EXISTENCE OF TRAVELING WAVE SOLUTIONS TO A NONLOCAL SCALAR EQUATION WITH SIGN-CHANGING KERNEL

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ABSTRACT. In this paper, we study the existence of traveling wave solutions connecting two constant states to a nonlocal scalar equation with sign-changing kernel. A typical example of such kernel in the neural fields is the Mexican hat type function. We first introduce a new notion of upper-lower-solution for the equation of wave profile for a given wave speed. Then, with the help of Schauder's fixed point theorem, we construct two different pairs of upper-lower-solutions to obtain traveling waves for a continuum of wave speeds under two different assumptions. Due to the sign-changing nature of the kernel, the wave profiles may take both positive and negative values. Finally, we analyze the limit of the right-hand tail of wave profiles. Under some further condition on the wave speeds, we prove that the right-hand tail limit of the wave profile does exist. In particular, we obtain the existence of nonnegative traveling waves connecting the unstable state 0 and the stable state 1 for wave speeds large enough.

1. INTRODUCTION

In this paper, we consider the following nonlocal evolution equation

$$u_t(x,t) = (K * u)(x,t) - \alpha u(x,t) + f(u(x,t)),\ x \in \mathbb{R},\ t > 0, \tag{1.1}$$

where the $*$ denotes the convolution operator with respective to the spatial variable,

$$(K * u)(x,t) := \int_{\mathbb{R}} K(y)u(x-y,t)dy,$$

in which K is a continuous function satisfying

$$K(-y) = K(y),\ \forall\, y \in \mathbb{R}, \quad \int_{\mathbb{R}} K(y)dy = \alpha \tag{1.2}$$

for some nonnegative constant α. The nonlinearity f is a locally Lipschitz continuous function defined in $\mathbb{R}$ such that

$$\begin{cases} f(-a) = f(0) = f(1) = 0,\ f < 0 \text{ in } (-a,0),\ f > 0 \text{ in } (0,1), \\ f'(-a) < 0,\ f'(0) > 0,\ f'(1) < 0 \end{cases} \tag{1.3}$$

for some positive constant a.

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A classical solution u to (1.1) is called a traveling wave solution if there exist a constant c (the *wave speed*) and a continuously differentiable function ϕ (the *wave profile*) such that $u(x,t) = \phi(x+ct)$. Setting $\xi := x + ct$, we see from (1.1) that ϕ satisfies

$$c\phi'(\xi) = (K * \phi)(\xi) - \alpha\phi(\xi) + f(\phi(\xi)),\ \xi \in \mathbb{R}, \tag{1.4}$$

where

$$(K * \phi)(\xi) := \int_{\mathbb{R}} K(y)\phi(\xi - y)dy.$$

We are interested in the waves connecting the state 0 and a constant state. So we also impose the boundary condition

$$\phi(-\infty) = 0. \tag{1.5}$$

We leave the right-hand boundary condition free. In fact, when a wave profile ϕ has a limit at $\xi = \infty$, we have $\phi(\infty) \in \{-a, 0, 1\}$. This is discussed in Section 4.

The nonlocal equation (1.1) has appeared as model equations in, e.g., neural fields ([3, 15]), dispersal motions of cells or organisms ([25]), pattern formation in biology ([28]) and material sciences ([5, 10]). In all such studies, localized patterns are sorts of main interests from the pattern formation point of view. As one typical important example of localized patterns, traveling wave solutions are concerned in this paper.

When the kernel function K is nonnegative (so that $\alpha > 0$), the nonlocal operator models a long range migration of the species. It is very important in real applications, since it takes into account the long-distance interactions and describes the dispersion via a dispersal kernel. For example, Medlock and Kot [32] investigated the effects of population dispersal using a nonlocal epidemic model. Lutscher et al. [30] has analyzed a stream population nonlocal model. In fact, when $f(u) = ru(1-u)$ with $r > 0$, the spatial propagation dynamics for (1.1) has been investigated extensively. We refer the reader to [35, 13, 16, 17, 40, 23] for the existence and uniqueness of monotone traveling wave solutions for (1.1). For the biological and theoretical backgrounds on nonlocal operators, we also refer the reader to, e.g., [21, 33, 26, 4]. For more results on nonlocal equations, we refer to [9, 6, 7, 5, 19, 11, 27, 36, 37, 38, 18, 22, 34, 1, 12, 29] and the references cited therein.

In the above mentioned works, nonnegative kernels are dealt with which can be regarded as the generalization of the diffusion process ([5]). In fact, similar results to the case with usual diffusion terms hold such as the monotonicity of solutions together with the comparison principle. But, from the biological point of view, kernels with negative parts are essential while there have been very few works for the case with sign-changing kernel (e.g., [8]). The Mexican hat type kernel is a typical example of it, which has been frequently used in neural fields ([3, 15]) and pattern formation problems ([28]). One of the main reasons is that the Mexican hat type kernel has been believed to express the property of local activation and long range inhibition related to activator-inhibitor systems causing Turing instability ([24]). Recently, this fact was theoretically shown in [20] in the sense that some class of activator-inhibitor systems causing Turing instability can be reduced to model equations (1.1) with the Mexican hat shape kernel $K(x)$.

Concerning equation (1.1) with a sign-changing kernel, our aim of this paper is to show the existence of a traveling wave solution. Due to the sign-changing nature of K, equation

(1.1) does not have a comparison principle. Therefore, the standard method of monotone iteration cannot be applied to derive the existence of traveling waves. This is one of the difficulties in dealing with equation (1.1). On the other hand, since the state 0 is unstable and the state 1 is stable in the sense of ordinary differential equation $u_t = f(u)$, the traveling waves are of monostable type and we expect to have a continuum of wave speeds. We shall apply Schauder's fixed point theorem with the help of (generalized) upper-lower-solutions to derive the existence of traveling waves (cf., e.g., [31, 14]). To the best of our knowledge, the definition of upper-lower-solutions introduced in this paper is new. Although the method of applying Schauder's fixed point theorem is by now very standard, to find a suitable pair of upper-lower-solutions is by no means trivial.

Throughout this paper, besides (1.3) we also assume that

$$|f(u)| \le f'(0)|u| \text{ for } u \in [u^-, u^+] \tag{1.6}$$

for some constants $u^\pm$ with $-\infty < u^- < -a < 1 < u^+ < \infty$. A typical example of f is the cubic function $f(u) = u(1-u^2)$, where $a = 1$ and $u^\pm = \pm\sqrt{2}$.

In the sequel, we set $K^+(y) := \max\{K(y), 0\}$ and $K^-(y) := \max\{-K(y), 0\}$. In this paper, we always assume that the kernel K has a compact support. Then the functions

$$I^\pm(\lambda) := \int_{\mathbb{R}} K^\pm(y) e^{-\lambda y} dy$$

are well-defined for all $\lambda \in [0, \infty)$. We also define the following two quantities

$$c^* := \inf_{\lambda \in (0,\hat{\lambda})} \frac{Q(\lambda)}{\lambda}, \quad Q(\lambda) := I^+(\lambda) - I^-(\lambda) - \alpha + f'(0), \tag{1.7}$$

$$c^{**} := \inf_{\lambda \in (0,\infty)} \frac{R(\lambda)}{\lambda}, \quad R(\lambda) := I^+(\lambda) + I^-(\lambda) - \alpha + f'(0), \tag{1.8}$$

where $\hat{\lambda}$ is defined to be the first positive zero of $Q(\lambda)$, if it exists, otherwise, set $\hat{\lambda} := \infty$. Note that $Q(0) = f'(0) > 0$. Hence $Q(\lambda) > 0$ for $\lambda > 0$ small, by the continuity of Q. Therefore, c^* is well-defined and $c^* \ge 0$. In particular, $c^* = 0$ when $\hat{\lambda} < \infty$. Also, it is easy to see that $R(\lambda)$ is a strictly convex function for $\lambda \ge 0$. Hence c^{**} is well-defined and $c^{**} > 0$. Moreover, $Q(\lambda) < R(\lambda)$ for all $\lambda \in (0, \hat{\lambda})$ and $c^* < c^{**}$.

Before stating our main results, let us introduce the following assumptions on $\{f, K\}$.

Assumption 1.1. *There is a small constant $\eta \in (0,1)$ such that*

$$f(u) = f'(0)u \quad \text{for } u \in [0, \eta]. \tag{1.9}$$

Moreover, there are constants $\delta \in (0,\infty)$ and $\gamma \in (a, \infty)$ such that

$$f(1+\delta) < 0,\ f(-\gamma) > 0,\ u^- \le -\gamma,\ 1+\delta \le u^+. \tag{1.10}$$

With these constants δ and γ, K^- satisfies

$$\int_{\mathbb{R}} K^-(y) dy \le \min\left\{ \frac{-f(1+\delta)}{1+\delta+\gamma}, \frac{f(-\gamma)}{1+\delta+\gamma} \right\}. \tag{1.11}$$

An example of $\{f, K\}$ such that Assumption 1.1 holds is that f is piecewise linear, e.g.,

$$f(u) = \begin{cases} -(u+1), & u \le -1/2, \\ u, & u \in [-1/2, 1/2], \\ -(u-1), & u \ge 1/2 \end{cases} \tag{1.12}$$

and K satisfies (a priori) the condition

$$\int_{\mathbb{R}} K^-(y)dy < 1. \tag{1.13}$$

In fact, since

$$\sup_{\delta>0} \frac{-f(1+\delta)}{1+\delta+\gamma} = 1,\ \forall\, \gamma > 1, \quad \sup_{\gamma>1} \frac{f(-\gamma)}{1+\delta+\gamma} = 1,\ \forall\, \delta > 0,$$

we can choose $\delta > 0$ and $\gamma > 1$ such that (1.10) and (1.11) hold. Clearly, (1.9) holds.

Assumption 1.2. *Besides* (1.9), f *satisfies*

$$f'(0) > \alpha. \tag{1.14}$$

Moreover, K *satisfies*

$$\int_{\mathbb{R}} K^-(y)dy \le \min\left\{ \frac{-f(1+\delta)}{1+\delta}, \frac{(f'(0)-\alpha)\eta}{1+\delta} \right\} \tag{1.15}$$

for some positive constant $\delta \in (0, u^+ - 1)$ *such that* $f(1+\delta) < 0$.

An example of $\{f, K\}$ such that Assumption 1.2 holds is that f is piecewise linear defined by (1.12) and K satisfies

$$\int_{\mathbb{R}} K^+(y)dy = \int_{\mathbb{R}} K^-(y)dy \le \frac{1}{3}. \tag{1.16}$$

In fact, since

$$\frac{-f(1+\delta)}{1+\delta} = \frac{\delta}{1+\delta}, \quad \frac{f'(0)\eta}{1+\delta} = \frac{1}{2(1+\delta)},$$

we can choose $\delta = 1/2$ such that (1.15) holds. Clearly, (1.14) holds with $\alpha = 0$.

Our first main result reads

Theorem 1.3. *Let Assumption 1.1 be enforced. Then for any* $c > c^{**}$ *equation* (1.4) *has a solution* ϕ *such that* $\phi \not\equiv 0$ *and* $\phi(-\infty) = 0$.

By some numerical simulations, the wave profile ϕ obtained in Theorem 1.3 may take both positive and negative values. However, our second result excludes this sign-changing nature of wave profile as follows.

Theorem 1.4. *Let Assumption 1.2 be enforced. Then for any* $c > c^{**}$ *equation* (1.4) *has a nonnegative solution* ϕ *such that* $\phi \not\equiv 0$ *and* $\phi(-\infty) = 0$.

The rest of this paper is organized as follows. In Section 2, we give several preliminaries including the definition of upper-lower-solution and a proposition to ensure the existence of solution to (1.4). Then, in Section 3, we construct two different pairs of upper-lower-solutions under Assumptions 1.1 and 1.2, respectively. Combining these upper-lower-solutions with the existence theory from Section 2, we give the proofs of Theorems 1.3 and 1.4. Finally, in Section 4, we analyze the right-hand tail of the wave profiles obtained in Theorems 1.3 and 1.4. Under some further restriction on the wave speeds, we derive that the right-hand tail limit of the wave profile does exist. In particular, we obtain nonnegative traveling wave solutions of (1.1) connecting the unstable state 0 and the stable state 1 for wave speeds large enough, under Assumption 1.2. Unfortunately, we were unable to determine the minimal speed (if it exists). We leave it as an open problem for a future study.

Two more remarks are made as follows. First, in [23], they consider the doubly nonlocal Fisher-KPP equation and the strict monotonicity of wave profiles are derived. Due to the positivity of the kernels, the comparison principle holds for their model. This important property is one of the keys to derive the monotonicity of wave profiles. However, our model does not have the comparison principle and so the theory of monotone semiflow cannot be applied. On the other hand, from our numerical simulations, we have observed non-monotone traveling waves. Therefore, we are not sure whether there are monotone traveling waves for our model with sign-changing kernel, even under some additional assumptions on the nonlinearity.

Secondly, we suspect that the very restricted condition (1.9) is just technical and it might be possible to replace this condition by some more general assumptions. However, our construction of upper-lower-solutions relies heavily on the condition (1.9). For the sign-changing upper-lower-solutions (for Theorem 1.3), one should notice that the nonlinearity is assumed to be of bistable type which is crucial for the kernel to have a nontrivial negative part. On the other hand, under Assumption 1.2, the constant η in condition (1.9) is needed for our kernel with a nontrivial negative part. It would be very interesting either to remove the assumption (1.9) or to replace it by some more general conditions. We leave this important question to be open.

2. Preliminaries

In this section, we introduce the notion of upper-lower-solution and provide a derivation of the existence of solution to (1.4). First, due to the sign-changing nature of K, we define the upper-lower-solutions of (1.4) as follows.

Definition 2.1. *Given a constant $c > 0$. A pair of continuous functions $\{\overline{\phi}, \underline{\phi}\}$ are upper and lower solutions of* (1.4) *if*

$$c\overline{\phi}'(\xi) \geq (K^+ * \overline{\phi})(\xi) - (K^- * \underline{\phi})(\xi) - \alpha\overline{\phi}(\xi) + f(\overline{\phi}(\xi)),\ \forall\, \xi \in \mathbb{R} \setminus A, \tag{2.1}$$

$$c\underline{\phi}'(\xi) \leq (K^+ * \underline{\phi})(\xi) - (K^- * \overline{\phi})(\xi) - \alpha\underline{\phi}(\xi) + f(\underline{\phi}(\xi)),\ \forall\, \xi \in \mathbb{R} \setminus A, \tag{2.2}$$

for some finite set $A \subset \mathbb{R}$.

Let $\kappa := \max_{u\in[u^-,u^+]}|f'(u)|$. The constant κ is well-defined, since f is a locally Lipschitz continuous function. To apply Schauder's fixed point theorem, we introduce the following integral operator

$$P[z](\xi) := \frac{1}{c}\int_{-\infty}^{\xi} e^{-(\alpha+\kappa)(\xi-y)/c}[(K*z)(y)+\kappa z(y)+f(z(y))]dy \tag{2.3}$$

for any continuous function z on $\mathbb{R}$ with range in $[u^-,u^+]$.

Note that, by a differentiating $z(\xi)=P[z](\xi)$ with respect to ξ, we obtain

$$z'(\xi)=\frac{1}{c}\{(K*z)(\xi)+\kappa z(\xi)+f(z(\xi))\}-\frac{\kappa+\alpha}{c}z(\xi).$$

Hence it is easy to see that ϕ is a fixed point of the mapping P if and only if ϕ satisfies (1.4). Therefore, to find a solution of (1.4) is equivalent to finding a fixed point of P.

Although the proof of the following proposition is very standard (cf. [31]), for reader's convenience we provide some details here.

Proposition 2.2. *Suppose that there exists a pair of upper-lower-solution* $\{\overline{\phi},\underline{\phi}\}$ *with range in* $[u^-,u^+]$ *such that* $\underline{\phi}\le\overline{\phi}$ *in* $\mathbb{R}$. *Then* (1.4) *has a solution* ϕ *such that* $\underline{\phi}\le\phi\le\overline{\phi}$ *in* $\mathbb{R}$.

Proof. Choose a constant $\mu\in(0,(\alpha+\kappa)/c)$. Define

$$B_\mu(\mathbb{R}) := \{z\in C(\mathbb{R})\mid ||z||_\mu<\infty\},\quad \|z\|_\mu := \sup_{\xi\in\mathbb{R}}|z(\xi)|e^{-\mu|\xi|}.$$

Then $(B_\mu(\mathbb{R}),||\cdot||_\mu)$ is a Banach space. Also, we set

$$\Gamma := \{z\in C(\mathbb{R})\mid \underline{\phi}(y)\le z(y)\le\overline{\phi}(y),\ \forall\, y\in\mathbb{R}\}.$$

Then Γ is a nonempty convex bounded closed set with respect to the weighted norm $\|\cdot\|_\mu$.

First, we show that $P(\Gamma)\subset\Gamma$. For a given $z\in\Gamma$, we have

$$\begin{aligned} P[z](\xi) &= \frac{1}{c}\int_{-\infty}^{\xi} e^{-(\alpha+\kappa)(\xi-y)/c}[(K^+*z)(y)-(K^-*z)(y)+\kappa z(y)+f(z(y))]dy \\ &\ge \frac{1}{c}\int_{-\infty}^{\xi} e^{-(\alpha+\kappa)(\xi-y)/c}[(K^+*\underline{\phi})(y)-(K^-*\overline{\phi})(y)+\kappa\underline{\phi}(y)+f(\underline{\phi}(y))]dy, \end{aligned}$$

using the fact $\kappa u+f(u)$ is increasing for $u\in[u^-,u^+]$. It then follows from (2.2) that

$$P[z](\xi)\ge\frac{1}{c}\int_{-\infty}^{\xi} e^{-(\alpha+\kappa)(\xi-y)/c}[c\underline{\phi}'(y)+(\alpha+\kappa)\underline{\phi}(y)]dy=\underline{\phi}(\xi)$$

for all $\xi\in\mathbb{R}$. Similarly, one can easily derive that $P[z](\xi)\le\overline{\phi}(\xi)$ for all $\xi\in\mathbb{R}$.

Next, a similar argument as that in [31] implies that the mapping $P:\Gamma\to\Gamma$ is completely continuous with respect to the weighted norm $\|z\|_\mu$. We omit the details here. Therefore, the proposition is proved by applying Schauder's fixed point theorem. □

3. Constructions of upper-lower-solution

This section is devoted to the construction of suitable pair of upper-lower-solutions.

For a given $c > c^{**}$, since $c > c^*$, it follows from (1.7) that there exists the smallest positive $\lambda_1 \in (0, \hat{\lambda})$ such that

$$Q(\lambda_1) = c\lambda_1, \quad Q(\lambda) > c\lambda \ \text{ for all } \lambda \in [0, \lambda_1). \tag{3.1}$$

Also, by the definition of c^{**}, the equation $R(\lambda) = c\lambda$ has two positive roots λ_2 and λ_3 with $\lambda_2 < \lambda_3$ such that

$$R(\lambda) < c\lambda, \ \forall\, \lambda \in (\lambda_2, \lambda_3); \quad R(\lambda) > c\lambda, \ \forall\, \lambda \in [0, \lambda_2) \cup (\lambda_3, \infty). \tag{3.2}$$

Note that $\lambda_2 > \lambda_1$. Indeed, if $\lambda_2 \leq \lambda_1$, then

$$R(\lambda_2) = c\lambda_2 \leq Q(\lambda_2),$$

a contradiction. Hence $\lambda_2 > \lambda_1$.

3.1. **Case under Assumption 1.1.** Choose $\nu > 1$ such that $\nu\lambda_1 \in (\lambda_2, \lambda_3)$. For a given constant $h > 1$, set

$$\psi(\xi) := e^{\lambda_1 \xi} - h e^{\nu\lambda_1 \xi}, \ \xi_0 := \frac{-\ln h}{(\nu - 1)\lambda_1}, \ \xi_M := \frac{-\ln(h\nu)}{(\nu - 1)\lambda_1}.$$

Then $\psi(\xi)$ is positive if and only if $\xi < \xi_0$, $\psi(\xi_0) = 0$ and $\psi < 0$ in (ξ_0, ∞). Moreover,

$$\psi(\xi) \leq \psi(\xi_M) = C(\nu) h^{-1/(\nu-1)}, \ \forall\, \xi \in \mathbb{R}, \tag{3.3}$$

for some positive constant $C = C(\nu) := \nu^{-1/(\nu-1)}(1 - 1/\nu)$. Then we choose a constant h large enough such that $\psi(\xi_M) \leq \eta$, where η is defined in (1.9).

With these λ_1, ν and h, we introduce the functions

$$\overline{\phi}(\xi) = \min\{e^{\lambda_1 \xi} + h e^{\nu\lambda_1 \xi}, 1 + \delta\}, \quad \underline{\phi}(\xi) = \max\{e^{\lambda_1 \xi} - h e^{\nu\lambda_1 \xi}, -\gamma\}, \tag{3.4}$$

where constants $\delta \in (0, 1)$ and $\gamma \in (a, 2a)$ are given so that (1.10) holds.

Now, we verify the functions $\{\overline{\phi}, \underline{\phi}\}$ are upper and lower solutions of (1.4). For convenience, we introduce

$$N_1(\xi) := -c\overline{\phi}'(\xi) + (K^+ * \overline{\phi})(\xi) - (K^- * \underline{\phi})(\xi) - \alpha\overline{\phi}(\xi) + f(\overline{\phi}(\xi)),$$
$$N_2(\xi) := -c\underline{\phi}'(\xi) + (K^+ * \underline{\phi})(\xi) - (K^- * \overline{\phi})(\xi) - \alpha\underline{\phi}(\xi) + f(\underline{\phi}(\xi)).$$

Then (2.1) ((2.2), resp.) is equivalent to $N_1(\xi) \leq 0$ ($N_2(\xi) \geq 0$, resp.) in $\mathbb{R} \setminus \{\xi_1, \xi_2\}$.

Lemma 3.1. *Let Assumption 1.1 be enforced. If $c > c^{**}$, then the functions $\{\overline{\phi}, \underline{\phi}\}$ defined by* (3.4) *are upper and lower solutions of* (1.4).

Proof. Set ξ_i, $i = 1, 2$, to be

$$e^{\lambda_1 \xi_1} + h e^{\nu\lambda_1 \xi_1} = 1 + \delta, \quad e^{\lambda_1 \xi_2} - h e^{\nu\lambda_1 \xi_2} = -\gamma.$$

Then we have

$$\overline{\phi}(\xi) = \begin{cases} e^{\lambda_1 \xi} + h e^{\nu\lambda_1 \xi}, & \xi \leq \xi_1, \\ 1 + \delta, & \xi \geq \xi_1, \end{cases} \quad \underline{\phi}(\xi) = \begin{cases} e^{\lambda_1 \xi} - h e^{\nu\lambda_1 \xi}, & \xi \leq \xi_2, \\ -\gamma, & \xi \geq \xi_2. \end{cases}$$

We first consider $N_1(\xi)$. When $\xi < \xi_1$, we have $\overline{\phi}(\xi) = e^{\lambda_1\xi} + he^{\nu\lambda_1\xi}$. Using $\overline{\phi}(\xi) \le e^{\lambda_1\xi} + he^{\nu\lambda_1\xi} \le 1+\delta \le u^+$ and $\underline{\phi}(\xi) \ge e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}$ for all $\xi \in \mathbb{R}$, we obtain

$$\begin{aligned} N_1(\xi) &\le -c(\lambda_1 e^{\lambda_1\xi} + h\nu\lambda_1 e^{\nu\lambda_1\xi}) + \int_{\mathbb{R}} K^+(y)[e^{\lambda_1(\xi-y)} + he^{\nu\lambda_1(\xi-y)}]dy \\ &\quad - \int_{\mathbb{R}} K^-(y)[e^{\lambda_1(\xi-y)} - he^{\nu\lambda_1(\xi-y)}]dy - \alpha(e^{\lambda_1\xi} + he^{\nu\lambda_1\xi}) + f(e^{\lambda_1\xi} + he^{\nu\lambda_1\xi}) \\ &= e^{\lambda_1\xi}\left\{-c\lambda_1 + I^+(\lambda_1) - I^-(\lambda_1) - \alpha\right\} \\ &\quad + he^{\nu\lambda_1\xi}\left\{-c\nu\lambda_1 + I^+(\nu\lambda_1) + I^-(\nu\lambda_1) - \alpha\right\} + f(e^{\lambda_1\xi} + he^{\nu\lambda_1\xi}) \\ &= e^{\lambda_1\xi}\left\{-c\lambda_1 + Q(\lambda_1) - f'(0)\right\} \\ &\quad + he^{\nu\lambda_1\xi}\left\{-c\nu\lambda_1 + R(\nu\lambda_1) - f'(0)\right\} + f(e^{\lambda_1\xi} + he^{\nu\lambda_1\xi}) \\ &< f(e^{\lambda_1\xi} + he^{\nu\lambda_1\xi}) - f'(0)(e^{\lambda_1\xi} + he^{\nu\lambda_1\xi}) \le 0 \end{aligned}$$

for $\xi < \xi_1$, by the conditions (3.1), (3.2) and (1.6).

When $\xi > \xi_1$, $\overline{\phi}(\xi) = 1+\delta$. Using $\overline{\phi}(\xi) \le 1+\delta$ and $\underline{\phi}(\xi) \ge -\gamma$ for all $\xi \in \mathbb{R}$, we compute

$$\begin{aligned} N_1(\xi) &\le 0 + (1+\delta)\int_{\mathbb{R}} K^+(y)dy + \gamma\int_{\mathbb{R}} K^-(y)dy - \alpha(1+\delta) + f(1+\delta) \\ &= (1+\delta+\gamma)\int_{\mathbb{R}} K^-(y)dy + f(1+\delta) \le 0 \end{aligned}$$

for $\xi > \xi_1$, due to (1.11). Therefore, we obtain $N_1(\xi) \le 0$ for all $\xi \in \mathbb{R} \setminus \{\xi_1\}$.

Next, we consider $N_2(\xi)$. When $\xi < \xi_2$, we have $\underline{\phi}(\xi) = e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}$. Then as before we obtain

$$\begin{aligned} N_2(\xi) &\ge -c(\lambda_1 e^{\lambda_1\xi} - h\nu\lambda_1 e^{\nu\lambda_1\xi}) + \int_{\mathbb{R}} K^+(y)[e^{\lambda_1(\xi-y)} - he^{\nu\lambda_1(\xi-y)}]dy \\ &\quad - \int_{\mathbb{R}} K^-(y)[e^{\lambda_1(\xi-y)} + he^{\nu\lambda_1(\xi-y)}]dy - \alpha(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) + f(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) \\ &= e^{\lambda_1\xi}\left\{-c\lambda_1 + I^+(\lambda_1) - I^-(\lambda_1) - \alpha\right\} \\ &\quad - he^{\nu\lambda_1\xi}\left\{-c\nu\lambda_1 + I^+(\nu\lambda_1) + I^-(\nu\lambda_1) - \alpha\right\} + f(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) \\ &= e^{\lambda_1\xi}\left\{-c\lambda_1 + Q(\lambda_1) - f'(0)\right\} \\ &\quad - he^{\nu\lambda_1\xi}\left\{-c\nu\lambda_1 + R(\nu\lambda_1) - f'(0)\right\} + f(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) \\ &> f(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) - f'(0)(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) \end{aligned}$$

for $\xi < \xi_2$, by using condition (3.1) and (3.2).

If $e^{\lambda_1\xi} - he^{\nu\lambda_1\xi} \le 0$, then $f(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) - f'(0)(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) \ge 0$ using conditions (1.6) and (1.10). Here $u^- \le -\gamma \le e^{\lambda_1\xi} - he^{\nu\lambda_1\xi} \le 0$ were used. On the other hand, suppose that $e^{\lambda_1\xi} - he^{\nu\lambda_1\xi} \ge 0$. By (3.3) and the choice of h, we have $\psi(\xi) \le \eta$ for all $\xi \in \mathbb{R}$. Then, we deduce from (1.9) that $f(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) - f'(0)(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) = 0$. Hence $N_2(\xi) \ge 0$ for all $\xi < \xi_2$.

When $\xi > \xi_2$, we know that $\underline{\phi}(\xi) = -\gamma$. Then we have

$$\begin{aligned} N_2(\xi) &\geq 0 - \gamma \int_{\mathbb{R}} K^+(y)dy - (1+\delta)\int_{\mathbb{R}} K^-(y)dy + \alpha\gamma + f(-\gamma) \\ &\geq -(1+\delta+\gamma)\int_{\mathbb{R}} K^-(y)dy + f(-\gamma) \geq 0 \end{aligned}$$

for $\xi > \xi_2$, due to (1.11). Therefore, $N_2(\xi) \geq 0$ for all $\xi \in \mathbb{R} \setminus \{\xi_2\}$.

We conclude that $\{\overline{\phi}, \underline{\phi}\}$ is a pair of upper-lower-solution to (1.4). □

Hence Theorem 1.3 is proved by applying Lemma 3.1 and Proposition 2.2.

3.2. **Case under Assumption 1.2.** Recall the function ψ defined in Subsection 3.1. With λ_1 in (3.1), $\nu > 1$ such that $\nu\lambda_1 \in (\lambda_2, \lambda_3)$ and the constants δ and η in Assumption 1.2, we introduce the functions

$$\overline{\phi}(\xi) = \begin{cases} e^{\lambda_1\xi} + he^{\nu\lambda_1\xi}, & \xi \leq \xi_1, \\ 1+\delta, & \xi \geq \xi_1, \end{cases} \quad \underline{\phi}(\xi) = \begin{cases} e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}, & \xi \leq \xi_M, \\ \eta, & \xi \geq \xi_M. \end{cases} \tag{3.5}$$

where the constants ξ_1 and h are chosen so that

$$e^{\lambda_1\xi_1} + he^{\nu\lambda_1\xi_1} = 1+\delta, \quad \psi(\xi_M) = \eta.$$

Now, we verify the functions $\{\overline{\phi}, \underline{\phi}\}$ are upper and lower solutions of (1.4).

Lemma 3.2. *Let Assumption 1.2 be enforced. Assume that $c > c^{**}$. Then the functions $\{\overline{\phi}, \underline{\phi}\}$ defined by* (3.5) *are upper and lower solutions of* (1.4).

Proof. As in Subsection 3.1, we have $N_1(\xi) \leq 0$ for all $\xi < \xi_1$.

When $\xi > \xi_1$, $\overline{\phi}(\xi) = 1+\delta$. Using $\overline{\phi}(\xi) \leq 1+\delta$ and $\underline{\phi}(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, we compute

$$\begin{aligned} N_1(\xi) &\leq 0 + (1+\delta)\int_{\mathbb{R}} K^+(y)dy - \alpha(1+\delta) + f(1+\delta) \\ &= (1+\delta)\int_{\mathbb{R}} K^-(y)dy + f(1+\delta) \leq 0 \end{aligned}$$

for $\xi > \xi_1$, due to (1.15). Therefore, $N_1(\xi) \leq 0$ for all $\xi \in \mathbb{R} \setminus \{\xi_1\}$.

For $\xi < \xi_M$, we have $\underline{\phi}(\xi) = e^{\lambda_1\xi} - he^{\nu\lambda_1\xi} \in (0, \eta]$. Then as in Subsection 3.1 we obtain

$$\begin{aligned} N_2(\xi) &\geq -c(\lambda_1 e^{\lambda_1\xi} - h\nu\lambda_1 e^{\nu\lambda_1\xi}) + e^{\lambda_1\xi} I^+(\lambda_1) - he^{\nu\lambda_1\xi} I^+(\nu\lambda_1) \\ &\qquad - e^{\lambda_1\xi} I^-(\lambda_1) - he^{\nu\lambda_1\xi} I^-(\nu\lambda_1) - \alpha(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) + f(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) \\ &= e^{\lambda_1\xi}\left\{-c\lambda_1 + I^+(\lambda_1) - I^-(\lambda_1) - \alpha\right\} \\ &\qquad - he^{\nu\lambda_1\xi}\left\{-c\nu\lambda_1 + I^+(\nu\lambda_1) + I^-(\nu\lambda_1) - \alpha\right\} + f(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) \\ &> f(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) - f'(0)(e^{\lambda_1\xi} - he^{\nu\lambda_1\xi}) = 0 \end{aligned}$$

for $\xi < \xi_M$, by using conditions (3.1), (3.2) and (1.9).

When $\xi > \xi_M$, we have $\underline{\phi}(\xi) = \eta$. Then we have

$$\begin{aligned} N_2(\xi) &\geq 0 + 0 - (1+\delta)\int_{\mathbb{R}} K^-(y)dy - \alpha\eta + f(\eta) \\ &= -(1+\delta)\int_{\mathbb{R}} K^-(y)dy - \alpha\eta + f'(0)\eta \geq 0 \end{aligned}$$

for $\xi > \xi_M$, due to (1.9) and (1.15). Therefore, $N_2(\xi) \geq 0$ for all $\xi \in \mathbb{R} \setminus \{\xi_M\}$. We conclude that $\{\overline{\phi}, \underline{\phi}\}$ is a pair of upper-lower-solution to (1.4). □

Finally, applying Lemma 3.2 and Proposition 2.2, Theorem 1.4 follows with a nonnegative wave profile ϕ, since $\underline{\phi} \geq 0$.

4. Right-hand tail of wave profile

In this section, we investigate the behavior of a solution to (1.4) at $\xi = \infty$. In the sequel, we denote by $\|g\|_\infty$ the supremum of $|g|$ over $\mathbb{R}$.

Proposition 4.1. *Let ϕ be a solution to* (1.4) *obtained in Theorems 1.3 and 1.4. Suppose that the limit $l := \lim_{\xi\to\infty} \phi(\xi)$ exists. Then $f(l) = 0$.*

Proof. First, by assumption, we can find a sequence $\{\xi_n\}$ with $\xi_n \to \infty$ and $\phi'(\xi_n) \to 0$ as $n \to \infty$.

To prove the proposition, it suffices to prove that

$$(K * \phi)(\xi_n) \to \alpha l \quad \text{as} \quad n \to \infty. \tag{4.1}$$

Now, given $\epsilon > 0$ small enough. Since $\phi(\xi) \to l$ as $\xi \to \infty$ and by (1.2), there is $M \gg 1$ such that

$$|\phi(\xi) - l| < \epsilon/\{4[I^+(0) + I^-(0)]\}, \quad \forall\, \xi \geq M, \tag{4.2}$$

$$\int_{|y|\geq M} K^\pm(y)dy < \epsilon/[4\|\phi\|_\infty], \tag{4.3}$$

$$\Big|\int_{|y|\geq M} K(y)dy\Big| < \epsilon/[4(|l|+1)]. \tag{4.4}$$

Next, we choose $N \gg 1$ such that $\xi_n \geq 2M$ for all $n \geq N$. Then, by (4.2),

$$|\phi(\xi_n - y) - l| < \epsilon/\{4[I^+(0) + I^-(0)]\}, \quad \forall\, n \geq N,\ \forall\, y \in [-M, M], \tag{4.5}$$

since $\xi_n - y \geq 2M - M = M$.

We compute

$$\begin{aligned} \Big|\int_{\mathbb{R}} K(y)\phi(\xi_n - y)dy - \alpha l\Big| &\leq \Big|\int_{|y|\geq M} K(y)\phi(\xi_n - y)dy\Big| \\ &\quad + \Big|\int_{-M}^{M} K(y)[\phi(\xi_n - y) - l]dy\Big| + \Big|\int_{-M}^{M} K(y)dy - \alpha\Big| \cdot |l|. \end{aligned}$$

Since, for $n \geq N$,

$$\Big| \int_{|y|\geq M} K(y)\phi(\xi_n - y)dy \Big| \leq \left(\int_{|y|\geq M} K^+(y)dy + \int_{|y|\geq M} K^-(y)dy \right) \|\phi\|_\infty < \epsilon/2,$$
$$\Big| \int_{-M}^{M} K(y)[\phi(\xi_n - y) - l]dy \Big| \leq \int_{-M}^{M} |K(y)|dy \cdot \epsilon/\{4[I^+(0) + I^-(0)]\} \leq \epsilon/4,$$
$$\Big| \int_{-M}^{M} K(y)dy - \alpha \Big| \cdot |l| = \Big| \int_{|y|\geq M} K(y)dy \Big| \cdot |l| < \epsilon/4,$$

by using (4.3), (4.5) and (4.4). Hence (4.1) follows.

Finally, putting ξ_n into (1.4) and letting $n \to \infty$, we deduce that $f(l) = 0$. Hence the proposition is proved. □

To prove that the right-hand tail converges, we find a condition for the derivative of a solution to (1.4) converges to 0 at $\xi = \infty$, using L^2 estimates based on a method of [2]. For convenience, we define

$$m_i := \int_{\mathbb{R}} |y^i K(y)|dy,\ i = 0, 1, 2.$$

Note that m_i is well-defined, since K has a compact support. Also, we define the set

$$C_b^k(\mathbb{R}) := \{g \in C^k(\mathbb{R}) \mid ||g^{(j)}||_\infty < \infty,\ j = 0, 1, ..., k\}.$$

Lemma 4.2. *Let $(c, \phi) \in \mathbb{R} \times C_b^2(\mathbb{R})$ be a solution to (1.4). Suppose that $c > \sqrt{m_0 m_2}$. Then $\phi' \in L^2(\mathbb{R})$ and $\phi'(\infty) = 0$.*

Proof. Let us define

$$F(u) := \int_0^u f(s)ds,\ M := ||\phi||_\infty,\ M' := ||\phi'||_\infty,\ M_f := \max_{u\in[-M,M]} |F(u)|.$$

We multiply (1.4) by ϕ' and then integrate from $-q < 0$ to $p > 0$ to get

$$\begin{aligned} 0 \leq c\int_{-q}^{p} (\phi')^2(\xi)d\xi &= \int_{-q}^{p} \{\phi'[(K * \phi) - \alpha\phi + f(\phi)]\}(\xi)d\xi \\ &= \int_{-q}^{p} [\phi'\{(K * \phi) - \alpha\phi\} + (F(\phi))'] (\xi)d\xi \\ &\leq \left(\int_{-q}^{p} (\phi')^2(\xi)d\xi \right)^{1/2} \left(\int_{-q}^{p} \{[(K * \phi) - \alpha\phi](\xi)\}^2 d\xi \right)^{1/2} + 2M_f. \end{aligned}$$

For $\xi \in \mathbb{R}$, we write

$$\begin{aligned} [(K * \phi) - \alpha\phi](\xi) &= \int_{\mathbb{R}} K(\xi - y)(\phi(y) - \phi(\xi))dy \\ &= \int_{\mathbb{R}} \int_0^1 K(\xi - y)(y - \xi)\phi'(\xi + s(y - \xi))dsdy. \end{aligned}$$

Applying the Cauchy-Schwartz inequality yields

$$\begin{aligned}
&\{[(K*\phi)-\alpha\phi](\xi)\}^2\\
\le\ &\left(\int_{\mathbb{R}}\int_0^1 |K(y-\xi)(y-\xi)\phi'(\xi+s(y-\xi))|dsdy\right)^2\\
\le\ &\left(\int_{\mathbb{R}}\int_0^1 |K(y-\xi)|(y-\xi)^2dsdy\right)\left(\int_{\mathbb{R}}\int_0^1 |K(y-\xi)|[\phi'(\xi+s(y-\xi))]^2dsdy\right)\\
=\ &m_2\left(\int_{\mathbb{R}}\int_0^1 |K(z)|(\phi'(\xi+sz))^2dsdz\right).
\end{aligned}$$

Therefore, we compute

$$\begin{aligned}
\int_{-q}^{p}\{[(K*\phi)-\alpha\phi](\xi)\}^2d\xi &\le m_2\int_{-q}^{p}\int_{\mathbb{R}}\int_0^1 |K(z)|[\phi'(\xi+sz)]^2dsdzd\xi\\
&= m_2\int_{\mathbb{R}}\int_0^1 |K(z)|\int_{-q}^{p}[\phi'(\xi+sz)]^2d\xi dsdz\\
&= m_2\int_{\mathbb{R}}\int_0^1 |K(z)|\int_{-q+sz}^{p+sz}[\phi'(\xi)]^2d\xi dsdz.
\end{aligned}$$

Now, using $|u'|\le M'$, we get

$$\begin{aligned}
\int_{-q+sz}^{p+sz}[\phi'(\xi)]^2d\xi &= \int_{-q+sz}^{-q}[\phi'(\xi)]^2d\xi+\int_{-q}^{p}[\phi'(\xi)]^2d\xi+\int_{p}^{p+sz}[\phi'(\xi)]^2d\xi\\
&\le \int_{-q}^{p}[\phi'(\xi)]^2d\xi+2(M')^2s|z|,\ \forall\, s\in[0,1],\ z\in\mathbb{R}.
\end{aligned}$$

This implies that

$$\int_{-q}^{p}\{[(K*\phi)-\alpha\phi](\xi)\}^2d\xi\le m_2\left\{m_0\int_{-q}^{p}(\phi')^2+(M')^2m_1\right\}.$$

Therefore, we see that

$$c\int_{-q}^{p}(\phi')^2 \ \le\ \sqrt{m_2}\left\{m_0\left(\int_{-q}^{p}(\phi')^2\right)^2+(M')^2m_1\left(\int_{-q}^{p}(\phi')^2\right)\right\}^{1/2}+2M_f.$$

If $c>\sqrt{m_0m_2}$, then $\{\int_{-q}^{p}(\phi')^2\mid p>0,\ q>0\}$ is uniformly bounded and so $\phi'\in L^2(\mathbb{R})$. Since ϕ' is uniformly continuous on $\mathbb{R}$, this implies $\phi'(\infty)=0$. The lemma is proved. □

Remark 4.3. *Let ϕ be a solution obtained in Theorems 1.3 and 1.4. Since ϕ is continuous and bounded in $\mathbb{R}$, by* (1.4)*, ϕ' is also continuous and bounded in $\mathbb{R}$. In the case when $f\in C^1$, by differentiating* (1.4) *once, we see easily that ϕ'' is also continuous and bounded in $\mathbb{R}$, i.e., $\phi\in C_b^2(\mathbb{R})$.*

Proposition 4.4. *Let $(c,\phi)\in\mathbb{R}\times C_b^2(\mathbb{R})$ be a solution to* (1.4)*. Suppose that $c>\sqrt{m_0m_2}$ and $\{u\in[-||\phi||_\infty,||\phi||_\infty]\mid f(u)=0\}=\{-a,0,1\}$. Then $\phi(\infty)$ exists and belongs to $\{-a,0,1\}$.*

Proof. Denote by $\mathcal{A}$ the set of accumulation points of ϕ at $+\infty$. Since ϕ is bounded, $\mathcal{A}$ is not empty. Let $l \in \mathcal{A}$. Then there exists a sequence $\xi_n \to +\infty$ such that $\phi(\xi_n) \to l$ as $n \to \infty$. Then $\psi_n(\xi) := \phi(\xi + \xi_n)$ satisfies

$$c\psi_n'(\xi) = (K * \psi_n)(\xi) - \alpha\psi_n(\xi) + f(\psi_n(\xi)),\ \xi \in \mathbb{R}.$$

For all $L > 0$ and all $1 < p < \infty$, the sequence $\{\psi_n\}$ is bounded in $W^{2,p}([-L, L])$. From the Sobolev embedding theorem, there exists a subsequence $\{\psi_{n(k)}\}$ of $\{\psi_n\}$ such that $\psi_{n(k)} \to \psi$ as $k \to \infty$ strongly in $C^1_{loc}(\mathbb{R})$ and weakly in $W^{1,p}_{loc}(\mathbb{R})$. It follows Lemma 4.2 that

$$\psi'(\xi) = \lim_{k\to\infty} \phi'(\xi + \xi_{n(k)}) = 0,\ \forall\, \xi \in \mathbb{R}.$$

Combining this with the fact that ψ solves

$$c\psi'(\xi) = (K * \psi)(\xi) - \alpha\psi(\xi) + f(\psi(\xi)),\ \xi \in \mathbb{R},$$

we deduce that $\psi(\xi) \in \{u \in [-||\phi||_\infty, ||\phi||_\infty] \mid f(u) = 0\}$ for all $\xi \in \mathbb{R}$. In particular,

$$l = \lim_{k\to\infty} \phi(\xi_{n(k)}) = \psi(0) \in \{u \in [-||\phi||_\infty, ||\phi||_\infty] \mid f(u) = 0\}.$$

Hence, by assumption, $l \in \{-a, 0, 1\}$. Since ϕ is a continuous function, $\mathcal{A}$ is connected. Therefore, $\phi(\infty)$ exists and belongs to $\{-a, 0, 1\}$. The proof is complete. □

Applying this proposition and using (3.5), we obtain the following corollary.

Corollary 4.5. *Let Assumption 1.2 be enforced. Also, assume that* $f \in C^1(\mathbb{R})$ *and* $\{u \in [u^-, u^+] \mid f(u) = 0\} = \{-a, 0, 1\}$*. Then for any* $c > \max\{c^{**}, \sqrt{m_0 m_2}\}$ *equation* (1.4) *has a nonnegative solution* ϕ *such that* $\phi(-\infty) = 0$ *and* $\phi(+\infty) = 1$*.*

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