the proof is done. $\square$

In the sequel, we set

$$\phi_i^+ := \limsup_{z\to-\infty} \phi_i(z), \quad \phi_i^- := \liminf_{z\to-\infty} \phi_i(z), \ i = 1, 2, 3,$$

for a solution (ϕ_1, ϕ_2, ϕ_3) of (1.3). Since $\phi_i \geq \underline{\phi}_i$, we have

$$\phi_i^- \geq \gamma_i \text{ for } i = 1, 2, 3, \tag{3.4}$$

where

$$\gamma_1 := 1 - h - a(2b - 1),\ \gamma_2 := 1 - k - a(2b - 1),$$
$$\gamma_3 := -1 + b\gamma_1 + b\gamma_2 = -1 + b[2 - h - k - 2a(2b - 1)]$$

are all positive due to (1.2) and (1.4). Then the following lemma can be proved by a similar argument to that in [7, 15, 6] with some modifications.

Lemma 3.2. *Assume the condition* (1.2) *is enforced. Let* (ϕ_1, ϕ_2, ϕ_3) *be a solution of* (1.3) *obtained from Lemma 3.1. Then* $(\phi_1, \phi_2, \phi_3)(-\infty) = E_4 =: (u^*, v^*, w^*)$.

Proof. Consider the following functions

$$m_1(\theta) := \theta u^* + (1 - \theta)(\gamma_1 - \varepsilon),\ M_1(\theta) := \theta u^* + (1 - \theta)(1 + \varepsilon),\ \theta \in [0, 1],$$
$$m_2(\theta) := \theta v^* + (1 - \theta)(\gamma_2 - \varepsilon),\ M_2(\theta) := \theta v^* + (1 - \theta)(1 + \varepsilon),\ \theta \in [0, 1],$$
$$m_3(\theta) := \theta w^* + (1 - \theta)(\gamma_3 - \tau_1\varepsilon),\ M_3(\theta) := \theta w^* + (1 - \theta)(2b - 1 + \tau_2\varepsilon),\ \theta \in [0, 1],$$

where

$$\tau_1 := \max\{3b, 2(1 - h)/a, 2(1 - k)/a\},\ \tau_2 := (2b + \min\{(1 - h)/a, (1 - k)/a\})/2$$

and ε is chosen to satisfy

$$0 < \varepsilon < \min\left\{\gamma_1, \gamma_2, \frac{\gamma_3}{\tau_1}, \frac{h\gamma_2 + a\gamma_3}{a\tau_1 - 1 + h}, \frac{k\gamma_1 + a\gamma_3}{a\tau_1 - 1 + k}\right\}. \tag{3.5}$$

Note that

$$\tau_2 \in (2b, \min\{(1 - h)/a, (1 - k)/a\}) \tag{3.6}$$

due to (1.2).

By (3.4), it is obvious that

$$m_i(\theta) < \phi_i^- \leq \phi_i^+ < M_i(\theta), \tag{3.7}$$

holds for $\theta = 0$ for $i = 1, 2, 3$. Hence the quantity

$$\theta_0 := \sup\{\theta \in [0, 1) : (3.7) \text{ holds for } i = 1, 2, 3\}$$

is well-defined. Note that $u^* < 1, v^* < 1$ and $w^* < 2b - 1$. Since (u^*, v^*, w^*) satisfies

$$u^* = 1 - hv^* - aw^*,\ v^* = 1 - ku^* - aw^*,\ w^* = -1 + b(u^* + v^*),$$

we have $u^* > 1 - h - a(2b - 1) = \gamma_1$, $v^* > 1 - k - a(2b - 1) = \gamma_2$ and $w^* > -1 + b(\gamma_1 + \gamma_2) = \gamma_3$. Then the function $m_i(\theta)$ (resp. $-M_i(\theta)$) is increasing in $\theta \in [0, 1]$, $i = 1, 2, 3$. Moreover, $m_1(1) = M_1(1) = u^*$, $m_2(1) = M_2(1) = v^*$ and $m_3(1) = M_3(1) = w^*$. Therefore, we only need to show that $\theta_0 = 1$.

For contradiction, we suppose that $\theta_0 < 1$. By passing to limit, we have

$$m_i(\theta_0) \leq \phi_i^- \leq \phi_i^+ \leq M_i(\theta_0)$$

for $i = 1, 2, 3$. By the definition of θ_0 and the continuity of $m_i(\theta)$ and $M_i(\theta)$, (3.7) cannot hold for $\theta = \theta_0$ for all $i = 1, 2, 3$. This means that at least one of the following equalities holds:

$$\phi_i^- = m_i(\theta_0), \quad \phi_i^+ = M_i(\theta_0), \quad i = 1, 2, 3. \tag{3.8}$$

First, we assume that $\phi_1^- = m_1(\theta_0)$. If ϕ_1 is eventually monotone, then $\phi_1(-\infty)$ exists, and $\liminf_{z\to-\infty} \phi_1'(z) = 0$ or $\limsup_{z\to-\infty} \phi_1'(z) = 0$. Then there exists a sequence $\{z_n\}$ with $z_n \to -\infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \phi_1'(z_n) = 0$ and $\lim_{n\to\infty} \phi_1(z_n) = m_1(\theta_0)$. Since $\limsup_{n\to\infty} \phi_2(z_n) \le M_2(\theta_0)$ and $\limsup_{n\to\infty} \phi_3(z_n) \le M_3(\theta_0)$, we have

$$\begin{aligned}
& \liminf_{n\to\infty} [\alpha(-z_n) - \phi_1(z_n) - h\phi_2(z_n) - a\phi_3(z_n)] \\
\ge\ & 1 - [\theta_0 u^* + (1-\theta_0)(\gamma_1 - \varepsilon)] - h[\theta_0 v^* + (1-\theta_0)(1+\varepsilon)] \\
& \quad - a[\theta_0 w^* + (1-\theta_0)(2b - 1 + \tau_2\varepsilon)] \\
=\ & \varepsilon(1 - h - a\tau_2)(1-\theta_0) > 0.
\end{aligned}$$

The last inequality holds by the choice of τ_2 and (3.6). By integrating the ϕ_1-equation of (1.3) from 0 to z_n, we have

$$\phi_1'(z_n) - \phi_1'(0) - s(\phi_1(z_n) - \phi_1(0)) = -r_1 \int_0^{z_n} \phi_1(z)[\alpha(-z) - \phi_1(z) - h\phi_2(z) - a\phi_3(z)]dz.$$

Since the right hand side goes to $+\infty$ as $n \to \infty$ and the left hand side is bounded, we have a contradiction.

Next, we assume that ϕ_1 is oscillatory at $-\infty$. Then we can choose a sequence $\{z_n\}$ of minimal points of ϕ_1 with $z_n \to -\infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \phi_1(z_n) = m_1(\theta_0)$. Since z_n is a minimal point of ϕ_1, $\phi_1'(z_n) = 0$ and $\phi_1''(z_n) \ge 0$ for all n. Similarly, we obtain from the first equation of (1.3) that

$$0 = \liminf_{n\to\infty} s\phi_1'(z_n) \ge \liminf_{n\to\infty} \{r_1\phi_1(z_n)[\alpha(-z_n) - \phi_1(z_n) - h\phi_2(z_n) - a\phi_3(z_n)]\} > 0,$$

a contradiction. Hence $\phi_1^- = m_1(\theta_0)$ cannot happen.

The other cases in (3.8) can be treated similarly using the following inequalities:

(i) for $\phi_1^+ = M_1(\theta_0)$,

$$\begin{aligned}
& \limsup_{n\to\infty} [\alpha(-z_n) - \phi_1(z_n) - h\phi_2(z_n) - a\phi_3(z_n)] \\
\le\ & 1 - [\theta_0 u^* + (1-\theta_0)(1+\varepsilon)] - h[\theta_0 v^* + (1-\theta_0)(\gamma_2 - \varepsilon)] \\
& \quad - a[\theta_0 w^* + (1-\theta_0)(\gamma_3 - \tau_1\varepsilon)] \\
=\ & (1-\theta_0)[(a\tau_1 - 1 + h)\varepsilon - (h\gamma_2 + a\gamma_3)] < 0,
\end{aligned}$$

using (3.5);

(ii) for $\phi_2^- = m_2(\theta_0)$,

$$\begin{aligned}
&\liminf_{n\to\infty}[\alpha(-z_n) - \phi_2(z_n) - k\phi_1(z_n) - a\phi_3(z_n)] \\
\geq\ & 1 - [\theta_0 v^* + (1-\theta_0)(\gamma_2 - \varepsilon)] - k[\theta_0 u^* + (1-\theta_0)(1+\varepsilon)] \\
& -a[\theta_0 w^* + (1-\theta_0)(2b - 1 + \tau_2\varepsilon)] \\
=\ & \varepsilon(1 - k - a\tau_2)(1-\theta_0) > 0,
\end{aligned}$$

using (3.6);

(iii) for $\phi_2^+ = M_2(\theta_0)$,

$$\begin{aligned}
&\limsup_{n\to\infty}[\alpha(-z_n) - \phi_2(z_n) - k\phi_1(z_n) - a\phi_3(z_n)] \\
\leq\ & 1 - [\theta_0 v^* + (1-\theta_0)(1+\varepsilon)] - k[\theta_0 u^* + (1-\theta_0)(\gamma_1 - \varepsilon)] \\
& -a[\theta_0 w^* + (1-\theta_0)(\gamma_3 - \tau_1\varepsilon)] \\
=\ & (1-\theta_0)[(a\tau_1 - 1 + k)\varepsilon - (k\gamma_1 + a\gamma_3)] < 0,
\end{aligned}$$

using (3.5);

(iv) for $\phi_3^- = m_3(\theta_0)$,

$$\begin{aligned}
&\liminf_{n\to\infty}[-1 + b(\phi_1(z_n) + \phi_2(z_n)) - \phi_3(z_n)] \\
\geq\ & -1 + b[\theta_0 u^* + (1-\theta_0)(\gamma_1 - \varepsilon)] + b[\theta_0 v^* + (1-\theta_0)(\gamma_2 - \varepsilon)] \\
& -[\theta_0 w^* + (1-\theta_0)(\gamma_3 - \tau_1\varepsilon)] \\
=\ & \varepsilon(\tau_1 - 2b)(1-\theta_0) > 0,
\end{aligned}$$

using $\tau_1 \geq 3b$;

(v) for $\phi_3^+ = M_3(\theta_0)$,

$$\begin{aligned}
&\limsup_{n\to\infty}[-1 + b(\phi_1(z_n) + \phi_2(z_n)) - \phi_3(z_n)] \\
\leq\ & -1 + b[\theta_0 u^* + (1-\theta_0)(1+\varepsilon)] + b[\theta_0 v^* + (1-\theta_0)(1+\varepsilon)] \\
& -[\theta_0 w^* + (1-\theta_0)(2b - 1 + \tau_2\varepsilon)] \\
=\ & \varepsilon(2b - \tau_2)(1-\theta_0) < 0,
\end{aligned}$$

using (3.6).

Similarly to the case $\phi_1^- = m_1(\theta_0)$ by using these inequalities, we can show that all cases in (3.8) are impossible. Therefore, the lemma is proved. □

From Lemma 3.1, Proposition 2.2 and Lemma 3.2, we have proved Theorem 1.1.

4. Mixed front-pulse type forced waves

In this section, we show the existence of mixed front-pulse type forced waves connecting E_i to $(0,0,0)$ for $i = 1,2,3$. Construction of upper and lower solutions are motivated by [7, 6].

First, by the assumption ($\alpha 3$), there is a positive constant K such that $1-\alpha(z) \le Ce^{-\rho z}$ for all $z \ge K$. By choosing C larger (if necessary), we indeed have $1-\alpha(z) \le Ce^{-\rho z}$ for all $z \in \mathbb{R}$. Then we have

$$\alpha(-z+M) \ge 1 - Ce^{-\rho M}e^{\rho z}, \ \forall\, z \in \mathbb{R},$$

for any constant M. Hence, for a given small $\varepsilon > 0$, we can choose $M = M(\varepsilon)$ large enough such that

$$\alpha(-z+M) \ge 1 - \varepsilon e^{\rho z}, \ \forall\, z \in \mathbb{R}. \tag{4.1}$$

With this M, we consider the system

$$\begin{cases} s\phi_1'(z) = d_1\phi_1''(z) + r_1\phi_1(z)[\alpha(-z+M) - \phi_1(z) - h\phi_2(z) - a\phi_3(z)], \ z \in \mathbb{R}, \\ s\phi_2'(z) = d_2\phi_2''(z) + r_2\phi_2(z)[\alpha(-z+M) - k\phi_1(z) - \phi_2(z) - a\phi_3(z)], \ z \in \mathbb{R}, \\ s\phi_3'(z) = d_3\phi_3''(z) + r_3\phi_2(z)[-1 + b\phi_1(z) + b\phi_2(z) - \phi_3(z)], \ z \in \mathbb{R}. \end{cases} \tag{4.2}$$

To investigate the existence of mixed front-pulse type forced waves, we first show that (2.1)-(2.6) hold for $\alpha(-z)$ replaced by $\alpha(-z+M)$ for a suitably chosen small $\varepsilon > 0$ and its corresponding M. Then a solution (ϕ_1, ϕ_2, ϕ_3) of (4.2) renders a solution of (1.3) by a translation from z to $z+M$.

4.1. **Waves connecting E_1.**

Note that ($\alpha 3$) holds for any $\rho' \le \rho$. Hence, when $\rho \ge \rho_{**}$, ($\alpha 3$) also holds for ρ replaced by any $\rho \le \rho_{**}$.

Case 1. $s \ge s_0^*(\rho)$ and $s > s_3^*$.

In this case, there exist λ_i, $i = 1,2,3,4$, such that $0 < \lambda_1 < \lambda_2$ and $0 < \lambda_3 < \lambda_4$ for which

$$\begin{cases} A_1(\lambda_i) := d_2\lambda_i^2 - s\lambda_i + r_2(1-k) = 0, \ i = 1,2, \\ A_2(\lambda_i) := d_3\lambda_i^2 - s\lambda_i + r_3(-1+b) = 0, \ i = 3,4. \end{cases}$$

Recall from (1.5) that $A_1 = A_2$ so that $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_4$. Choose

$$\begin{cases} \varepsilon \in (0, [r_3(b-1)]/r_1), \\ \tilde{\mu} \in (\lambda_3, \min\{2\lambda_3, \lambda_4\}), \ \mu' \in (\lambda_3, \min\{2\lambda_3, \lambda_4\}). \end{cases}$$

Since $\tilde{\mu} \in (\lambda_3, \lambda_4)$ and $\mu' \in (\lambda_3, \lambda_4)$, we have $A_2(\tilde{\mu}) < 0$ and $A_2(\mu') < 0$. We define

$$\begin{cases} \overline{\phi}_1(z) \equiv 1, \ \underline{\phi}_1(z) = \max\{1 - e^{\lambda_3 z}, 0\}, \\ \overline{\phi}_2(z) = \min\{e^{\lambda_3 z}, 1\}, \ \underline{\phi}_2(z) = \max\{e^{\lambda_3 z} - q_1 e^{\tilde{\mu} z}, 0\}, \\ \overline{\phi}_3(z) = \min\{(2b-1)e^{\lambda_3 z}, 2b-1\}, \ \underline{\phi}_3(z) = \max\{(2b-1)e^{\lambda_3 z} - q_2 e^{\mu' z}, 0\}, \end{cases}$$

where

$$q_1 > \max\left\{1, \frac{r_2[\varepsilon + 1 + a(2b-1)]}{-A_2(\tilde{\mu})}\right\}, q_2 > \max\left\{2b-1, \frac{r_3(2b-1)(3b-1)}{-A_2(\mu')}\right\}. \tag{4.3}$$

Note that, by (4.3), there exist $z_1 < 0$ and $z_2 < 0$ such that

$$e^{\lambda_3 z_1} - q_1 e^{\tilde{\mu} z_1} = (2b-1)e^{\lambda_3 z_2} - q_2 e^{\mu' z_2} = 0.$$

Then we have

Lemma 4.1. *Suppose $s \geq s_0^*(\rho)$, $s > s_3^*$ and condition (1.5) holds. Then there exists a solution (ϕ_1, ϕ_2, ϕ_3) of (1.3) such that $\underline{\phi}_1 \leq \phi_1 \leq \overline{\phi}_1$, $\underline{\phi}_2 \leq \phi_2 \leq \overline{\phi}_2$ and $\underline{\phi}_3 \leq \phi_3 \leq \overline{\phi}_3$ in $\mathbb{R}$.*

Proof. We consider system (4.2) for the corresponding constant $M(\varepsilon)$ and show that (2.1)-(2.6) hold for $\alpha(-z)$ replaced by $\alpha(-z+M)$. Note that, since $s \geq s_0^*(\rho)$, we have $\rho \geq \lambda_3$. Hence we may choose $\rho = \lambda_3$.

Since $\overline{\phi} = 1$ for $z \in \mathbb{R}$ and $\overline{\phi}_2 = 1, \overline{\phi}_3 = 2b-1$ for $z > 0$, (2.1) holds for $z \in \mathbb{R}$ and (2.2)-(2.3) hold for $z > 0$. For $z < 0$, we have

$$\begin{aligned} & d_2\overline{\phi}_2'' - s\overline{\phi}_2' + r_2\overline{\phi}_2[\alpha(-z+M) - k\underline{\phi}_1 - \overline{\phi}_2 - a\underline{\phi}_3] \\ \leq\ & e^{\lambda_3 z}[d_2\lambda_3^2 - s\lambda_3 + r_2(1 - k + ke^{\lambda_3 z} - e^{\lambda_3 z} - a\underline{\phi}_3)] \leq 0, \end{aligned}$$

using $d_2 = d_3$, $r_2(1-k) = r_3(b-1)$ and $k < 1$. Thus (2.2) holds for $z \neq 0$. Also, for $z < 0$,

$$d_3\overline{\phi}_3'' - s\overline{\phi}_3' + r_3\overline{\phi}_3[-1 + b(\overline{\phi}_1 + \overline{\phi}_2) - \overline{\phi}_3] = r_3\overline{\phi}_3 e^{\lambda_3 z}(1-b) \leq 0.$$

Hence (2.3) holds for $z \neq 0$.

Next, it is trivial that (2.4) holds for $z > 0$. For $z < 0$, we compute

$$\begin{aligned} & d_1\underline{\phi}_1'' - s\underline{\phi}_1' + r_1\underline{\phi}_1[\alpha(-z+M) - \underline{\phi}_1 - h\overline{\phi}_2 - a\overline{\phi}_3] \\ \geq\ & -(d_1\lambda_3^2 - s\lambda_3)e^{\lambda_3 z} + r_1(1 - e^{\lambda_3 z})[1 - \varepsilon e^{\rho z} - (1 - e^{\lambda_3 z}) - he^{\lambda_3 z} - a(2b-1)e^{\lambda_3 z}] \\ \geq\ & -(d_3\lambda_3^2 - s\lambda_3)e^{\lambda_3 z} - r_1\varepsilon e^{\rho z} = e^{\lambda_3 z}[r_3(b-1) - r_1\varepsilon] \geq 0, \end{aligned}$$

using $a(2b-1) < 2ab < 1-h$ (due to (1.2)), $d_1 \leq d_3$, $\rho = \lambda_3$ and the choice of ε. Hence (2.4) holds for $z \neq 0$.

Now, for $z > z_1$, (2.5) immediately hold. Thus we only need to consider $z < z_1$. For $z < z_1$, we compute, using $\rho = \lambda_3$, $d_2 = d_3$, $r_2(1-k) = r_3(b-1)$,

$$\begin{aligned} & d_2\underline{\phi}_2'' - s\underline{\phi}_2' + r_2\underline{\phi}_2[\alpha(-z+M) - k\overline{\phi}_1 - \underline{\phi}_2 - a\overline{\phi}_3] \\ \geq\ & e^{\lambda_3 z}[d_2\lambda_3^2 - s\lambda_3 + r_2(1-k)] - q_1e^{\tilde{\mu}z}[d_2\tilde{\mu}^2 - s\tilde{\mu} + r_2(1-k)] \\ & \quad + r_2\underline{\phi}_2(z)[-\varepsilon e^{\rho z} - (e^{\lambda_3 z} - q_1e^{\tilde{\mu}z}) - a(2b-1)e^{\lambda_3 z}] \\ =\ & -q_1e^{\tilde{\mu}z}A_2(\tilde{\mu}) + r_2e^{\lambda_3 z}[-\varepsilon e^{\rho z} - e^{\lambda_3 z} - a(2b-1)e^{\lambda_3 z}] \\ =\ & e^{\tilde{\mu}z}\{-q_1A_1(\tilde{\mu}) - r_2e^{(2\lambda_3 - \tilde{\mu})z}[\varepsilon + 1 + a(2b-1)]\} \geq 0, \end{aligned}$$

by the choice of $\tilde{\mu}$ and (4.3). Hence (2.5) holds for $z \neq z_1$. Similarly, for $z < z_2$ we compute

$$\begin{aligned} & d_3\underline{\phi}_3'' - s\underline{\phi}_3' + r_3\underline{\phi}_3[-1 + b(\underline{\phi}_1 + \underline{\phi}_2) - \underline{\phi}_3] \\ =\ & (2b-1)e^{\lambda_3 z}[d_3\lambda_3^2 - s\lambda_3 + r_3(-1+b)] - q_2e^{\mu' z}[d_3(\mu')^2 - s\mu' + r_3(-1+b)] \\ & \quad + r_3\underline{\phi}_3(z)[-be^{\lambda_3 z} + b\underline{\phi}_2 - (2b-1)e^{\lambda_3 z} + q_2e^{\mu' z}] \\ \geq\ & -q_2e^{\mu' z}A_2(\mu') + r_3(2b-1)e^{\lambda_3 z}[-be^{\lambda_3 z} - (2b-1)e^{\lambda_3 z}] \\ =\ & -q_2e^{\mu' z}A_2(\mu') + r_3(2b-1)e^{2\lambda_3 z}[-b - (2b-1)] \\ \geq\ & e^{\mu' z}[-q_2A_2(\mu') - r_3(2b-1)(3b-1)e^{(2\lambda_3 - \mu')z}] \geq 0, \end{aligned}$$

by the choice of μ' and (4.3). Hence (2.6) holds for $z \neq z_2$. Therefore, the lemma follows by applying Lemma 2.1. $\square$

Case 2. $s = s_0^*(\rho) = s_3^*$ and $s_3^* = s_2^*$.

For $s = s_3^*$, the equation $A_2(\lambda) = 0$ has a double root $\lambda_0 = \rho_{**}$. We assume that $d_1 = d_2 = d_3$. Note that λ_0 satisfies

$$d_i\lambda_0^2 - s\lambda_0 + r_2(1-k) = d_i\lambda_0^2 - s\lambda_0 + r_3(b-1) = 0,\ 2d_i\lambda_0 = s,\ i = 1,2,3.$$

Let $L_0 = \lambda_0 e$. Choose $\varepsilon \in (0, e[1-h-a(2b-1)])$ and

$$\zeta_1 > \max\left\{e(\lambda_0)^{1/2}, 4r_2L_0\left[\varepsilon\left(\frac{5}{2L_0}\right)^{5/2} + (1+a(2b-1))L_0\left(\frac{7}{2L_0}\right)^{7/2}\right]\Big/d_2\right\}, \tag{4.4}$$

$$\zeta_2 > \max\left\{(2b-1)e(\lambda_0)^{1/2}, 4r_3(3b-1)(2b-1)L_0^2\left(\frac{7}{2L_0}\right)^{7/2}\Big/d_3\right\}. \tag{4.5}$$

Set $z_3 := -(\zeta_1/L_0)^2$ and $z_4 := -\{\zeta_2/[(2b-1)L_0]\}^2$. Note that $z_3 < -1/\lambda_0$, by (4.4), and $z_4 < -1/\lambda_0$, by (4.5). We define

$$\overline{\phi}_1(z) \equiv 1,\ \underline{\phi}_1(z) = \begin{cases} 1 - L_0(-z)e^{\lambda_0 z},\ z < -1/\lambda_0, \\ 0,\ z \geq -1/\lambda_0, \end{cases}$$

$$\overline{\phi}_2(z) = \begin{cases} L_0(-z)e^{\lambda_0 z},\ z < -1/\lambda_0, \\ 1,\ z \geq -1/\lambda_0, \end{cases} \qquad \underline{\phi}_2(z) = \begin{cases} L_0(-z)e^{\lambda_0 z} - \zeta_1(-z)^{1/2}e^{\lambda_0 z},\ z < z_3, \\ 0,\ z \geq z_3, \end{cases}$$

$$\overline{\phi}_3(z) = \begin{cases} (2b-1)L_0(-z)e^{\lambda_0 z},\ z < -1/\lambda_0, \\ 2b-1,\ z \geq -1/\lambda_0, \end{cases}$$

$$\underline{\phi}_3(z) = \begin{cases} (2b-1)L_0(-z)e^{\lambda_0 z} - \zeta_2(-z)^{1/2}e^{\lambda_0 z},\ z < z_4, \\ 0,\ z \geq z_4. \end{cases}$$

Then we have

Lemma 4.2. *Suppose, in addition to* (1.5), $s = s_0^*(\rho) = s_3^*$ *and* $d_1 = d_2 = d_3$. *Then there exists a solution* (ϕ_1, ϕ_2, ϕ_3) *of* (1.3) *such that* $\underline{\phi}_1 \leq \phi_1 \leq \overline{\phi}_1$, $\underline{\phi}_2 \leq \phi_2 \leq \overline{\phi}_2$ *and* $\underline{\phi}_3 \leq \phi_3 \leq \overline{\phi}_3$ *in* $\mathbb{R}$.

Proof. Consider (4.2) with the corresponding $M(\varepsilon)$. We show that (2.1)-(2.6) hold for $\alpha(-z)$ replaced by $\alpha(-z+M)$.

First, (2.1) is trivial and (2.2)-(2.3) immediately hold for $z > -1/\lambda_0$. For $z < -1/\lambda_0$, we compute, using $\underline{\phi}_3 \geq 0$, $\alpha \leq 1$, $d_2 = d_3$, $2d_3\lambda_0 = s$, $k < 1$ and $A_2(\lambda_0) = 0$,

$$\begin{aligned} & d_2\overline{\phi}_2'' - s\overline{\phi}_2' + r_2\overline{\phi}_2[\alpha(-z+M) - k\underline{\phi}_1 - \overline{\phi}_2 - a\underline{\phi}_3] \\ \leq\ & \overline{\phi}_2(z)(d_2\lambda_0^2 - s\lambda_0) + L_0e^{\lambda_0 z}(s - 2d_2\lambda_0) + r_2\overline{\phi}_2(z)[(1-k) + (k-1)L_0(-z)e^{\lambda_0 z}] \\ \leq\ & 0. \end{aligned}$$

Hence (2.2) holds for $z \neq -1/\lambda_0$. For $z < -1/\lambda_0$, we have

$$\begin{aligned} & d_3\overline{\phi}_3'' - s\overline{\phi}_3' + r_3\overline{\phi}_3[-1 + b(\overline{\phi}_1 + \overline{\phi}_2) - \overline{\phi}_3] \\ = \; & (2b-1)L_0(-z)e^{\lambda_0 z}[d_3\lambda_0^2 - s\lambda_0 + r_3(-1+b)] + (2b-1)L_0e^{\lambda_0 z}(-2d_3\lambda_0 + s) \\ & + r_3\overline{\phi}_3(1-b)L_0(-z)e^{\lambda_0 z} \leq 0, \end{aligned}$$

using $A_2(\lambda_0) = 0$, $s = 2d_3\lambda_0$ and $b > 1$. Hence (2.3) holds for $z \neq -1/\lambda_0$.

For (2.4), we only need to consider $z < -1/\lambda_0$. Since $s = s_3^*$ can happen only when $\rho \geq \rho_{**}$, we may set $\rho = \rho_{**} = \lambda_0$ in (α3). Hence $\alpha(-z+M) \geq 1 - \varepsilon e^{\lambda_0 z}$ for all $z < 0$. For $z < -1/\lambda_0$, we compute

$$\begin{aligned} & d_1\underline{\phi}_1'' - s\underline{\phi}_1' + r_1\underline{\phi}_1[\alpha(-z+M) - \underline{\phi}_1 - h\overline{\phi}_2 - a\overline{\phi}_3] \\ \geq \; & -L_0(-z)e^{\lambda_0 z}(d_1\lambda_0^2 - s\lambda_0) + L_0e^{\lambda_0 z}(2d_1\lambda_0 - s) \\ & \quad + r_1\underline{\phi}_1 e^{\lambda_0 z}\{-\varepsilon + L_0(-z)[1 - h - a(2b-1)]\} \\ \geq \; & r_3L_0(-z)e^{\lambda_0 z}(b-1) + r_1\underline{\phi}_1 e^{\lambda_0 z}\{-\varepsilon + e[1 - h - a(2b-1)]\} > 0, \end{aligned}$$

using $d_1 = d_3$, $A_2(\lambda_0) = 0$, $2d_3\lambda_0 = s$ and by the choice of ε. Hence (2.4) holds for $z \neq -1/\lambda_0$.

Next, (2.5) and (2.6) immediately hold for $z > z_3$ and $z > z_4$, respectively. Note that, due to $s = s_0^*(\rho) = s_3^*$, we have

$$\alpha(-z+M) \geq 1 - \varepsilon e^{\lambda_0 z} \text{ for all } z < 0.$$

Hereafter we use the fact that

$$\sup_{z \leq 0}\{(-z)^\nu e^{\gamma z}\} = \left(\frac{\nu}{\gamma e}\right)^\nu \tag{4.6}$$

for any given positive constants ν and γ.

For $z < z_3$, we compute

$$\begin{aligned} & d_2\underline{\phi}_2'' - s\underline{\phi}_2' + r_2\underline{\phi}_2[\alpha(-z+M) - k\overline{\phi}_1 - \underline{\phi}_2 - a\overline{\phi}_3] \\ \geq \; & \frac{1}{4}d_2\zeta_1(-z)^{-3/2}e^{\lambda_0 z} + \underline{\phi}_2[d_2\lambda_0^2 - s\lambda_0 + r_2(1-k)] + (2d_2\lambda_0 - s)[-L_0 + \frac{1}{2}(-z)^{-1/2}\zeta_1]e^{\lambda_0 z} \\ & \quad + r_2\underline{\phi}_2(z)[-\varepsilon e^{\lambda_0 z} - L_0(-z)e^{\lambda_0 z} + \zeta_1(-z)^{1/2}e^{\lambda_0 z} - a(2b-1)L_0(-z)e^{\lambda_0 z}] \\ \geq \; & \frac{1}{4}d_2\zeta_1(-z)^{-3/2}e^{\lambda_0 z} - r_2L_0(-z)e^{\lambda_0 z}\{\varepsilon e^{\lambda_0 z} + [1 + a(2b-1)]L_0(-z)e^{\lambda_0 z}\} \\ \geq \; & \frac{1}{4}(-z)^{-3/2}e^{\lambda_0 z}\{d_2\zeta_1 - 4r_2L_0[\varepsilon(-z)^{5/2}e^{\lambda_0 z} + (1 + a(2b-1))L_0(-z)^{7/2}e^{\lambda_0 z}]\} \\ \geq \; & \frac{1}{4}(-z)^{-3/2}e^{\lambda_0 z}\left\{d_2\zeta_1 - 4r_2L_0\left[\varepsilon\left(\frac{5}{2L_0}\right)^{5/2} + (1 + a(2b-1))L_0\left(\frac{7}{2L_0}\right)^{7/2}\right]\right\} \geq 0, \end{aligned}$$

by using (4.6) and (4.4). Hence (2.5) holds for $z \neq z_3$.

Similarly, for $z < z_4$, we compute

$$\begin{aligned}
& d_3\underline{\phi}_3'' - s\underline{\phi}_3' + r_3\underline{\phi}_3[-1 + b(\underline{\phi}_1 + \underline{\phi}_2) - \underline{\phi}_3] \\
\geq\ & \frac{1}{4}d_3\zeta_2(-z)^{-3/2}e^{\lambda_0 z} - r_3(3b-1)(2b-1)L_0^2(-z)^2e^{2\lambda_0 z} \\
=\ & \frac{1}{4}(-z)^{-3/2}e^{\lambda_0 z}\left[d_3\zeta_2 - 4r_3(3b-1)(2b-1)L_0^2(-z)^{7/2}e^{\lambda_0 z}\right] \\
\geq\ & \frac{1}{4}(-z)^{-3/2}e^{\lambda_0 z}\left[d_3\zeta_2 - 4r_3(3b-1)(2b-1)L_0^2\Big(\frac{7}{2L_0}\Big)^{7/2}\right] \geq 0,
\end{aligned}$$

by (4.6) and (4.5). Hence (2.6) holds for $z \neq z_4$. Therefore, the lemma follows by applying Lemma 2.1. □

As a summary, we have proved Theorem 1.2.

For the non-existence of forced waves, we have

Proposition 4.3. *Suppose $s > 0$, $k < 1$ and $b > 1$. Then* (1.3) *has a positive solution such that* (1.6) *holds only if $s \geq \max\{s_2^*, s_3^*\}$.*

Proof. Without loss of generality, we may assume that $s_3^* \geq s_2^*$. Suppose that there exists a positive solution (ϕ_1, ϕ_2, ϕ_3) of (1.3) satisfying (1.6) for some $s > 0$. Set $\zeta(z) := \phi_3'(z)/\phi_3(z)$. Then ζ satisfies

$$d_3\zeta'(z) + d_3\zeta^2(z) - s\zeta(z) + r_3\Phi(z) = 0,\ z \in \mathbb{R}, \tag{4.7}$$

where $\Phi(z) := -1 + b\phi_1(z) + b\phi_2(z) - \phi_3(z)$. Since $(\phi_1, \phi_2, \phi_3)(-\infty) = (1, 0, 0)$ and $b > 1$, we can choose a constant K such that $\Phi(z) > 0$ for all $z \in (-\infty, K]$. Hence, by ϕ_3-equation in (1.3), any critical point of ϕ_3 in $(-\infty, K]$ must be a strict maximal point. Since $\phi_3(-\infty) = 0$ and $\phi_3 > 0$ in $\mathbb{R}$, this implies that $\phi_3'(z) > 0$ for all $z \in (-\infty, K_1]$ for some $K_1 < K$ and so $\zeta(z) > 0$ for $z \in (-\infty, K_1]$. On the other hand, from

$$[d_3\phi_3'(z) - s\phi_3(z)]' = d_3\phi_3''(z) - s\phi_3'(z) = -r_3\phi_3(z)\Phi(z) < 0,\ \forall\, z \in (-\infty, K],$$

and $d_3\phi_3'(z) - s\phi_3(z) \to 0$ as $z \to -\infty$, it follows that $d_3\phi_3'(z) - s\phi_3(z) < 0$ for all $z \in (-\infty, K]$. Hence ζ is bounded above by s/d_3 in $(-\infty, K]$.

Now, if $\zeta(z)$ is monotone ultimately at $z = -\infty$, then the limit $\lambda := \lim_{z\to-\infty}\zeta(z)$ exists and is finite, due to the boundedness of ζ. Hence we can find a sequence $\{z_n\}$ tending to $-\infty$ such that $\zeta'(z_n) \to 0$ as $n \to \infty$. Passing to the limit in (4.7), λ satisfies

$$d_3\lambda^2 - s\lambda + r_3(b-1) = 0. \tag{4.8}$$

On the other hand, if $\zeta(z)$ is oscillatory at $z = -\infty$, then there is a sequence $\{z_n\}$ such that z_n is a maximal point of ζ for each n and

$$\zeta(z_n) \to \limsup_{z\to-\infty}\zeta(z) =: \zeta_+ \in [0, s/d_3].$$

It follows from (4.7) that ζ_+ also satisfies (4.8). Since (4.8) has a nonnegative root only if $s \geq s_3^*$, the proposition is proved. □

4.2. **Waves connecting E_2.**

Recall

$$\beta_2 = -1 + b(u_c + v_c),\ (u_c, v_c) = \left(\frac{1-h}{1-hk}, \frac{1-k}{1-hk}\right),\ s_3^{**} = 2\sqrt{d_3 r_3 \beta_2}.$$

Since ($\alpha 3$) holds for any $\rho' \le \rho$, ($\alpha 3$) also holds for $\rho = \rho_*$ when $\rho \ge \rho_*$.

Case 1. $s \ge s_c^*(\rho)$ and $s > s_3^{**}$.
For $s > s_3^{**}$, the equation

$$A_3(\lambda) := d_3\lambda^2 - s\lambda + r_3\beta_2 = 0$$

has two positive roots λ_1 and λ_2 such that $0 < \lambda_1 < \lambda_2$. We define

$$\begin{cases} \overline{\phi}_1(z) = \min\{u_c + hv_c e^{\lambda_1 z}, 1\},\ \underline{\phi}_1(z) = \max\{u_c(1 - e^{\lambda_1 z}), 0\}, \\ \overline{\phi}_2(z) = \min\{v_c + ku_c e^{\lambda_1 z}, 1\},\ \underline{\phi}_2(z) = \max\{v_c(1 - e^{\lambda_1 z}), 0\}, \\ \overline{\phi}_3(z) = \min\{(2b-1)e^{\lambda_1 z}, 2b-1\},\ \underline{\phi}_3(z) = \max\{(2b-1)e^{\lambda_1 z} - qe^{\tilde{\lambda} z}, 0\}, \end{cases}$$

where $\tilde{\lambda} \in (\lambda_1, \min\{2\lambda_1, \lambda_2\})$ and

$$q > \max\left\{2b-1, \frac{r_3(2b-1)[b(u_c+v_c) + 2b - 1]}{-A_3(\tilde{\lambda})}\right\}. \tag{4.9}$$

Note that $A_3(\tilde{\lambda}) < 0$, since $\tilde{\lambda} \in (\lambda_1, \lambda_2)$. Then we have

Lemma 4.4. *Suppose $d_3 \ge \max\{d_1, d_2\}/2$, $s \ge s_c^*(\rho)$ and $s > s_3^{**}$. Then there exists a solution (ϕ_1, ϕ_2, ϕ_3) of (1.3) such that $\underline{\phi}_1 \le \phi_1 \le \overline{\phi}_1$, $\underline{\phi}_2 \le \phi_2 \le \overline{\phi}_2$ and $\underline{\phi}_3 \le \phi_3 \le \overline{\phi}_3$ in $\mathbb{R}$.*

Proof. We choose a positive constant ε such that

$$\varepsilon < 1 - \max\{h, k\} - a(2b-1) \tag{4.10}$$

and consider system (4.2) with the corresponding $M(\varepsilon)$. We verify that $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ satisfy (2.1)-(2.6) for $\alpha(-z)$ replaced by $\alpha(-z+M)$.

First, for $z > 0$, $\overline{\phi}_1 = 1, \overline{\phi}_2 = 1, \overline{\phi}_3 = 2b-1$, so (2.1)-(2.3) immediately holds. For $z < 0$, since $d_3 \ge d_1/2$, we can easily show that $d_1\lambda_1^2 - s\lambda_1 \le 0$. Then, using $\alpha \le 1$ and $1 - u_c - hv_c = 0$,

$$d_1\overline{\phi}_1'' - s\overline{\phi}_1' + r_1\overline{\phi}_1[\alpha(-z+M) - \overline{\phi}_1 - h\underline{\phi}_2 - a\underline{\phi}_3] \le hv_c e^{\lambda_1 z}(d_1\lambda_1^2 - s\lambda_1) - ar_1\overline{\phi}_1\underline{\phi}_3 < 0.$$

Thus (2.1) holds for $z \ne 0$. Similarly, (2.2) holds for $z \ne 0$.

For $z < 0$, we compute

$$\begin{aligned} & d_3\overline{\phi}_3'' - s\overline{\phi}_3' + r_3\overline{\phi}_3[-1 + b(\overline{\phi}_1 + \overline{\phi}_2) - \overline{\phi}_3] \\ = \ & (2b-1)e^{\lambda_1 z}(d_3\lambda_1^2 - s\lambda_1 + r_3\beta_2) + r_3\overline{\phi}_3 e^{\lambda_1 z}[b(hv_c + ku_c) - 2b + 1] \\ \le \ & r_3\overline{\phi}_3 e^{\lambda_1 z}[-b(2-h-k)/(1-hk) + 1] < 0. \end{aligned}$$

The last inequality holds due to $b > 1$ and $h, k < 1$. Hence (2.3) holds for $z \ne 0$.

Next, for $z<0$, using (4.1) we compute

$$\begin{aligned}
&d_1\underline{\phi}_1''-s\underline{\phi}_1'+r_1\underline{\phi}_1[\alpha(-z+M)-\underline{\phi}_1-h\overline{\phi}_2-a\overline{\phi}_3]\\
\geq\ & -u_ce^{\lambda_1 z}(d_1\lambda_1^2-s\lambda_1)+r_1\underline{\phi}_1[1-\varepsilon e^{\rho z}-u_c-hv_c+e^{\lambda_1 z}(u_c(1-hk)-a(2b-1))]\\
\geq\ & r_1\underline{\phi}_1e^{\lambda_1 z}[-\varepsilon e^{(\rho-\lambda_1)z}+(1-h-a(2b-1))]>0,
\end{aligned}$$

due to $\rho\geq\lambda_1$, since $s\geq s_c^*(\rho)$, and (4.10). Hence (2.4) holds for $z\neq 0$. Similarly, (2.5) holds for $z\neq 0$.

Finally, it remains to check (2.6). Since $q>2b-1$, there exists $z_0<0$ such that

$$\underline{\phi}_3(z)=\begin{cases}0, & z\geq z_0,\\ (2b-1)e^{\lambda_1 z}-qe^{\tilde{\lambda}z}, & z<z_0.\end{cases}$$

Clearly, (2.6) holds for $z>z_0$. For $z<z_0$, we compute

$$\begin{aligned}
&d_3\underline{\phi}_3''-s\underline{\phi}_3'+r_3\underline{\phi}_3[-1+b(\underline{\phi}_1+\underline{\phi}_2)-\underline{\phi}_3]\\
=\ & (2b-1)e^{\lambda_1 z}(d_3\lambda_1^2-s\lambda_1+r_3\beta_2)-qe^{\tilde{\lambda}z}(d_3\tilde{\lambda}^2-s\tilde{\lambda}+r_3\beta_2)\\
&\quad +r_3\underline{\phi}_3(z)[-b(u_c+v_c)e^{\lambda_1 z}-(2b-1)e^{\lambda_1 z}+qe^{\tilde{\lambda}z}]\\
\geq\ & -qe^{\tilde{\lambda}z}A_3(\tilde{\lambda})+r_3(2b-1)e^{2\lambda_1 z}[-b(u_c+v_c)-(2b-1)]\\
=\ & e^{\tilde{\lambda}z}[-qA_3(\tilde{\lambda})-r_3(2b-1)e^{(2\lambda_1-\tilde{\lambda})z}[b(u_c+v_c)+2b-1]\geq 0.
\end{aligned}$$

The last inequality holds due to (4.9) and the choice of $\tilde{\lambda}$. Hence (2.6) holds for $z\neq z_0$. The proof is thus complete by applying Lemma 2.1. $\square$

Case 2. $s=s_c^*(\rho)=s_3^{**}$.

For $s=s_3^{**}$, the equation $A_3(\lambda)=0$ has a double root $\lambda_*=\rho_*$. Since this case only happen when $\rho\geq\rho_*$, we may set $\rho=\lambda_*$ in (α3). Let $L_1=hv_c\lambda_*e$, $L_2=ku_c\lambda_*e$, $L_3=(2b-1)\lambda_*e$ and

$$q_*>\max\left\{(2b-1)e(\lambda_*)^{1/2},\left[4r_3L_3(\lambda_*e)\Big(\frac{7}{2e\lambda_*}\Big)^{7/2}(\beta_2+2b)\right]\Big/d_3\right\}. \tag{4.11}$$

We define

$$\overline{\phi}_1(z)=\begin{cases}u_c+L_1(-z)e^{\lambda_* z},\ z<-1/\lambda_*,\\ 1,\ z\geq-1/\lambda_*,\end{cases}\qquad \underline{\phi}_1(z)=\begin{cases}u_c-u_c\lambda_*e(-z)e^{\lambda_* z},\ z<-1/\lambda_*,\\ 0,\ z\geq-1/\lambda_*,\end{cases}$$

$$\overline{\phi}_2(z)=\begin{cases}v_c+L_2(-z)e^{\lambda_* z},\ z<-1/\lambda_*,\\ 1,\ z\geq-1/\lambda_*,\end{cases}\qquad \underline{\phi}_2(z)=\begin{cases}v_c-v_c\lambda_*e(-z)e^{\lambda_* z},\ z<-1/\lambda_*,\\ 0,\ z\geq-1/\lambda_*,\end{cases}$$

$$\overline{\phi}_3(z)=\begin{cases}L_3(-z)e^{\lambda_* z},\ z<-1/\lambda_*,\\ 2b-1,\ z\geq-1/\lambda_*,\end{cases}\qquad \underline{\phi}_3(z)=\begin{cases}L_3(-z)e^{\lambda_* z}-q_*(-z)^{1/2}e^{\lambda_* z},\ z<z_*,\\ 0,\ z\geq z_*,\end{cases}$$

where, by the choice of q_*, there is a unique $z_*<-1/\lambda_*$ such that

$$L_3(-z_*)e^{\lambda_* z_*}-q_*(-z_*)^{1/2}e^{\lambda_* z_*}=0.$$

Then we have

Lemma 4.5. *Suppose* $\max\{d_1, d_2\}/2 \le d_3 \le \min\{d_1, d_2\}$ *and* $s = s_c^*(\rho) = s_3^{**}$. *Then there exists a solution* (ϕ_1, ϕ_2, ϕ_3) *of* (1.3) *such that* $\underline{\phi}_1 \le \phi_1 \le \overline{\phi}_1$, $\underline{\phi}_2 \le \phi_2 \le \overline{\phi}_2$ *and* $\underline{\phi}_3 \le \phi_3 \le \overline{\phi}_3$ *in* $\mathbb{R}$.

Proof. Choose $\varepsilon < e[1 - \max\{h, k\} - a(2b-1)]$ and consider (4.2) with the corresponding $M(\varepsilon)$. We show that (2.1)-(2.6) hold for $\alpha(-z)$ replaced by $\alpha(-z+M)$.

First, for $z > -1/\lambda_*$, since $\overline{\phi}_1 = 1, \overline{\phi}_2 = 1, \overline{\phi}_3 = 2b-1$, (2.1)-(2.3) immediately hold. For $z < -1/\lambda_*$,

$$\overline{\phi}_1'(z) = -L_1 e^{\lambda_* z} + L_1(-z)\lambda_* e^{\lambda_* z},\ \overline{\phi}_1'' = -2L_1\lambda_* e^{\lambda_* z} + L_1(-z)\lambda_*^2 e^{\lambda_* z}.$$

Then we obtain from $2d_3\lambda_* - s = 0$ that

$$\begin{aligned}
& d_1\overline{\phi}_1'' - s\overline{\phi}_1' + r_1\overline{\phi}_1[\alpha(-z+M) - \overline{\phi}_1 - h\underline{\phi}_2 - a\underline{\phi}_3] \\
\le\ & L_1(-2d_1\lambda_* + s)e^{\lambda_* z} + L_1(-z)[d_1\lambda_*^2 - s\lambda_*]e^{\lambda_* z} + r_1\overline{\phi}_1[-L_1(-z)e^{\lambda_* z} + hv_c\lambda_* e(-z)e^{\lambda_* z}] \\
=\ & 2L_1\lambda_*(-d_1 + d_3)e^{\lambda_* z} + L_1(-z)\lambda_*^2(d_1 - 2d_3)e^{\lambda_* z} \le 0,
\end{aligned}$$

using $d_3 \in [d_1/2, d_1]$. Thus (2.1) holds for $z \ne -1/\lambda_*$. Similarly, (2.2) holds for $z \ne -1/\lambda_*$.

For $z < -1/\lambda_*$, we compute

$$\begin{aligned}
& d_3\overline{\phi}_3'' - s\overline{\phi}_3' + r_3\overline{\phi}_3[-1 + b(\overline{\phi}_1 + \overline{\phi}_2) - \overline{\phi}_3] \\
=\ & L_3(-z)e^{\lambda_* z}(d_3\lambda_*^2 - s\lambda_* + r_3\beta_2) + L_3 e^{\lambda_* z}(-2d_3\lambda_* + s) + r_3\overline{\phi}_3(-z)e^{\lambda_* z}(L_1 b + L_2 b - L_3) \\
=\ & r_3\lambda_* e\overline{\phi}_3(-z)e^{\lambda_* z}\Big[-\frac{b(2-h-k)}{1-hk} + 1\Big] \le 0,
\end{aligned}$$

using $b > 1$ and $h, k < 1$. Hence (2.3) holds for $z \ne -1/\lambda_*$.

Next, for $z < -1/\lambda_*$, using (4.1), $2\lambda_* d_3 = s$ and $d_3 \in [d_1/2, d_1]$, we compute

$$\begin{aligned}
& d_1\underline{\phi}_1'' - s\underline{\phi}_1' + r_1\underline{\phi}_1[\alpha(-z+M) - \underline{\phi}_1 - h\overline{\phi}_2 - a\overline{\phi}_3] \\
\ge\ & d_1[2u_c e\lambda_*^2 - u_c e\lambda_*^3(-z)]e^{\lambda_* z} - s[u_c e\lambda_* - u_c e\lambda_*^2(-z)]e^{\lambda_* z} \\
& \quad + r_1\underline{\phi}_1[1 - \varepsilon e^{\lambda_* z} - u_c - hv_c + u_c e\lambda_*(-z)e^{\lambda_* z} - hL_2(-z)e^{\lambda_* z} - aL_3(-z)e^{\lambda_* z}] \\
=\ & -u_c e\lambda_*^2(-z)[d_1\lambda_* - s] + u_c e\lambda_* e^{\lambda_* z}[2d_1\lambda_* - s] \\
& \quad + r_1\underline{\phi}_1 e^{\lambda_* z}[-\varepsilon + e\lambda_*(-z)((1-hk)u_c - a(2b-1))] \\
\ge\ & r_1\underline{\phi}_1 e^{\lambda_* z}[-\varepsilon + e[1 - h - a(2b-1)] > 0,
\end{aligned}$$

due to the choice of ε. Hence (2.4) holds for all $z \ne -1/\lambda_*$. The case for (2.5) can be shown similarly.

Finally, (2.6) immediately holds for $z > z_*$. For $z < z_*$, we have

$$\begin{aligned}
& d_3\underline{\phi}_3'' - s\underline{\phi}_3' + r_3\underline{\phi}_3[-1 + b(\underline{\phi}_1 + \underline{\phi}_2) - \underline{\phi}_3] \\
= & \frac{1}{4}d_3q_*(-z)^{-3/2}e^{\lambda_* z} + \underline{\phi}_3[d_3\lambda_*^2 - s\lambda_* + r_3\beta_2] + (2d_3\lambda_* - s)[-L_3 + \frac{1}{2}(-z)^{-1/2}q_*]e^{\lambda_* z} \\
& \quad + r_3\underline{\phi}_3(z)[-b(u_c + v_c)e\lambda_*(-z)e^{\lambda_* z} - L_3(-z)e^{\lambda_* z} + q_*(-z)^{1/2}e^{\lambda_* z}] \\
\geq & \frac{1}{4}d_3q_*(-z)^{-3/2}e^{\lambda_* z} - r_3e\lambda_*L_3(-z)^2e^{2\lambda_* z}[\beta_2 + 2b] \\
\geq & \frac{d_3}{4}(-z)^{-3/2}e^{\lambda_* z}\left[q_* - 4r_3L_3(\lambda_* e)(-z)^{7/2}e^{\lambda_* z}(\beta_2 + 2b)/d_3\right] \\
\geq & \frac{d_3}{4}(-z)^{-3/2}e^{\lambda_* z}\left[q_* - 4r_3L_3(\lambda_* e)\Big(\frac{7}{2e\lambda_*}\Big)^{7/2}(\beta_2 + 2b)/d_3\right] \geq 0,
\end{aligned}$$

by (4.11) and the fact that

$$(-z)^{7/2}e^{\lambda_* z} \leq \left(\frac{7}{2e\lambda_*}\right)^{7/2} \text{ for all } z \leq 0.$$

Hence (2.6) holds for $z \neq z_*$ and the lemma follows by applying Lemma 2.1. □

Hence Theorem 1.3 is proved.

A similar proof to that of Proposition 4.3, we obtain

Proposition 4.6. *Suppose $s > 0$, $b > 1$, and $0 < h, k < 1$. Then* (1.3) *has a positive solution such that* (1.7) *holds only if $s \geq s_3^{**}$.*

4.3. **Waves connecting E_3.**

Recall

$$\delta_2 = 1 - ku_p - aw_p,\ (u_p, w_p) = \left(\frac{1+a}{1+ab}, \frac{b-1}{1+ab}\right),\ s_2^{**} = 2\sqrt{d_2r_2\delta_2}.$$

Note that (u_p, w_p) satisfies

$$1 - u_p - aw_p = 0, \quad -1 + bu_p - w_p = 0.$$

Case 1. $s \geq s_p^*(\rho)$ and $s > s_2^{**}$. In this case,

$$A_4(\lambda) := d_2\lambda^2 - s\lambda + r_2\delta_2 = 0$$

has two positive roots λ_3 and λ_4 such that $\lambda_3 < \lambda_4$. Note that $\rho \geq \lambda_3$ due to $s \geq s_p^*(\rho)$. Then, recalling (1.8),

$$d_1\lambda_3^2 - s\lambda_3 + r_1a(2b-1) < 0,\ d_3\lambda_3^2 - s\lambda_3 + r_3 < 0. \tag{4.12}$$

We construct

$$\begin{cases}
\overline{\phi}_1(z) = \min\{u_p + aw_pe^{\lambda_3 z}, 1\},\ \underline{\phi}_1(z) = \max\{u_p(1 - e^{\lambda_3 z}), 0\} \\
\overline{\phi}_2(z) = \min\{e^{\lambda_3 z}, 1\},\ \underline{\phi}_2(z) = \max\{e^{\lambda_3 z} - \eta_2e^{\lambda' z}, 0\}, \\
\overline{\phi}_3(z) = \min\{w_p + Be^{\lambda_3 z}, 2b-1\},\ \underline{\phi}_3(z) = \max\{w_p(1 - e^{\lambda_3 z}), 0\},
\end{cases}$$

where $B := 2b - 1 - w_p$, $\lambda' \in (\lambda_3, \min\{2\lambda_3, \lambda_4\})$,

$$\eta_2 > \max\left\{1, \frac{r_2[1 + (akw_p + 1 + aB)]}{-A_4(\lambda')}\right\}. \tag{4.13}$$

Then we have

Lemma 4.7. *Suppose* (1.8) *holds,* $s \ge s_p^*(\rho)$ *and* $s > s_2^{**}$*. Then there exists a solution* (ϕ_1, ϕ_2, ϕ_3) *of* (1.3) *such that* $\underline{\phi}_1 \le \phi_1 \le \overline{\phi}_1$, $\underline{\phi}_2 \le \phi_2 \le \overline{\phi}_2$ *and* $\underline{\phi}_3 \le \phi_3 \le \overline{\phi}_3$ *in* $\mathbb{R}$.

Proof. Choose $\varepsilon \in (0, \min\{1, r_2\delta_2/r_1 - a(2b-1)\})$. We consider system (4.2) for the corresponding constant $M(\varepsilon)$ and show that (2.1)-(2.6) hold for $\alpha(-z)$ replaced by $\alpha(-z+M)$.

First, (2.1)-(2.3) immediately holds for $z > 0$. For $z < 0$, we compute

$$\begin{aligned}
& d_1\overline{\phi}_1'' - s\overline{\phi}_1' + r_1\overline{\phi}_1[\alpha(-z+M) - \overline{\phi}_1 - h\underline{\phi}_2 - a\underline{\phi}_3] \\
\le\ & aw_pe^{\lambda_3 z}(d_1\lambda_3^2 - s\lambda_3) + r_1(u_p + aw_pe^{\lambda_3 z})[1 - u_p - aw_pe^{\lambda_3 z} - aw_p + aw_pe^{\lambda_3 z}] \\
=\ & aw_pe^{\lambda_3 z}(d_1\lambda_3^2 - s\lambda_3) \le 0,
\end{aligned}$$

due to (4.12). Hence (2.1) holds for $z \ne 0$.

For $z < 0$, we have

$$\begin{aligned}
& d_2\overline{\phi}_2'' - s\overline{\phi}_2' + r_2\overline{\phi}_2[\alpha(-z+M) - k\underline{\phi}_1 - \overline{\phi}_2 - a\underline{\phi}_3] \\
\le\ & e^{\lambda_3 z}[d_2\lambda_3^2 - s\lambda_3 + r_2(1 - ku_p + ku_pe^{\lambda_3 z} - e^{\lambda_3 z} - aw_p + aw_pe^{\lambda_3 z})] \\
=\ & r_2e^{2\lambda_3 z}(ku_p + aw_p - 1) \le 0,
\end{aligned}$$

due to $1 - ku_p - aw_p > 0$. Hence (2.2) holds for $z \ne 0$.

For $z < 0$, we compute

$$\begin{aligned}
& d_3\overline{\phi}_3'' - s\overline{\phi}_3' + r_3\overline{\phi}_3[-1 + b(\overline{\phi}_1 + \overline{\phi}_2) - \overline{\phi}_3] \\
=\ & Be^{\lambda_3 z}(d_3\lambda_3^2 - s\lambda_3) + r_3\overline{\phi}_3[-1 + bu_p - w_p + e^{\lambda_3 z}((1+ab)w_p - (b-1))] \\
=\ & Be^{\lambda_3 z}(d_3\lambda_3^2 - s\lambda_3) \le 0,
\end{aligned}$$

by (4.12). Hence (2.3) holds for $z \ne 0$.

Next, it is clear that (2.4) holds for $z > 0$. For $z < 0$, we compute, using $u_p + aw_p = 1$, $h < 1$, $d_1 \le d_2$ and $A_4(\lambda_3) = 0$,

$$\begin{aligned}
& d_1\underline{\phi}_1'' - s\underline{\phi}_1' + r_1\underline{\phi}_1[\alpha(-z+M) - \underline{\phi}_1 - h\overline{\phi}_2 - a\overline{\phi}_3] \\
\ge\ & -u_pe^{\lambda_3 z}(d_1\lambda_3^2 - s\lambda_3) + r_1\underline{\phi}_1[1 - \varepsilon e^{\rho z} - u_p(1 - e^{\lambda_3 z}) - he^{\lambda_3 z} - a(w_p + Be^{\lambda_3 z})] \\
\ge\ & -u_pe^{\lambda_3 z}(d_2\lambda_3^2 - s\lambda_3) + r_1u_p(1 - e^{\lambda_3 z})e^{\lambda_3 z}[-\varepsilon e^{(\rho-\lambda_3)z} - a(2b-1)] \\
\ge\ & u_pe^{\lambda_3 z}[r_2\delta_2 - r_1a(2b-1) - r_1\varepsilon] \ge 0,
\end{aligned}$$

by the choice of ε. Hence (2.4) holds for $z \ne 0$.

Now, since $\eta_2 > 1$, there is $z_2 < 0$ such that

$$\underline{\phi}_2(z) = \begin{cases} 0, & z \ge z_2, \\ e^{\lambda_3 z} - \eta_2 e^{\lambda' z}, & z < z_2. \end{cases}$$

For $z < z_2$, we compute, using (4.1),

$$\begin{aligned}
& d_2\underline{\phi}_2'' - s\underline{\phi}_2' + r_2\underline{\phi}_2[\alpha(-z+M) - k\overline{\phi}_1 - \underline{\phi}_2 - a\overline{\phi}_3] \\
\geq\ & e^{\lambda_3 z}(d_2\lambda_3^2 - s\lambda_3 + r_2\delta_2) - \eta_2 e^{\lambda' z}\{d_2(\lambda')^2 - s\lambda' + r_2\delta_2\} \\
& \quad + r_2\underline{\phi}_2(z)[-\varepsilon e^{\rho z} - akw_p e^{\lambda_3 z} - e^{\lambda_3 z} + \eta_2 e^{\lambda' z} - aBe^{\lambda_3 z}] \\
\geq\ & -\eta_2 e^{\lambda' z} A_4(\lambda') + r_2 e^{2\lambda_3 z}[-\varepsilon e^{(\rho-\lambda_3)z} - akw_p - 1 - aB] \\
\geq\ & e^{\lambda' z}[-\eta_2 A_4(\lambda') - r_2 e^{(2\lambda_3 - \lambda')z}(1 + (akw_p + 1 + aB))] \geq 0,
\end{aligned}$$

by the choices of λ' and η_2. Hence (2.5) holds for $z \neq z_2$.

Finally, for $z < 0$, we have

$$\begin{aligned}
& d_3\underline{\phi}_3'' - s\underline{\phi}_3' + r_3\underline{\phi}_3[-1 + b(\underline{\phi}_1 + \underline{\phi}_2) - \underline{\phi}_3] \\
\geq\ & -w_p e^{\lambda_3 z}(d_3\lambda_3^2 - s\lambda_3) + r_3\underline{\phi}_3[-1 + bu_p(1 - e^{\lambda_3 z}) - w_p(1 - e^{\lambda_3 z})] \\
\geq\ & -w_p e^{\lambda_3 z}(d_2\lambda_3^2 - s\lambda_3) + r_3\underline{\phi}_3 e^{\lambda_3 z}(-bu_p + w_p) \\
\geq\ & -w_p e^{\lambda_3 z}[(d_2\lambda_3^2 - s\lambda_3) + r_3] > 0,
\end{aligned}$$

due to (4.12). Hence (2.6) holds for $z \neq 0$.

Thus the lemma is proved by applying Lemma 2.1. □

Case 2. $s = s_p^*(\rho) = s_2^{**}$. This case happens only when $\rho \geq \rho^*$ and we thus set $\rho = \rho^*$ in (α3). Since $s = s_2^{**}$, the equation $A_4(\lambda) = 0$ has a double root $\lambda_5 = \rho^* > 0$. Let $L_* = \lambda_5 e^2/2$, $B = 2b - 1 - w_p$, and η_4 satisfies

$$\eta_4 > \max\left\{4r_2 L_*\Big[\Big(\frac{5}{2e\lambda_5}\Big)^{5/2} + L_*(1 + a(2b-1))\Big(\frac{7}{2e\lambda_5}\Big)^{7/2}\Big]/d_2, L_*\sqrt{\frac{2}{\lambda_5}}\right\}. \tag{4.14}$$

Note that, from the choice η_4, there exists $z_4 < -2/\lambda_5$ such that

$$L_*(-z_4)e^{\lambda_5 z_4} - \eta_4(-z_4)^{1/2}e^{\lambda_5 z_4} = 0.$$

We define

$$\overline{\phi}_1(z) = \begin{cases} u_p + L_* a w_p(-z)e^{\lambda_5 z}, \ z < -2/\lambda_5, \\ 1, \ z \geq -2/\lambda_5, \end{cases} \quad \underline{\phi}_1(z) = \begin{cases} u_p(1 - L_*(-z)e^{\lambda_5 z}), \ z < -2/\lambda_5, \\ 0, \ z \geq -2/\lambda_5, \end{cases}$$

$$\overline{\phi}_2(z) = \begin{cases} L_*(-z)e^{\lambda_5 z}, \ z < -2/\lambda_5, \\ 1, \ z \geq -2/\lambda_5, \end{cases} \quad \underline{\phi}_2(z) = \begin{cases} L_*(-z)e^{\lambda_5 z} - \eta_4(-z)^{1/2}e^{\lambda_5 z}, \ z < z_4, \\ 0, \ z \geq z_4, \end{cases}$$

$$\overline{\phi}_3(z) = \begin{cases} w_p + L_* B(-z)e^{\lambda_5 z}, \ z < -2/\lambda_5, \\ 2b - 1, \ z \geq -2/\lambda_5, \end{cases} \quad \underline{\phi}_3(z) = \begin{cases} w_p(1 - L_*(-z)e^{\lambda_5 z}), \ z < -2/\lambda_5, \\ 0, \ z \geq -2/\lambda_5. \end{cases}$$

Then we have

Lemma 4.8. *Suppose* (1.8) *and* (1.10) *hold. If* $s = s_p^*(\rho) = s_2^{**}$, *then there exists a solution* (ϕ_1, ϕ_2, ϕ_3) *of* (1.3) *such that* $\underline{\phi}_1 \leq \phi_1 \leq \overline{\phi}_1$, $\underline{\phi}_2 \leq \phi_2 \leq \overline{\phi}_2$ *and* $\underline{\phi}_3 \leq \phi_3 \leq \overline{\phi}_3$ *in* $\mathbb{R}$.

Proof. Choose $\varepsilon > 0$ such that

$$\varepsilon < \min\{1, e^2[r_2\delta_2/r_1 - a(2b-1)]\}$$

and consider system (4.2) for the corresponding constant $M(\varepsilon)$. We show that (2.1)-(2.6) hold with $\alpha(-z)$ replaced by $\alpha(-z+M)$.

First, for $z>-2/\lambda_5$, (2.1)-(2.3) hold, since $\overline{\phi}_1=1, \overline{\phi}_2=1, \overline{\phi}_3=2b-1$. For $z<-2/\lambda_5$, we have

$$\begin{aligned}
& d_1\overline{\phi}_1''-s\overline{\phi}_1'+r_1\overline{\phi}_1[\alpha(-z+M)-\overline{\phi}_1-h\underline{\phi}_2-a\underline{\phi}_3] \\
\leq\ & L_*aw_p(-2d_1\lambda_5+s)e^{\lambda_5 z}+L_*aw_p(-z)(d_1\lambda_5^2-s\lambda_5)e^{\lambda_5 z}+r_1\overline{\phi}_1[1-\overline{\phi}_1-a\underline{\phi}_3] \\
=\ & -r_2\delta_2L_*aw_p(-z)e^{\lambda_5 z}\leq 0,
\end{aligned}$$

using $d_1=d_2$ and $-2d_2\lambda_5+s=0$. Hence (2.1) holds for $z\neq-2/\lambda_5$.

For $z<-2/\lambda_5$, we compute

$$\begin{aligned}
& d_2\overline{\phi}_2''-s\overline{\phi}_2'+r_2\overline{\phi}_2[\alpha(-z+M)-k\underline{\phi}_1-\overline{\phi}_2-a\underline{\phi}_3] \\
\leq\ & L_*(-2d_2\lambda_1+s)e^{\lambda_5 z}+L_*(-z)[d_2\lambda_5^2-s\lambda_1+r_2\delta_2]e^{\lambda_5 z} \\
& +r_2\overline{\phi}_2[-L_*(-z)e^{\lambda_5 z}+ku_pL_*(-z)e^{\lambda_5 z}+aw_pL_*(-z)e^{\lambda_5 z}] \\
\leq\ & -r_2L_*(-z)e^{\lambda_5 z}\overline{\phi}_2(1-ku_p-aw_p)\leq 0,
\end{aligned}$$

due to $1-ku_p-aw_p>0$. Hence (2.2) holds for $z\neq-2/\lambda_5$.

For $z<-2/\lambda_5$, we have

$$\begin{aligned}
& d_3\overline{\phi}_3''-s\overline{\phi}_3'+r_3\overline{\phi}_3[-1+b(\overline{\phi}_1+\overline{\phi}_2)-\overline{\phi}_3] \\
=\ & L_*B(-2d_3\lambda_5+s)e^{\lambda_5 z}+L_*B(-z)(d_3\lambda_5^2-s\lambda_5)e^{\lambda_5 z} \\
& +r_3\overline{\phi}_3[-1+b(u_p+L_*aw_p(-z)e^{\lambda_5 z}+L_*(-z)e^{\lambda_5 z})-w_p-L_*B(-z)e^{\lambda_5 z}] \\
=\ & L_*B(-2d_3\lambda_5+s)e^{\lambda_5 z}+L_*B(-z)(d_3\lambda_5^2-s\lambda_5)e^{\lambda_5 z} \\
& +r_3L_*(-z)e^{\lambda_5 z}\overline{\phi}_3[(ab+1)w_p-(b-1)] \\
=\ & L_*B(-2d_3\lambda_5+s)e^{\lambda_5 z}+L_*B\lambda_5^2(-z)(d_3-2d_2)e^{\lambda_5 z}\leq 0,
\end{aligned}$$

using $(ab+1)w_p=b-1$, $s=2d_2\lambda_5$ and $d_2\leq d_3<2d_2$. Hence (2.3) holds for $z\neq-2/\lambda_5$.

Next, it is trivial that (2.4) holds for $z>-2/\lambda_5$. Recall $d_1=d_2$. For $z<-2/\lambda_5$, we compute, using $u_p+aw_p=1$ and $h<1$,

$$\begin{aligned}
& d_1\underline{\phi}_1''-s\underline{\phi}_1'+r_1\underline{\phi}_1[\alpha(-z+M)-\underline{\phi}_1-h\overline{\phi}_2-a\overline{\phi}_3] \\
\geq\ & -u_pL_*e^{\lambda_5 z}(-2d_1\lambda_5+s)-u_pL_*(-z)e^{\lambda_5 z}(d_1\lambda_5^2-s\lambda_5) \\
& +r_1\underline{\phi}_1\{1-\varepsilon e^{\lambda_5 z}-u_p[1-L_*(-z)e^{\lambda_5 z}]-hL_*(-z)e^{\lambda_5 z}-a[w_p+L_*B(-z)e^{\lambda_5 z}]\} \\
\geq\ & -u_pL_*(-z)e^{\lambda_5 z}(d_2\lambda_5^2-s\lambda_5)-r_1\underline{\phi}_1\left\{a(2b-1)L_*(-z)e^{\lambda_5 z}+\varepsilon e^{\lambda_5 z}\right\} \\
\geq\ & u_pL_*(-z)e^{\lambda_5 z}(r_2\delta_2)-r_1u_p\left\{a(2b-1)L_*(-z)e^{\lambda_5 z}+\varepsilon e^{\lambda_5 z}\right\} \\
=\ & u_pL_*(-z)e^{\lambda_5 z}\left\{r_2\delta_2-r_1a(2b-1)-r_1\varepsilon/[L_*(-z)]\right\} \\
\geq\ & u_pL_*(-z)e^{\lambda_5 z}[r_2\delta_2-r_1a(2b-1)-r_1\varepsilon/e^2]\geq 0,
\end{aligned}$$

using $L_*(-z)\geq e^2$ for $z<-2/\lambda_5$, and the choice of ε. Hence (2.4) holds for $z\neq-2/\lambda_5$.

For (2.5), we only need to consider $z < z_4$. For $z < z_4$, since $z_4 < -2/\lambda_5$ and $k < 1$, we compute

$$\begin{aligned}
& d_2\underline{\phi}_2'' - s\underline{\phi}_2' + r_2\underline{\phi}_2[\alpha(-z+M) - k\overline{\phi}_1 - \underline{\phi}_2 - a\overline{\phi}_3] \\
\geq\ & \frac{1}{4}\eta_4 d_2(-z)^{-3/2}e^{\lambda_5 z} + \underline{\phi}_2(d_2\lambda_5^2 - s\lambda_5 + r_2\delta_2) + (2d_2\lambda_5 - s)[-L_* + \frac{1}{2}(-z)^{-1/2}\eta_4]e^{\lambda_5 z} \\
& \quad + r_2\underline{\phi}_2(z)[-\varepsilon e^{\lambda_5 z} - akL_* w_p(-z)e^{\lambda_5 z} - L_*(-z)e^{\lambda_5 z} + \eta_4(-z)^{1/2}e^{\lambda_5 z} - aL_* B(-z)e^{\lambda_5 z}] \\
\geq\ & \frac{1}{4}\eta_4 d_2(-z)^{-3/2}e^{\lambda_5 z} + r_2 L_*(-z)e^{2\lambda_5 z}\{-1 - L_*[1 + a(2b-1)](-z)\} \\
\geq\ & \frac{1}{4}(-z)^{-3/2}e^{\lambda_5 z}\Big\{d_2\eta_4 - 4r_2L_*\Big[(-z)^{5/2}e^{\lambda_5 z} + L_*(1 + a(2b-1))(-z)^{7/2}e^{\lambda_5 z}\Big]\Big\} \\
\geq\ & \frac{1}{4}(-z)^{-3/2}e^{\lambda_5 z}\Big\{\eta_4 - 4r_2L_*\Big[\Big(\frac{5}{2e\lambda_5}\Big)^{5/2} + L_*(1 + a(2b-1))\Big(\frac{7}{2e\lambda_5}\Big)^{7/2}\Big]\Big\} \geq 0,
\end{aligned}$$

due to the choice of η_4 in (4.14). Hence (2.5) holds for $z \neq z_4$.

It remains to consider (2.6). It is trivial for $z > -2/\lambda_5$. For $z < -2/\lambda_5$, we have

$$\begin{aligned}
& d_3\underline{\phi}_3'' - s\underline{\phi}_3' + r_3\underline{\phi}_3[-1 + b(\underline{\phi}_1 + \underline{\phi}_2) - \underline{\phi}_3] \\
\geq\ & -w_pL_*e^{\lambda_5 z}(-2d_3\lambda_5 + s) - w_pL_*(-z)e^{\lambda_5 z}(d_3\lambda_5^2 - s\lambda_5) \\
& \quad + r_3\underline{\phi}_3[-1 + bu_p(1 - L_*(-z)e^{\lambda_5 z}) - w_p(1 - L_*(-z)e^{\lambda_5 z})] \\
=\ & -w_pL_*(-z)e^{\lambda_5 z}(d_3\lambda_5^2 - s\lambda_5) + r_3\underline{\phi}_3 L_*(-z)e^{\lambda_5 z}(-bu_p + w_p) \\
\geq\ & -w_pL_*(-z)e^{\lambda_5 z}(d_3\lambda_5^2 - s\lambda_5) - r_3w_pL_*(-z)e^{\lambda_5 z}(bu_p) \\
\geq\ & -w_pL_*(-z)e^{\lambda_5 z}(d_3\lambda_5^2 - s\lambda_5 + r_3bu_p) \\
\geq\ & -w_pL_*(-z)e^{\lambda_5 z}[r_2\delta_2(d_3/d_2 - 2) + r_3bu_p] \geq 0,
\end{aligned}$$

by using $-1 + bu_p - w_p = 0$, $\lambda_5 = \sqrt{r_2\delta_2/d_2}$ and (1.10). Hence (2.6) holds for $z \neq -2/\lambda_5$. The proof is thus complete. ☐

Therefore, Theorem 1.4 is proved. Moreover, we also obtain

Proposition 4.9. *Suppose $s > 0$, $b > 1$ and $k < 1$. Then* (1.3) *has a positive solution such that* (1.9) *holds only if $s \geq s_2^{**}$.*

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