UNIQUENESS AND STABILITY OF FORCED WAVES FOR THE FISHER-KPP EQUATION IN A SHIFTING ENVIRONMENT

JONG-SHENQ GUO, KAREN GUO, AND MASAHIKO SHIMOJO

ABSTRACT. In this paper, we investigate the existence, uniqueness and stability of forced waves for the Fisher-KPP equation in a shifting environment without imposing the monotonicity condition on the shifting intrinsic growth term. First, the existence of forced waves for some range of shifting speeds is proved. Then we prove the uniqueness of saturation forced waves. Moreover, a new method is introduced to derive the non-existence of forced waves. Finally, we derive the stability of forced waves under certain perturbation of a class of initial data.

1. INTRODUCTION

Consider the Fisher-KPP equation [14, 20] in a shifting environment

$$u_t = u_{xx} + u[h(x - st) - u],\ x \in \mathbb{R},\ t > 0, \tag{1.1}$$

in which u is the density of a species and the intrinsic growth rate is given by a function h of spatial-temporal variable $x - st$ with a real constant s representing the shifting speed of the environment. In this paper, we shall always assume the following conditions on h:

(h1) h is bounded and continuous in $\mathbb{R}$ such that $h(-\infty) = 1$;
(h2) $h(z) < 0$ for all $z \geq K$ for some positive constant K.

Therefore, the favorable habitat of species u, $\{(x,t) \mid h(x - st) > 0\}$, can be expanding or shrinking in time in the horizontal axis depending on the sign of s. Notice that the monotonicity condition is not imposed on h.

A forced wave is a traveling wave solution of (1.1) in the form

$$u(x,t) = \phi(z),\ z := x - st,$$

for some function ϕ (the wave profile). Specifically, the wave speed s is the same as the environmental shifting speed. Note that ϕ satisfies

$$\phi''(z) + s\phi'(z) + \phi(z)[h(z) - \phi(z)] = 0,\ z \in \mathbb{R}. \tag{1.2}$$

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Throughout this paper, we shall focus on the forced waves $u(x,t) = \phi(x-st)$ of (1.1) satisfying (1.2) and

$$\phi(\infty) = 0 < \phi(z) < 1 = \phi(-\infty),\ \forall\, z \in \mathbb{R}. \tag{1.3}$$

In this setting, there are two different types of forced waves as follows. When $s < 0$, since $u(x,t) = \phi(x-st) \to 0$ as $t \to \infty$ for all $x \in \mathbb{R}$, it is the *extinction* type. While, when $s > 0$, it is the *saturation* type, since $u(x,t) \to 1$ as $t \to \infty$ for all $x \in \mathbb{R}$.

For the study of forced waves, we refer the reader to, e.g., [6, 7, 12, 19, 2, 16] for the scalar equation, [1, 25, 11, 23, 9] for two-species systems and [10, 13, 15] for a three-species system. In particular, under the extra monotonicity condition on h, the existence of extinction forced waves of (1.1) for any $s < 0$ was derived in [12] for $h \in C^1(\mathbb{R})$ and in [19] for $h \in C^0(\mathbb{R})$; while the saturation forced waves of (1.1) exists if any only if $s \in (0,2)$ was proved in [12] for $h \in C^1(\mathbb{R})$. Note that, without the monotonicity condition on h, under the assumptions (h1)-(h2) the existence of forced waves was proved in [16] for $s \in (0,1)$ and in [13] for $s < 0$ with an extra exponential decay condition for $h(-\infty)$.

For the uniqueness and stability of the extinction forced waves ($s < 0$), we refer the reader to [24] for both classical diffusion and nonlocal dispersal cases under monotonicity condition on h. For the existence, multiplicity and attractivity of saturation forced waves ($s > 0$), we refer the reader to [2] for a more general nonlinearity of KPP type which includes (1.1) as a special case. In [2], the nonlinearity is assumed to be C^1 and the monotonicity condition on the shifting term h is also imposed. By a classification of the generalized eigenvalue $\lambda_1(s)$ of the associated elliptic operator $\phi'' + s\phi' + h(z)\phi$ in $\mathbb{R}$, they [2] obtained a complete description of the global dynamics of (1.2). For the study of the generalized eigenvalues in unbounded domains, we also refer the reader to, e.g., [4, 5, 3, 8, 12].

However, biologically the intrinsic growth rate h of the species is not necessarily monotone. The main purpose of this paper is to remove the monotonicity condition on h imposed in, e.g., [1, 12, 19, 2, 24] for the existence, uniqueness and stability of forced waves of (1.1).

In this paper, we first derive the existence of forced waves for any $s \in (-2,2)$ without assuming the monotonicity condition on h. More precisely, we have

Theorem 1.1. *Under the assumptions* (h1)-(h2), (1.2)-(1.3) *has a solution for any* $s \in (-2,2)$.

Next, for the uniqueness and non-existence of forced waves we have

Theorem 1.2. *Assume, in addition to* (h1)-(h2), *that*

$$h(\infty) \text{ exists and } h(\infty) \in (-\infty, 0). \tag{1.4}$$

Then there exists at most one solution ϕ *of* (1.2)-(1.3) *for any* $s > 0$. *Moreover, no solution of* (1.2)-(1.3) *exists when* $s \geq 2$, *if we assume further that* $h(z) \leq h(-\infty) = 1$ *for all* $z \in \mathbb{R}$.

Finally, we study the stability of forced waves. Let u be a positive solution of (1.1) for a given s. Then, using the moving coordinate $z = x - st$, $u = u(z,t)$ satisfies

$$u_t = u_{zz} + su_z + u[h(z) - u],\ z \in \mathbb{R},\ t > 0. \tag{1.5}$$

Recall the classical Lyapunov function $g(y) := y - 1 - \ln(y)$, $y > 0$. Note that $g(y) \geq 0$ for all $y > 0$ and $g(y) = 0$ if and only if $y = 1$. Note also that ϕ is a stationary solution of (1.5). Then we have the following stability theorem for extinction forced waves.

Theorem 1.3. *Let* (h1)-(h2) *be enforced. Let ϕ be a solution of* (1.2)-(1.3) *for some $s < 0$ and let u be a solution of* (1.5) *with a positive initial data u_0 at $t = 0$. If $\exp(sz/2)\phi g(u_0/\phi) \in L^2(\mathbb{R})$, then $u(z,t) \to \phi(z)$ as $t \to \infty$ locally uniformly for $z \in \mathbb{R}$ and uniformly for $z \in (-\infty, l]$ for any $l \in \mathbb{R}$.*

For the stability of saturation forced waves, we have

Theorem 1.4. *Assume, in addition to* (h1)-(h2)*, that* (1.4) *is enforced. Let ϕ be a solution of* (1.2)-(1.3) *for some $s \in (0,2)$ and let u be a solution of* (1.5) *with a positive initial data u_0 at $t = 0$. Suppose that there exist a constant $\varepsilon \in (0,1)$ and a sufficiently large $R > 0$ such that $u_0(z) \geq \varepsilon$ for all $z \leq -R$. Then $u(z,t) \to \phi(z)$ as $t \to \infty$ for every $z \in \mathbb{R}$.*

Note that the convergence in Theorem 1.4 is only locally uniformly in $\mathbb{R}$.

Remark 1.1. For the reader's convenience, we quote some results in [2] for (1.2) with $s \geq 0$ and h is a bounded monotone C^1 function in $\mathbb{R}$ as follows. Set $\alpha := h(-\infty) = 1$ and $\beta := h(+\infty)$. We only concern the case when $\beta < 0$. Recall from [12] that $\lambda_1(s) = -\alpha + s^2/4$. Hence $s \in [0,2)$ implies that $\lambda_1(s) < 0$. Then [2, Theorem 1.3 (i)] tells us that (1.2) has a unique positive bounded solution ϕ satisfying $\phi(+\infty) = 0$. On the other hand, [2, Theorem 1.4] gives the non-existence of positive bounded solutions to (1.2) when $s \geq 2$. Lastly, [2, Theorem 1.5 (ii)] provides the convergence (locally uniformly in $\mathbb{R}$) to the positive bounded solution of (1.2) as $t \to \infty$ for any solution of (1.5) with a positive bounded continuous initial data at $t = 0$, when $s \in [0,2)$.

The rest of this paper is organized as follows. A proof of Theorem 1.1 is given in §2. As remarked in [16], the key of the proof of Theorem 1.1 is to construct a suitable sub-solution of (1.2). A simple sub-solution is constructed here. This answers a question left in [16]. Then we provide a proof of Theorem 1.2 in §3. The proof relies on a key lemma (Lemma 3.1) on the uniform boundedness of ϕ'/ϕ in $\mathbb{R}$ for any solution ϕ of (1.2)-(1.3). Also, using the limit of the ratio ϕ'/ϕ at ∞ (see (3.3) below), we provide a new proof of the non-existence of forced waves for $s \geq 2$ without monotonicity condition on h. Finally, in §4, we first prove Theorem 1.3 by a spectrum analysis. Then a proof of Theorem 1.4 is given. The proof of Theorem 1.4 is different from that of Theorem 1.3 and it relies on the uniqueness of forced waves (Theorem 1.2).

2. Existence of forced waves

In this section, we provide a proof of the existence of forced waves for $s \in (-2,2)$.

Proof of Theorem 1.1. Given $s \in (-2,2)$. As remarked in [16], all we need is to construct a nontrivial sub-solution $\underline{\phi}$ to (1.2) such that $\liminf_{z \to -\infty} \underline{\phi}(z) > 0$.

First, since $|s| < 2$, there is a sufficiently small $\varepsilon \in (0,1)$ such that $1 - 2\varepsilon > s^2/4$. It then follows from condition (h1) that there exists $\bar{z} \in \mathbb{R}$ such that

$$h(z) \geq 1 - \varepsilon \ \text{ for all } z \leq \bar{z}. \tag{2.1}$$

Motivated by [1], we introduce

$$\hat{\phi}(z) = e^{-\frac{s}{2}z} \cos(\omega z),\ z \in (-\pi/(2\omega), \pi/(2\omega)), \quad \omega := \sqrt{(1-2\varepsilon) - s^2/4} > 0.$$

Note that $\hat{\phi}$ satisfies

$$\hat{\phi}'' + s\hat{\phi}' + (1 - 2\varepsilon)\hat{\phi} = 0 \ \text{ in } (-\pi/(2\omega), \pi/(2\omega)). \tag{2.2}$$

Also, since

$$\hat{\phi}'(z) = -e^{-\frac{s}{2}z}\left[\frac{s}{2}\cos(\omega z) + \omega \sin(\omega z)\right] = 0,\ z \in (-\pi/(2\omega), \pi/(2\omega)),$$

if and only if $z = -\gamma/\omega := \hat{z}$, where $\gamma \in (-\pi/2, \pi/2)$ is uniquely determined by

$$\sin\gamma = \frac{s/2}{\sqrt{1-2\varepsilon}},$$

we have

$$\|\hat{\phi}\|_{L^\infty(-\frac{\pi}{2\omega}, \frac{\pi}{2\omega})} = \hat{\phi}(\hat{z}) = e^{\frac{s\gamma}{2\omega}} \cos(-\gamma).$$

Next, we choose a constant z_0 such that $z_0 + \frac{\pi}{2\omega} \leq \bar{z}$, and define

$$\underline{\phi}(z) = \begin{cases} \varepsilon, & z \leq z_0 + \hat{z}, \\ \varepsilon\hat{\phi}(z - z_0)/\hat{\phi}(\hat{z}), & z_0 + \hat{z} < z < z_0 + \frac{\pi}{2\omega}, \\ 0, & z \geq z_0 + \frac{\pi}{2\omega}. \end{cases}$$

Then one can check that $\underline{\phi}$ is continuous in $\mathbb{R}$ and, using (2.1), $\underline{\phi} \leq \varepsilon$, and (2.2), $\underline{\phi}$ satisfies

$$\begin{aligned} & \underline{\phi}''(z) + s\underline{\phi}'(z) + \underline{\phi}(z)[h(z) - \underline{\phi}(z)] \\ \geq\ & \underline{\phi}''(z) + s\underline{\phi}'(z) + \underline{\phi}(z)[1 - \varepsilon - \underline{\phi}(z)] \\ \geq\ & \underline{\phi}''(z) + s\underline{\phi}'(z) + \underline{\phi}(z)(1 - 2\varepsilon) \geq 0,\ \forall\, z \in \mathbb{R} \setminus \{z_0 + \hat{z}, z_0 + \pi/(2\omega)\}. \end{aligned}$$

Note that

$$\underline{\phi}'(z_0 + \hat{z}) = 0, \ \lim_{z \to (z_0 + \pi/(2\omega))^-} \underline{\phi}'(z) < 0 = \lim_{z \to (z_0 + \pi/(2\omega))^+} \underline{\phi}'(z).$$

This shows that $\underline{\phi}$ is a sub-solution of (1.2). By the results of [16], we conclude that there exists a solution ϕ to (1.2) such that $\phi(\infty) = 0$ and $\phi(-\infty) = 1$. The proof is complete. □

3. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. First, we prepare the following lemma which plays an essential role in the latter proofs.

Lemma 3.1. *Assume, in addition to* (h1)-(h2)*, that* (1.4) *holds. Let ϕ be a solution of* (1.2)-(1.3) *for a given $s \in \mathbb{R}$. Then the function $\phi'(z)/\phi(z)$ is uniformly bounded in $\mathbb{R}$.*

Proof. To prove this lemma, we first set $\psi(z) := \phi'(z)/\phi(z)$. Then it is easy to check that ψ satisfies

$$\psi'(z) + \psi^2(z) + s\psi(z) + [h(z) - \phi(z)] = 0,\ z \in \mathbb{R}. \tag{3.1}$$

Recall that

$$h(\infty) < 0,\ h(-\infty) = 1,\ \phi(-\infty) = 1,\ \phi(\infty) = 0,\ 0 < \phi(z) < 1,\ \forall\, z \in \mathbb{R}.$$

From (1.2) and (h2) it follows that $\phi''(z) > 0$ for any point $z \in [K, \infty)$ such that $\phi'(z) = 0$. On the other hand, it follows from

$$\int_K^\infty \phi'(z)dz = -\phi(K) < 0$$

that $\phi'(z_1) < 0$ for some $z_1 > K$. Similarly, for a given z_n with $n \geq 1$, it follows from

$$\int_{z_n+1}^\infty \phi'(z)dz = -\phi(z_n + 1) < 0$$

that $\phi'(z_{n+1}) < 0$ for some $z_{n+1} > z_n + 1$. By induction, we can find a strictly increasing sequence $\{z_n\}$ such that $\phi'(z_n) < 0$ for each $n \geq 1$ and $z_n \to \infty$ as $n \to \infty$.

We claim that $\phi'(z) < 0$ for all $z \geq z_1$. For contradiction, we assume that $\phi'(y_0) \geq 0$ for some $y_0 \in (z_n, z_{n+1})$ for some $n \geq 1$. Then there must be a point $y_1 \in (y_0, z_{n+1})$ such that $\phi'(y_1) > 0$, due to $\phi''(y_0) > 0$. It follows from the continuity of ϕ' that $\phi'(y_2) = 0$ for some $y_2 \in (y_1, z_{n+1})$. Without loss of generality, we may assume that $\phi'(z) < 0$ for all $z \in (y_2, z_{n+1})$, by choosing

$$y_2 = \max\{z \in [y_1, z_{n+1}] \mid \phi'(z) \geq 0\}.$$

But, this contradicts that $\phi''(y_2) > 0$. Hence we conclude that $\phi'(z) < 0$ for all $z \geq z_1$. As a by-product, ψ is bounded above by 0 over $[z_1, \infty)$.

To obtain a lower bound for ψ, we write (3.1) as

$$\psi'(z) = -\psi^2(z) - s\psi(z) + \phi(z) - h(z) \leq -\psi^2(z) - s\psi(z) + \alpha,$$

where $\alpha := \sup_{z\in\mathbb{R}}[\phi(z) - h(z)]$. Note that $\alpha \in (0, \infty)$. Then we choose $M \gg 1$ such that

$$-sy + \alpha \leq y^2/2,\ \forall\, y \leq -M.$$

We claim that $\psi(z) \geq -M$ for all $z \in [K, \infty)$. For contradiction, we assume that there is $z_0 \in [K, \infty)$ such that $\psi(z_0) < -M$. Then ψ is decreasing for $z > z_0$ and $\psi(z) < -M$ for all $z \geq z_0$. It follows that ψ satisfies

$$\psi'(z) \leq -\psi^2(z)/2,\ \forall\, z \geq z_0.$$

This implies that $\psi(z)$ tends to $-\infty$ as $z \uparrow z_1$ for some $z_1 \in (z_0, \infty)$. This is a contradiction, since $\psi(z)$ is defined for all $z \in \mathbb{R}$. Hence we have proved that ψ is bounded below on $[K, \infty)$. We conclude that ψ is bounded on $[K, \infty)$.

Now, let λ_0 be the unique negative root of $\lambda^2 + s\lambda + h(\infty) = 0$, i.e.,

$$\lambda_0^2 + s\lambda_0 + h(\infty) = 0,\ \lambda_0 = \frac{-s - \sqrt{s^2 - 4h(\infty)}}{2}. \tag{3.2}$$

Using $h(z)-\phi(z)\to h(\infty)<0$ as $z\to\infty$, by (1.4), we claim that

$$\lim_{z\to\infty}\psi(z) \text{ exists and } \lim_{z\to\infty}\psi(z)=\lambda_0. \tag{3.3}$$

Since $\psi<0$ in $[z_1,\infty)$, it is clear (3.3) holds when ψ is monotone ultimately at $z=\infty$. When ψ is oscillatory near $z=\infty$, by (3.1) any sequence $\{z_n\}$ of critical points of ψ satisfy

$$\psi^2(z_n)+s\psi(z_n)+[h(z_n)-\phi(z_n)]=0,\ \forall\, n.$$

Hence the quantities

$$\psi_-:=\liminf_{z\to\infty}\psi(z),\ \psi_+:=\limsup_{z\to\infty}\psi(z)$$

are solutions of $\lambda^2+s\lambda+h(\infty)=0$. Since $\psi_\pm\le 0$, by (3.2) we must have $\psi_-=\lambda_0=\psi_+$. Thus (3.3) is proved.

To proceed further, we let

$$\zeta(z):=\phi(-z),\ \theta(z):=\frac{\zeta'(z)}{\zeta(z)}.$$

Then θ satisfies

$$\theta'(z)=-\theta^2(z)+s\theta(z)-h(-z)+\zeta(z)\le-\theta^2(z)+s\theta(z)+\alpha, \tag{3.4}$$

where $\alpha=\sup_{z\in\mathbb{R}}[\zeta(z)-h(-z)]=\sup_{z\in\mathbb{R}}[\phi(-z)-h(-z)]$ as defined before. Note that

$$\zeta(-\infty)=0,\ \theta(-\infty)=-\lambda_0>0.$$

Define

$$\mu_\pm:=\frac{s\pm\sqrt{s^2+4\alpha}}{2},$$

which are two roots of $\mu^2-s\mu-\alpha=0$. Since $\alpha\ge-h(\infty)$, it holds $-\lambda_0\le\mu_+$. By comparison, $\theta(z)\le\mu_+$ for all $z\in\mathbb{R}$, since $y(z)\equiv\mu_+$ is a solution of

$$y'(z)=-y^2(z)+sy(z)+\alpha=[y(z)-\mu_-]\cdot[\mu_+-y(z)].$$

On the other hand, there is a constant $L\gg 1$ such that

$$sy+\alpha\le y^2/2,\ \forall\, y\le -L.$$

Hence, by a similar argument as before, we can deduce that $\theta(z)\ge -L$ for all $z\in\mathbb{R}$. We conclude that θ (and so ϕ'/ϕ) is bounded in $\mathbb{R}$. Thus, the lemma is proved. $\square$

With Lemma 3.1, we are ready to prove Theorem 1.2 as follows.

Proof of Uniqueness. Let ϕ_1 and ϕ_2 be two solutions to (1.2)-(1.3) for some $s>0$. Define $\Phi_i=e^{\frac{s}{2}z}\phi_i$ for $i=1,2$. Then Φ_i satisfies

$$\Phi_i''+\Big(h-\frac{s^2}{4}\Big)\Phi_i-e^{-\frac{s}{2}z}\Phi_i^2=0. \tag{3.5}$$

Multiplying (3.5) with $i=1$ by Φ_2 and (3.5) with $i=2$ by Φ_1, by taking the difference we obtain

$$\Phi_1''\Phi_2-\Phi_2''\Phi_1=e^{-\frac{s}{2}z}\Phi_1\Phi_2(\Phi_1-\Phi_2). \tag{3.6}$$

First, we claim that

$$\Phi_i(\pm\infty) = 0,\ \Phi_i'(\pm\infty) = 0,\ i = 1, 2. \tag{3.7}$$

It is clear that $\Phi_i(-\infty) = 0$, since $\phi_i(-\infty) = 1$ and $s > 0$. Also, from

$$\Phi_i'(z) = \frac{s}{2}e^{\frac{s}{2}z}\phi_i(z) + e^{\frac{s}{2}z}\phi_i'(z) = \frac{s}{2}e^{\frac{s}{2}z}\phi_i(z) + e^{\frac{s}{2}z}\phi_i(z)\frac{\phi_i'(z)}{\phi_i(z)}$$

and Lemma 3.1 it follows that $\Phi_i'(-\infty) = 0$. On the other hand, we deduce from (3.3) that

$$e^{\frac{s}{2}z}\phi_i(z) \le Ce^{\frac{-\sqrt{s^2-4h(\infty)}}{4}z},\ \forall\, z \gg 1, \tag{3.8}$$

for some positive constant C. To see this, since

$$\lambda_0 < -\frac{s}{2} - \frac{\sqrt{s^2 - 4h(\infty)}}{4} := \lambda_1,$$

it follows from (3.3) that

$$[\ln(\phi_i(z)]' = \frac{\phi_i'(z)}{\phi_i(z)} \le \lambda_1,\ \forall\, z \ge M, \tag{3.9}$$

for some $M \gg 1$. By an integration of (3.9) from M to $z \ge M$, we obtain

$$\phi_i(z) \le \phi_i(M)e^{\lambda_1(z-M)} = \phi_i(M)e^{-\lambda_1 M}e^{\lambda_1 z},\ \forall\, z \ge M.$$

This implies (3.8). Thus $\Phi_i(\infty) = \Phi_i'(\infty) = 0$ and so (3.7) is proved.

Next, for a contradiction we assume that $\Phi_1 \not\equiv \Phi_2$. We divide our discussion into the following cases

Case 1. There are at least two intersection points of the graphs of Φ_1 and Φ_2. Without loss of generality, we may assume that there exist a, b with $a < b$ such that

$$\Phi_1(a) = \Phi_2(a),\ \Phi_1(b) = \Phi_2(b),\ \Phi_1(z) > \Phi_2(z),\ \forall\, z \in (a, b). \tag{3.10}$$

Then, by an integration of (3.6), we get

$$[\Phi_1'\Phi_2 - \Phi_2'\Phi_1](b) - [\Phi_1'\Phi_2 - \Phi_2'\Phi_1](a) = \int_a^b e^{-\frac{s}{2}z}\Phi_1\Phi_2(\Phi_1 - \Phi_2)\, dz. \tag{3.11}$$

However, from (3.10) we have

$$\Phi_1'(b)\Phi_2(b) - \Phi_2'(b)\Phi_1(b) = [\Phi_1'(b) - \Phi_2'(b)]\Phi_1(b) \le 0,$$
$$\Phi_1'(a)\Phi_2(a) - \Phi_2'(a)\Phi_1(a) = [\Phi_1'(a) - \Phi_2'(a)]\Phi_1(a) \ge 0.$$

Thus the left-hand side of (3.11) is non-positive. But, the right-hand side of (3.11) is positive, a contradiction.

Case 2. There is exactly one intersection point $a \in \mathbb{R}$ of the graphs of Φ_1 and Φ_2. Without loss of generality, we may assume that $[\Phi_1(z) - \Phi_2(z)](z - a) > 0$ for all $z \ne a$. Then we have $[\Phi_1'(a) - \Phi_2'(a)] \ge 0$. An integration of (3.6) from a to ∞ and using (3.7), we obtain

$$0 \ge -[\Phi_1'(a) - \Phi_2'(a)] = \int_a^\infty e^{-\frac{s}{2}z}\Phi_1\Phi_2(\Phi_1 - \Phi_2)\, dz > 0,$$

a contradiction.

Case 3. There is no intersection of the graphs of Φ_1 and Φ_2. Without loss of generality, we may assume that $\Phi_1 > \Phi_2$ in $\mathbb{R}$. Then an integration of (3.6) over $\mathbb{R}$, using (3.7), gives

$$0 = \int_{-\infty}^{\infty} e^{-\frac{s}{2}z}\Phi_1\Phi_2(\Phi_1 - \Phi_2)\, dz > 0,$$

a contradiction again.

We conclude that $\Phi_1 = \Phi_2$ and so $\phi_1 = \phi_2$. Thus, the uniqueness part of the forced waves in Theorem 1.2 is proved. $\square$

Next, we provide a proof of the non-existence of forced waves (cf. [12]), but without the monotonicity condition on h.

Proof of Non-existence. Assume on the contrary that there exists a solution $\phi(z)$ of (1.2)-(1.3) for some $s \geq 2$.

First, we define

$$Q(z) := e^{\left(-s/2+\sqrt{s^2/4-1}\right)z}.$$

Then Q satisfies

$$Q'' + sQ' + Q = 0.$$

Moreover, the function $AQ(z)$ is a super-solution of (1.2) for any $A > 0$, since

$$AQ'' + sAQ' + AQ(h(z) - AQ) \leq AQ'' + sAQ' + AQ = 0,$$

using the assumption that $h \leq h(-\infty) = 1$.

Next, note that $\phi(-\infty) = 1$, $Q(-\infty) = \infty$ and, by (3.8), $\phi(z) < AQ(z)$ for all $z \gg 1$ for any $A > 0$, since

$$-s/2 - \sqrt{s^2 - 4h(\infty)}/4 < -s/2 + \sqrt{s^2/4 - 1}.$$

Hence, by choosing $A > 0$ large enough, we have

$$\phi(z) \leq AQ(z),\ z \in \mathbb{R}.$$

Therefore, the quantity

$$A_* := \inf\{A > 0 \mid \phi(z) \leq AQ(z),\ z \in \mathbb{R}\}$$

is well-defined and $A_* \in (0, \infty)$.

Note that $\phi(z) \leq A_*Q(z)$ for all $z \in \mathbb{R}$, and $\phi(z_*) = A_*Q(z_*)$ for some $z_* \in \mathbb{R}$, by the definition of A_*. Thus, by the strong maximum principle, $\phi(z) = A_*Q(z)$ for all $z \in \mathbb{R}$. This contradicts $\phi(-\infty) = 1 < A_*Q(-\infty) = \infty$. Thereby, the non-existence part in Theorem 1.2 is proved. $\square$

4. Stability of forced waves

This section is devoted to the proof for the stability of forced waves.

First, we provide a proof of Theorem 1.3 as follows.

Proof of Theorem 1.3. Let ϕ be a solution of (1.2)-(1.3) for some $s<0$ and let u be a solution of (1.5) with a positive initial data u_0 at $t=0$.

First, we let

$$V(z,t) := u(z,t) - \phi(z) - \phi \ln \frac{u(z,t)}{\phi(z)}.$$

By a simple calculation, we have

$$\begin{aligned} V_t &= u_t - \frac{\phi}{u}u_t, \quad V_z = u_z - \frac{\phi}{u}u_z - \phi' \ln\frac{u}{\phi}, \\ V_{zz} &= u_{zz} - \frac{\phi}{u}u_{zz} - \phi'' \ln\frac{u}{\phi} + \phi\Big[\frac{u_z}{u} - \frac{\phi'}{\phi}\Big]^2. \end{aligned}$$

It follows from (1.2) and (1.5) that

$$\begin{aligned} V_t - V_{zz} - sV_z &\leq u(h-u) - \phi(h-u) - (h-\phi)\phi \ln\frac{u}{\phi} \\ &= (h-\phi)V - (u-\phi)^2 \leq (h-\phi)V, \end{aligned}$$

i.e., V satisfies

$$V_t - V_{zz} - sV_z \leq (h-\phi)V \ \text{ in } \mathbb{R}. \tag{4.1}$$

Now, we consider the linear equation

$$\bar{V}_t = \bar{V}_{zz} + s\bar{V}_z + (h-\phi)\bar{V} \ \text{ in } \mathbb{R}. \tag{4.2}$$

Note that $\phi(z)$ is a stationary solution of (4.2). Setting $W := e^{\frac{s}{2}z}\bar{V}$, (4.2) is transformed to

$$W_t = W_{zz} + \Big(h - \phi - \frac{s^2}{4}\Big)W \ \text{ in } \mathbb{R}. \tag{4.3}$$

Note that this equation has a stationary solution $e^{\frac{s}{2}z}\phi(z)$. We then define a linear differential operator $L : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$Lw = w'' + (h - \phi - \frac{s^2}{4})w,$$

which is a self-adjoint operator on $L^2(\mathbb{R})$. Hence all eigenvalues of the operator L are real.

Next, we show that 0 is not an eigenvalue of the operator L when $s<0$. Indeed, using that the function $e^{\frac{s}{2}z}\phi(z)$ is a solution of

$$v'' + \Big(h - \phi - \frac{s^2}{4}\Big)v = 0, \tag{4.4}$$

and the reduction of order, the general solution to equation (4.4) is represented as

$$v(z) = e^{\frac{s}{2}z}\phi(z)\Big(a\int_0^z \frac{dz}{e^{sz}\phi^2(z)} + b\Big), \ z \in \mathbb{R}, \tag{4.5}$$

for some constants a and b. Recall

$$e^{\frac{s}{2}z}\phi(z) = O(e^{\frac{s}{2}z}) \quad \text{as } z \to -\infty, \tag{4.6}$$

and, by (3.8),

$$e^{\frac{s}{2}z}\phi(z) = O\left(e^{-\frac{\sqrt{s^2-4h(\infty)}}{2}z}\right) \quad \text{as} \quad z \to +\infty. \tag{4.7}$$

Thus

$$e^{\frac{s}{2}z}\phi(z)\int_0^z \frac{dz}{e^{sz}\phi^2(z)} = O\left(e^{\frac{\sqrt{s^2-4h(\infty)}}{2}z}\right) \quad \text{as} \quad z \to +\infty.$$

Therefore, the only H^2 solution to equation (4.4) is $v \equiv 0$. We conclude that 0 is not an eigenvalue of L when $s < 0$.

To proceed further, we show that there does not exist any positive eigenvalue of the operator L. Assume, on the contrary, there exists an eigenvalue $\lambda > 0$ and a function $\psi \in H^2(\mathbb{R})$ such that $(L - \lambda)\psi = 0$. Then the asymptotic decaying behavior of the function ψ at space infinity is given by

$$\psi = O(e^{-\gamma_+ z}) \quad \text{as} \quad z \to +\infty, \tag{4.8}$$

$$\psi = O(e^{\gamma_- z}) \quad \text{as} \quad z \to -\infty, \tag{4.9}$$

where

$$\gamma_+ = \gamma_+(\lambda) := \sqrt{\lambda + s^2/4 - h(+\infty)}, \qquad \gamma_- = \gamma_-(\lambda) := \sqrt{\lambda + s^2/4},$$

Now we define a function

$$P(z) = \frac{\psi(z)}{e^{\frac{s}{2}z}\phi(z)}.$$

Then (4.6) and (4.9) imply that

$$P(z) = O(e^{(\gamma_-(\lambda)-\gamma_-(0))z}) \to 0 \quad \text{as} \quad z \to -\infty,$$

while (4.7) and (4.8) imply that

$$P(z) = O(e^{-(\gamma_+(\lambda)-\gamma_+(0))z}) \to 0 \quad \text{as} \quad z \to +\infty.$$

In addition, $P(z)$ satisfies

$$P''(z) + 2\frac{\frac{d}{dz}\left(e^{\frac{s}{2}z}\phi(z)\right)}{e^{\frac{s}{2}z}\phi(z)}P'(z) = \lambda P(z).$$

We claim that $P \equiv 0$, which implies $\psi \equiv 0$. If this does not hold, then there exists a positive maximal point $z_m \in \mathbb{R}$ of P such that

$$P''(z_m) \le 0 = P'(z_m) < P(z_m).$$

Here we have applied the condition $P(\pm\infty) = 0$. This and $\lambda > 0$ immediately give us

$$0 \ge P''(z_m) + 2\frac{\frac{d}{dz}\left(e^{\frac{s}{2}z_m}\phi(z_m)\right)}{e^{\frac{s}{2}z}\phi(z_m)}P'(z_m) = \lambda P(z_m) > 0,$$

which is a contradiction. As a conclusion, there does not exist any positive eigenvalue for the operator L.

Finally, we consider the essential spectrum of L. The associated limiting operators $L_{\pm}$ of L are defined by

$$L_+w = w'' + \Big(h(+\infty) - \frac{s^2}{4}\Big)w, \qquad L_-w = w'' - \frac{s^2}{4}w.$$

Then

$$S_+ := \{\lambda \in \mathbb{C} \mid \lambda = -k^2 + \Big(h(+\infty) - \frac{s^2}{4}\Big) \text{ for some } k \in \mathbb{R}\} = \big(-\infty, h(+\infty) - \frac{s^2}{4}\big],$$
$$S_- := \{\lambda \in \mathbb{C} \mid \lambda = -k^2 - \frac{s^2}{4} \text{ for some } k \in \mathbb{R}\} = \big(-\infty, -\frac{s^2}{4}\big],$$

and so the essential spectrum of L is contained in $(-\infty, -s^2/4]$, by Theorem A.2 in Chapter 5 of [18]. We conclude that there exists $\omega > 0$ such that $\sigma(L) \subset (-\infty, -\omega)$ when $s < 0$.

Furthermore, using this information on the spectrum of L, it follows from [21, Theorem 51.1] that there exist positive constants C' and C'' such that

$$\|e^{tL}g\|_{L^2} \le C'e^{-\omega t}\|g\|_{L^2}, \quad \|e^{tL}g\|_{H^1} \le C''t^{-\frac{1}{2}}e^{-\omega t}\|g\|_{L^2}, \quad g \in L^2,$$

for all $t > 0$. Then, using the inequality

$$\|f\|_{L^\infty} \le 2^{\frac{1}{2}}\|f'\|_{L^2}^{\frac{1}{2}}\|f\|_{L^2}^{\frac{1}{2}}, \quad f \in H^1(\mathbb{R}),$$

we conclude that there exists a positive constant C such that

$$\|e^{tL}\|_{\mathcal{L}(L^2,L^\infty)} \le Ct^{-\frac{1}{4}}e^{-\omega t} \text{ for all } t > 0.$$

Thus

$$\|W(\cdot,t)\|_{L^\infty} \le Ct^{-\frac{1}{4}}e^{-\omega t}\|W(\cdot,0)\|_{L^2} \to 0 \text{ as } t \to \infty,$$

provided that $W(\cdot,0) \in L^2(\mathbb{R})$ and $s < 0$. By the comparison principle, we conclude that $e^{sz/2}V(z,t) \to 0$ as $t \to \infty$ uniformly over $\mathbb{R}$. This implies that $V(z,t) \to 0$ locally uniformly in $\mathbb{R}$ as $t \to \infty$. Since $s < 0$, the convergence is uniformly over $(-\infty, l]$ for any $l \in \mathbb{R}$. This completes the proof of the theorem. □

Remark 4.1. In fact, by a simple approach we have another stability theorem, but only for some extinction forced waves and the perturbation of initial data is different from that in Theorem 1.3, as follows. Let (h1)-(h2) be enforced. Set $\hat{h} := \sup_{z\in\mathbb{R}} h(z)$. Let ϕ be a solution of (1.2)-(1.3) for some $s \le -2\sqrt{\hat{h}}$ and let u be a solution of (1.5) with a positive initial data u_0 at $t = 0$. Set $\mu := s/2 + \sqrt{s^2 - 4\hat{h}}/2$. If $\exp(\mu z)\phi g(u_0/\phi) \in L^1(\mathbb{R})$, then $u(z,t) \to \phi(z)$ as $t \to \infty$ locally uniformly for $z \in \mathbb{R}$ and uniformly for $z \in (-\infty, l]$ for any $l \in \mathbb{R}$.

Proof of Remark 4.1. Although the proof of Remark 4.1 is almost the same as that for [17, Theorem 1.1], we provide the details here for the reader's convenience.

Let ϕ be a solution of (1.2)-(1.3) for some $s \le -2\sqrt{\hat{h}}$ and let u be a solution of (1.5) with a positive initial data u_0 at $t = 0$.

First, since $\phi > 0$, $V > 0$ and $h(z) \le \hat{h}$ for all $z \in \mathbb{R}$, we obtain from (4.1) that V satisfies

$$V_t \le V_{zz} + sV_z + \hat{h}V, \; z \in \mathbb{R}, \, t > 0. \tag{4.10}$$

Let us define the function $Q(z,t) := e^{\mu z} V(z,t)$, where

$$\mu := \frac{s + \sqrt{s^2 - 4\hat{h}}}{2} < 0, \ \mu^2 - s\mu + \hat{h} = 0,$$

which is well-defined due to $s \le -2\sqrt{\hat{h}} < 0$. Then it follows from (4.10) that

$$Q_t \le Q_{zz} - \sqrt{s^2 - 4\hat{h}} Q_z, \ z \in \mathbb{R}, \ t > 0.$$

Now, let $\hat{Q}$ be the solution of

$$\hat{Q}_t = \hat{Q}_{zz} - \sqrt{s^2 - 4\hat{h}} \hat{Q}_z, \quad z \in \mathbb{R}, \ t > 0, \quad \hat{Q}(\cdot, 0) = Q(\cdot, 0).$$

It follows from the comparison principle that

$$\begin{aligned} 0 \le Q(z,t) \le \hat{Q}(z,t) &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp\Big\{ - \frac{(z + \sqrt{s^2 - 4\hat{h}}\, t - y)^2}{4t} \Big\} Q(y,0)\, dy \\ &\le \frac{\|Q(\cdot,0)\|_{L^1(\mathbb{R})}}{\sqrt{4\pi t}} \to 0 \quad \text{as } t \to \infty, \end{aligned}$$

using $Q(\cdot, 0) \in L^1(\mathbb{R})$. This implies that $V(z,t) \to 0$ locally uniformly in $\mathbb{R}$ as $t \to \infty$. Since $\mu < 0$, the convergence is uniformly over $(-\infty, l]$ for any $l \in \mathbb{R}$. This proves the remark. $\square$

Next, we consider the case $s \in (0,2)$ for the saturation forced waves.

Proof of Theorem 1.4. First, in the proof of Theorem 1.1, we can choose a sub-solution $\underline{\phi}$ of (1.2) with $\varepsilon > 0$ small enough and set $R = -(z_0 + \pi/(2\omega))$ with a sufficient large $-z_0$ such that $u_0(z) \ge \varepsilon$ for all $z \le -R$. It is easy to check that the constant function $\overline{\phi} \equiv M$ a super-solution of (1.2), if $M \ge \sup_{z \in \mathbb{R}} h(z)$. We also choose M so that $M \ge \|u_0\|_\infty$. Then we have $\overline{\phi}(z) \ge u_0(z) \ge \underline{\phi}(z)$ for all $z \in \mathbb{R}$.

Next, we denote the solution of (1.5) with initial data u_0 by $u(z,t;u_0)$. Then the solution $u(z,t;\bar{\phi})$ is monotone decreasing in t for $t \ge 0$ and $u(z,t;\underline{\phi})$ is monotone increasing in t for $t \ge 0$. By [22, Theorem 3.6] and Theorem 1.2, we deduce that

$$\lim_{t\to\infty} u(z,t;\underline{\phi}) = \phi(z) = \lim_{t\to\infty} u(z,t;\overline{\phi})$$

for every $z \in \mathbb{R}$. On the other hand, by the comparison principle, we obtain from $\underline{\phi} \le u_0 \le \overline{\phi}$ that

$$u(z,t;\underline{\phi}) \le u(z,t;u_0) \le u(z,t;\overline{\phi}), \ z \in \mathbb{R}, \ t \ge 0.$$

Letting $t \to \infty$, we conclude that $u(z,t;u_0) \to \phi(z)$ as $t \to \infty$ for every $z \in \mathbb{R}$. The theorem is thereby proved. $\square$

Remark 4.2. One should note that the perturbation of initial data in Theorem 1.4 can be arbitrarily near $z = \infty$. This is due to the uniqueness of forced waves for $s \in (0,2)$. When $h(z)$ is monotone decreasing, the uniqueness of forced wave with $s = 0$ is proved in [1]. Hence Theorem 1.4 also holds for $s = 0$, if we further assume that $h(z)$ is monotone decreasing.

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Department of Mathematics, Tamkang University, Tamsui, New Taipei City 251301, Taiwan
Email address: jsguo@mail.tku.edu.tw

Department of Data Science and Big Data Analytics, Providence University, Taichung, Taiwan
Email address: kguo2021@pu.edu.tw

Department of Mathematical Sciences, Tokyo Metropolitan University, Hachioji, Tokyo 192-0397, Japan
Email address: shimojo@tmu.ac.jp