STABILITY OF TRAVELING WAVES IN NON-COOPERATIVE MODELS WITH DISCRETE DIFFUSION

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ABSTRACT. In this paper, we investigate the stability of traveling waves in non-cooperative reaction-diffusion systems with discrete diffusion, where the classical comparison principle does not apply. We employ a Lyapunov-type relative entropy functional, inspired by the entropy method for ordinary differential equations, to discuss the stability of traveling waves without relying on spectral techniques for the linearized operator. Using this approach, we establish sufficient conditions for the convergence of solutions to traveling waves under certain perturbations of a class of initial data, in both equal and unequal diffusivity cases. We further provide a few specific biological models, including predator-prey dynamics and epidemic transmission models, to illustrate the application of our theorems.

1. INTRODUCTION

We consider the following reaction-diffusion system with discrete diffusion:

$$(\tilde{u}_i)_t(x,t) = d_i\mathcal{D}_2[\tilde{u}_i](x,t) + \tilde{u}_i(x,t)g_i(\tilde{u}(x,t)), \quad x \in \mathbb{R}, \quad t > 0, \quad i = 1,\dots,m, \tag{1.1}$$

where $\tilde{u} = (\tilde{u}_1,\dots,\tilde{u}_m)$, $d_i > 0$, and

$$\mathcal{D}_2[\tilde{u}_i](x,t) := \tilde{u}_i(x+1,t) - 2\tilde{u}_i(x,t) + \tilde{u}_i(x-1,t).$$

Here, m is a positive integer representing the number of interacting species or components. Note that the spatially discrete analog of system (1.1), namely,

$$\tilde{u}'_{i,j}(t) = d_i\mathcal{D}_{i,j}[\tilde{u}](t) + \tilde{u}_{i,j}(t)g_i((\tilde{u}_{1,j},\dots,\tilde{u}_{m,j})(t)),\ j \in \mathbb{Z},\ t > 0,\ i = 1,\dots,m, \tag{1.2}$$

where $\mathcal{D}_{i,j}[\tilde{u}](t) := \tilde{u}_{i,j+1}(t) - 2\tilde{u}_{i,j}(t) + \tilde{u}_{i,j-1}(t)$, is called a lattice dynamical system.

We assume that each g_i is smooth on an open subset $\Omega \subset \mathbb{R}^m$ with $\mathcal{I} = \prod_{i=1}^m[\underline{M}_i, \overline{M}_i] \subset \Omega$ for some $\overline{M}_i > \underline{M}_i \geq 0$. The corresponding kinetic system to (1.1), obtained by removing the spatial dependence from (1.1), reads

$$\frac{d\bar{u}_i}{dt}(t) = \bar{u}_i(t)g_i(\bar{u}(t)), \quad i = 1,\dots,m, \tag{1.3}$$

where $\bar{u} = (\bar{u}_1,\dots,\bar{u}_m)$. We assume that system (1.3) has two distinct equilibria $\{E_\pm\}$.

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A positive traveling wave of (1.1) connecting $\{E_\pm\}$ is a special solution of (1.1) in the form

$$\tilde{u}_i(x,t) = \phi_i(z),\ z := x - ct,\ i = 1, \dots, m,$$

for some profile function $\Phi := (\phi_1, \dots, \phi_m)$ such that $\phi_i(z) > 0$ for all $z \in \mathbb{R}$ and $i = 1, \dots, m$, together with the following limits at $z = \pm\infty$

$$\Phi(-\infty) = E_-,\ \Phi(\infty) = E_+. \tag{1.4}$$

Here c is a positive constant and is called the wave speed. Note that a traveling wave profile $\Phi = (\phi_1, \cdots, \phi_m)$ with the speed c satisfies the following system of equations

$$d_i \mathcal{D}_2[\phi_i](z) + c\phi_i'(z) + \phi_i g_i(\Phi(z)) = 0, \quad z \in \mathbb{R},\ i = 1, \dots, m, \tag{1.5}$$

where $\mathcal{D}_2[\phi_i](z) := \phi_i(z+1) - 2\phi_i(z) + \phi_i(z-1)$. As for the lattice dynamical system (1.2), a traveling wave with speed c is a solution in the form

$$\tilde{u}_{i,j}(t) = \phi_i(z),\ z := j - ct,\ j \in \mathbb{Z},\ i = 1, \dots, m.$$

One can check that the wave profile $\Phi = (\phi_1, \cdots, \phi_m)$ also satisfies the same system (1.5). Therefore, a traveling wave of (1.2) is the same as that of (1.1).

The study on the existence of traveling waves for system (1.2) can be traced back to the works by Zinner [22, 23] for scalar equations ($m = 1$). Since then, the existence of traveling waves for lattice dynamical systems has been extensively studied. We refer the reader to, e.g., [7, 2, 3, 4] for scalar equations, [12, 17, 16, 18] for cooperative multiple component systems, and [11, 5, 6, 8, 20] for non-cooperative systems. As far as the stability is concerned, there are many works on the stability of traveling waves in cooperative systems with discrete diffusion (see, e.g., [21, 2, 19, 1, 13]). However, for non-cooperative systems with discrete diffusion, the stability theory remains underdeveloped due to the lack of the comparison principle.

Our primary aim of this study is the stability of traveling waves in the context of discrete diffusion systems, particularly in the absence of the comparison principle due to non-cooperativity. To tackle this issue, we employ a Lyapunov-type relative entropy functional, inspired by the classical entropy method for ODE systems. This approach, recently developed in [14], provides a new route to study the stability of traveling waves without relying on spectral analysis of the linearized operator. In this paper, we first derive sufficient conditions for the convergence of solutions toward traveling waves when diffusion coefficients are equal (Theorem 1.1). We then generalize the approach to cover the case of non-equal diffusion coefficients under suitable assumptions (Theorem 1.2).

Let $\tilde{u} = (\tilde{u}_1, \cdots, \tilde{u}_m)$ be a positive solution of (1.1). We introduce the moving coordinate $z = x - ct$ as well as the functions $u = (u_1, \cdots, u_m)$ with

$$u_i(z,t) = \tilde{u}_i(z + ct, t)$$

so that the maps $\{u_i = u_i(z,t)\}_{i=1,..,m}$ satisfy

$$(u_i)_t = d_i \mathcal{D}_2[u_i] + c(u_i)_z + u_i g_i(u), \quad z \in \mathbb{R},\ t > 0,\ i = 1, \dots, m. \tag{1.6}$$

Note that Φ is a stationary solution of (1.6). Throughout this paper, we assume that (1.1) admits positive traveling waves $\{c, \Phi\}$ for all $c \geq c^*$ for some positive constant c^*. Hereafter Φ is positive means $\phi_i(z) > 0$ for all $z \in \mathbb{R}$ and for all $i = 1, .., m$.

Also, for a given positive constant R, we define the quantity

$$c_R^i := \min_{\lambda>0} G_i(\lambda), \quad G_i(\lambda) := \frac{d_i(e^\lambda + e^{-\lambda} - 2) + R}{\lambda}. \tag{1.7}$$

Note that the function $G_i : (0, \infty) \to (0, \infty)$ has only one minimum value $\lambda_{\min}^i$, and the function G_i is monotone decreasing on $(0, \lambda_{\min}^i)$, and it is monotone increasing on $(\lambda_{\min}^i, \infty)$. Moreover, one has $G_i(0^+) = G_i(\infty) = \infty$. Thus the quantity c_R^i is well-defined and we have $c_R^i = G_i(\lambda_{\min}^i) > 0$. In addition, due to these properties of the function G_i, the equation $G_i(\lambda) = c$ has two positive roots when $c > c_R^i$, and has only one positive root (namely $\lambda = \lambda_{\min}^i$) when $c = c_R^i$. Denote a maximum diffusion constant by

$$d_k := \max_{1\leq i\leq m} d_i,$$

and observe that the inequality $G_k(\lambda) \geq G_i(\lambda)$ holds for any $i = 1, .., m$, since the function $G_i(\lambda)$ is monotone increasing with respect to the parameter d_i. This yields the following relation

$$c_R := c_R^k = \max_{1\leq i\leq m} c_R^i. \tag{1.8}$$

In particular, if $c > c_R$, then there exists a constant $\mu > 0$ such that

$$d_i(e^\mu + e^{-\mu} - 2) - c\mu + R < 0, \qquad i = 1, 2, \cdots, m. \tag{1.9}$$

Let $\{\sigma_i \mid 1 \leq i \leq m\}$ be a set of positive constants. When $\bar{u}(t) = (\bar{u}_1(t), \cdots, \bar{u}_m(t))$ is a solution of (1.3), and $u^* = (u_1^*, \cdots, u_m^*)$ is a positive equilibrium of (1.3), the time derivative of

$$\mathcal{C}[\bar{u}(t)] := \sum_{i=1}^m \sigma_i \Big(\bar{u}_i(t) - u_i^* - u_i^* \log \frac{\bar{u}_i(t)}{u_i^*}\Big) \tag{1.10}$$

is computed as

$$\frac{d}{dt}\mathcal{C}(\bar{u}(t)) = \sum_{i=1}^m \sigma_i \Big(1 - \frac{u_i^*}{\bar{u}_i(t)}\Big) \frac{d\bar{u}_i(t)}{dt} = \sum_{i=1}^m \sigma_i \left(\bar{u}_i(t) - u_i^*\right) \{g_i(\bar{u}(t)) - g_i(u^*)\}, \tag{1.11}$$

since $g_i(u^*) = 0$ for all $i = 1, \dots, m$. There are many ordinary differential systems in physical and biological phenomena which have Lyapunov functionals of the form (1.10), in other words the right-hand side of (1.11) is non-positive.

Motivated from the form (1.10), for a given set of positive constant $\{\sigma_i \mid 1 \leq i \leq m\}$, we introduce the following relative entropy function

$$\mathcal{K}[U] := \sum_{i=1}^m \sigma_i \mathcal{K}_i[U_i], \quad \mathcal{K}_i[U_i] := U_i - \phi_i - \phi_i \ln \frac{U_i}{\phi_i}, \tag{1.12}$$

for a positive function $U(z) = (U_1(z), \dots, U_m(z))$ defined in $z \in \mathbb{R}$. Note that $\mathcal{K}[U](z) \geq 0$ for all $z \in \mathbb{R}$ and $\mathcal{K}[U](z) = 0$ if and only if $U(z) = \Phi(z)$ for some $z \in \mathbb{R}$. Then we define the related entropy function of $u(z,t)$ by

$$W(z,t) := \mathcal{K}[u(\cdot,t)](z),\ z \in \mathbb{R},\ t > 0. \tag{1.13}$$

In this paper, we denote the Fourier transformation and the Fourier inverse transform of a function $v \in L^1(\mathbb{R})$ by

$$\mathcal{F}[v](\xi) = \int_{\mathbb{R}} v(z) e^{-i\xi z}\, dz, \qquad \mathcal{F}^*[v](z) = \frac{1}{2\pi} \int_{\mathbb{R}} v(\xi) e^{iz\xi}\, d\xi.$$

Hereafter, we denote by $BUC(\mathbb{R}, F)$ the space of all bounded and uniformly continuous functions from $\mathbb{R}$ to a closed subset $F \subset \mathbb{R}^m$.

Now, we are ready to state the following convergence theorem for the traveling wave of system (1.6) when all the diffusion coefficients are equal. Note that $c_R := c_R^k = c_R^i$ for all $1 \leq i \leq m$ in this case.

Theorem 1.1. *Assume that system* (1.1) *has a bounded invariant set* $\mathcal{I} := \prod_{i=1}^m [\underline{M}_i, \overline{M}_i]$ *for some* $0 \leq \underline{M}_i < \overline{M}_i < \infty$ *with* $i = 1, 2, \cdots, m$*. Assume that* $d_i = 1$ *for all* $1 \leq i \leq m$*. Let* R *be a positive constant such that* $c_R \geq c^*$*, and let* $\{c, \Phi\}$ *be a positive traveling wave of* (1.1) *for some* $c \geq c_R$ *such that* $\Phi(z) \in \mathcal{I}$ *for all* $z \in \mathbb{R}$*, where* c_R *is a constant defined by* (1.8)*. Suppose that there exists a set of positive constants* $\{\sigma_i \mid 1 \leq i \leq m\}$ *such that*

$$\mathcal{M}(u,v) := \sum_{i=1}^m \sigma_i (u_i - v_i) \{g_i(u) - g_i(v)\} \leq 0 \quad \text{for} \quad u, v \in \mathcal{I}, \tag{1.14}$$

where $u = (u_1, \dots u_m)$ *and* $v = (v_1, \dots, v_m)$*. Moreover, assume that*

$$\max_{1 \leq i \leq m} \{\|g_i(\Phi)\|_{L^\infty(\mathbb{R})}\} \leq R.$$

Let u *be a solution of* (1.6) *with the positive initial data* $u^0 = (u_1^0, ..., u_m^0) \in BUC(\mathbb{R}, \mathcal{I})$ *such that*

$$z \mapsto e^{\lambda_c z} \mathcal{K}[u^0](z) \in L^1(\mathbb{R}) \text{ and } \mathcal{F}[e^{\lambda_c \cdot} \mathcal{K}[u^0]] \in L^1(\mathbb{R}),$$

where λ_c *is the smallest positive root of* $G_1(\lambda) = c$*. Then*

$$u(z,t) \to \Phi(z) \text{ as } t \to \infty \text{ locally uniformly for } z \in \mathbb{R}.$$

Theorem 1.1, motivated by [14, 15], provides the stability of traveling wave in reaction-diffusion systems with discrete diffusion in a certain sense. However, as in the case of classical and nonlocal diffusions, the above theorem can be applied only for discrete diffusion with the same diffusivities.

For the non-equal discrete diffusion case, we have the following theorem.

Theorem 1.2. *Assume that system* (1.1) *has a bounded invariant set* $\mathcal{I} := \prod_{i=1}^m [\underline{M}_i, \overline{M}_i] \subset \mathbb{R}^m$ *for some* $0 \leq \underline{M}_i < \overline{M}_i < \infty$ *with* $i = 1, 2, \cdots, m$*. Assume* (1.14) *holds for some positive constants* $\{\sigma_i \mid 1 \leq i \leq m\}$*. Let* R *be a positive constant such that* $c_R \geq c^*$*, and let* $\{c, \Phi\}$ *be* *a positive traveling wave of* (1.1) *for some* $c \geq c_R$ *such that* $\Phi(z) \in \mathcal{I}$ *for all* $z \in \mathbb{R}$*, where* c_R *is a constant defined by* (1.8)*. Moreover, we assume*

$$\phi_i(z) \geq Ce^{-\alpha_0|z|}, \quad \forall z \in \mathbb{R}, \quad 1 \leq i \leq m, \tag{1.15}$$

for some constants $\alpha_0 > 0$ *and* $C > 0$*; and there exists a constant* $\mu > 0$ *such that* (1.9) *holds, where* R *is a positive constant such that*

$$\max_{1 \leq i \leq m} \{\|g_i(\Phi)\|_{L^\infty(\mathbb{R})}\} \leq R. \tag{1.16}$$

Let u *be the solution of* (1.6) *with a positive initial data* $u^0 \in BUC(\mathbb{R}, \mathcal{I})$ *such that* $z \mapsto (u_i^0(z) - \phi_i(z))\, e^{-\alpha^+ z} \in H^1(\mathbb{R})$ *and* $z \mapsto (u_i^0(z) - \phi_i(z))\, e^{-\alpha^- z} \in H^1(\mathbb{R})$ *for some constants* $\alpha^+ > \alpha_0$ *and* $\alpha^- < -\max\{\alpha_0, (\alpha_0 + \mu)/2\}$*. Then*

$$u(z,t) \to \Phi(z) \quad in \quad L^p_{\mathrm{loc}}(\mathbb{R}) \quad as \quad t \to \infty$$

for any $p \in [1, \infty)$*.*

Note that Theorem 1.2 is in particular motivated by the arguments developed in [10, 9], but it addresses a different setting. The work [10] investigates the case of classical diffusion with non-equal diffusion coefficients, while [9] considers nonlocal diffusion models with continuous integral kernels. Our proof strategy builds upon the core ideas in both references, namely the use of relative entropy structures and weighted energy methods, but with substantial modifications. Indeed, the discrete diffusion operator in (1.1) introduces fundamentally different analytical challenges, unlike the continuous Laplacian or integral operators considered in [9, 10], the discrete second-difference operator lacks smoothness and interacts with exponential weight functions in a more delicate way. Consequently, while the overall conceptual framework of our proof parallels those earlier studies, the technical implementation diverges significantly. We emphasize that this distinction is essential, since the discrete diffusion setting is not a straightforward extension of the continuous or nonlocal cases, but rather requires a tailored treatment.

Our contribution can be summarized as follows. First, we formulate an entropy-type Lyapunov functional suitable for discrete diffusion systems. Second, we establish convergence results under both equal and non-equal diffusion settings. Thirdly, we apply the theory to non-cooperative models drawn from biological literature (e.g., [11, 6]). This work thus complements and extends the recent developments in [14, 15] from classical and nonlocal diffusions to discrete case. In doing so, it sheds new light on the stability mechanisms underlying spatially structured biological systems, especially, where traditional comparison principles are inapplicable.

The rest of this paper is organized as follows. In §2, we provide a proof of Theorem 1.1. Then, in §3, we prove Theorem 1.2. Although the proofs of Theorems 1.1 and 1.2 are motivated by the ideas used in [14, 15, 9, 10], there are certain nontrivial modifications. To be self-contained and for the reader's convenience, we present some details of proofs here. Finally, we provide an application of Theorem 1.1 and Theorem 1.2 to non-cooperative systems studied in [11, 6] in §4.

2. Proof of Theorem 1.1

We shall prove Theorem 1.1 in this section. Following [14], we set $W_i = \mathcal{K}_i[u_i]$, where $u = u(z,t)$ is a positive solution of (1.6). Then straightforward calculations yield

$$\begin{aligned}
&(W_i)_t - d_i\mathcal{D}_2[W_i] - c(W_i)_z \\
= \;& (u_i-\phi_i)(g_i(u)-g_i(\Phi)) - \frac{\phi_i}{u_i}d_i\mathcal{D}_2[u_i] + g_i(\Phi)(u_i-\phi_i) \\
&+ d_i\mathcal{D}_2[\phi_i] + d_i\mathcal{D}_2[\phi_i\log(u_i/\phi_i)] + c\phi_i'\log\frac{u_i}{\phi_i} \\
= \;& (u_i-\phi_i)(g_i(u)-g_i(\Phi)) - \frac{\phi_i}{u_i}d_i\mathcal{D}_2[u_i] + g_i(\Phi)\Big(W_i + \phi_i\log\frac{u_i}{\phi_i}\Big) \\
&+ d_i\mathcal{D}_2[\phi_i] + d_i\mathcal{D}_2[\phi_i\log(u_i/\phi_i)] + c\phi_i'\log\frac{u_i}{\phi_i}.
\end{aligned}$$

By substituting

$$\begin{aligned}
-\frac{\phi_i}{u_i}\mathcal{D}_2[u_i] &= -\frac{\phi_i}{u_i}[u_i(z+1,t)+u_i(z-1,t)] + 2\phi_i(z), \\
\mathcal{D}_2\phi_i &= [\phi_i(z+1)+\phi_i(z-1)] - 2\phi_i(z), \\
g_i(\Phi)\phi_i\log\frac{u_i}{\phi_i} &= -\Big\{\phi_i(z+1,t)+\phi_i(z-1,t)-2\phi_i(z,t)+c\phi_i'(z,t)\Big\}\Big(\log\frac{u_i}{\phi_i}\Big)(z,t), \\
\mathcal{D}_2[\phi_i\log(u_i/\phi_i)] &= \Big(\phi_i\log\frac{u_i}{\phi_i}\Big)(z+1,t) + \Big(\phi_i\log\frac{u_i}{\phi_i}\Big)(z-1,t) - 2\Big(\phi_i\log\frac{u_i}{\phi_i}\Big)(z,t),
\end{aligned}$$

and using (1.14), we deduce that

$$\begin{aligned}
&W_t - \sum_{i=1}^m \sigma_i\Big\{d_i\mathcal{D}_2[W_i] + c(W_i)_z + g_i(\Phi)W_i\Big\} \\
\le\;& \sum_{i=1}^m \sigma_i\Big[-\frac{\phi_i}{u_i}[u_i(z+1,t)+u_i(z-1,t)] + (\phi_i(z+1)+\phi_i(z-1)) \\
&-\{\phi_i(z+1)+\phi_i(z-1)\}\log\frac{u_i}{\phi_i} + \Big(\phi_i\log\frac{u_i}{\phi_i}\Big)(z+1,t) + \Big(\phi_i\log\frac{u_i}{\phi_i}\Big)(z-1,t)\Big].
\end{aligned}$$

We further compute that

$$\begin{aligned}
&-\frac{\phi_i}{u_i}u_i(z\pm1,t) + \phi_i(z\pm1) - \phi_i(z\pm1)\log\frac{u_i}{\phi_i} + \Big(\phi_i\log\frac{u_i}{\phi_i}\Big)(z\pm1,t) \\
=\;& \phi_i(z\pm1) - \frac{u_i(z\pm1,t)}{u_i(z,t)}\phi_i(z) - \phi_i(z\pm1)\log\frac{u_i(z,t)}{\phi_i(z)} + \phi_i(z\pm1)\log\frac{u_i(z\pm1,t)}{\phi_i(z\pm1)} \\
=\;& \Big\{\phi_i(y) - \frac{u_i(y,t)}{u_i(z,t)}\phi_i(z) + \phi_i(y)\log\frac{u_i(y,t)\phi_i(z)}{u_i(z,t)\phi_i(y)}\Big\} \\
\le\;& \phi_i(z\pm1) - \frac{u_i(z\pm1,t)}{u_i(z,t)}\phi_i(z) + \phi_i(z\pm1)\Big(\frac{u_i(z\pm1,t)\phi_i(z)}{u_i(z,t)\phi_i(z\pm1)} - 1\Big) = 0.
\end{aligned}$$

Here we used the inequality $\log X \le X-1$ for $X>0$, by setting

$$X = \frac{u(z\pm1,t)\phi_i(z)}{u(z,t)\phi_i(z\pm1)}.$$

Thus we obtain

$$W_t \leq \sum_{i=1}^{m} \sigma_i \Big\{ d_i \mathcal{D}_2[W_i] + c(W_i)_z + g_i(\Phi) W_i \Big\}. \tag{2.1}$$

In particular, when $d_i = 1$ for all $1 \leq i \leq m$, the following inequality holds:

$$W_t \leq \mathcal{D}_2[W] + cW_z + RW, \ z \in \mathbb{R}, \ t > 0, \tag{2.2}$$

for the related entropy function $W(z,t)$ of $u(z,t)$ defined by (1.13). Note that $w(z) = e^{-\lambda_c z}$ satisfies

$$\mathcal{D}_2[w] + cw_z + Rw = 0, \ z \in \mathbb{R}.$$

Define a function $V(z,t) = e^{\lambda_c z} W(z,t)$. Then, by (2.2), we obtain

$$\begin{aligned} V_t(z,t) &\leq e^{\lambda_c} V(z-1,t) + e^{-\lambda_c} V(z+1,t) + (R - 2 - c\lambda_c) V(z,t) + cV_z(z,t) \\ &= e^{\lambda_c}\{V(z-1,t) - V(z,t)\} + e^{-\lambda_c}\{V(z+1,t) - V(z,t)\} + cV_z(z,t). \end{aligned}$$

If we define the function $\tilde{U}(z+ct,t) = V(z,t)$, then

$$\tilde{U}_t(x,t) \leq e^{\lambda_c}\{\tilde{U}(x-1,t) - \tilde{U}(x,t)\} + e^{-\lambda_c}\{\tilde{U}(x+1,t) - \tilde{U}(x,t)\}, \qquad x = z + ct.$$

Now we introduce a new time variable τ by the relation $\frac{\tau}{t} := e^{\lambda_c} + e^{-\lambda_c} \in (1, \infty)$. Then

$$\tilde{U}_\tau(x,\tau) \leq \frac{e^{\lambda_c}}{e^{\lambda_c} + e^{-\lambda_c}} \tilde{U}(x-1,\tau) + \frac{e^{-\lambda_c}}{e^{\lambda_c} + e^{-\lambda_c}} \tilde{U}(x+1,\tau) - \tilde{U}(x,\tau).$$

We shall study the solution of

$$\begin{cases} \bar{U}_\tau(x,\tau) = \dfrac{e^{\lambda_c}}{e^{\lambda_c} + e^{-\lambda_c}} \bar{U}(x-1,\tau) + \dfrac{e^{-\lambda_c}}{e^{\lambda_c} + e^{-\lambda_c}} \bar{U}(x+1,\tau) - \bar{U}(x,\tau), \ x \in \mathbb{R}, \ \tau > 0, \\ \bar{U}(x,0) = \tilde{U}(x,0) + \tilde{U}(-x,0) \in L^1(\mathbb{R}), \ x \in \mathbb{R}. \end{cases}$$

In order to analyze this problem, we introduce the following linear operator

$$(\mathcal{L}v)(x) := \frac{e^{\lambda_c}}{e^{\lambda_c} + e^{-\lambda_c}} v(x-1) + \frac{e^{-\lambda_c}}{e^{\lambda_c} + e^{-\lambda_c}} v(x+1) - v(x), \quad v \in L^1(\mathbb{R}).$$

Then we have

$$\mathcal{F}[\mathcal{L}v](\xi) = \Big(\frac{e^{\lambda_c + i\xi} + e^{-\lambda_c - i\xi}}{e^{\lambda_c} + e^{-\lambda_c}} - 1 \Big) \mathcal{F}[v](\xi) = \Big\{ (\cos\xi - 1) + i \Big(\frac{e^{\lambda_c} - e^{-\lambda_c}}{e^{\lambda_c} + e^{-\lambda_c}} \Big) \sin\xi \Big\} \mathcal{F}[v](\xi).$$

Hence the function $\bar{U} \in L^1(\mathbb{R})$ satisfies

$$\frac{d}{d\tau} \mathcal{F}[\bar{U}](\xi,\tau) = \Big\{ (\cos\xi - 1) + i \Big(\frac{e^{\lambda_c} - e^{-\lambda_c}}{e^{\lambda_c} + e^{-\lambda_c}} \Big) \sin\xi \Big\} \mathcal{F}[\bar{U}](\xi,\tau).$$

Note that

$$\begin{aligned} \mathcal{F}[\bar{U}(\cdot,0)](\xi) = \mathcal{F}[\tilde{U}(x,0) + \tilde{U}(-x,0)](\xi) &= \int_{\mathbb{R}} \{\tilde{U}(x,0) + \tilde{U}(-x,0)\} e^{ix\xi}\, dx \\ &= \int_{\mathbb{R}} \tilde{U}(x,0) \{e^{ix\xi} + e^{-ix\xi}\}\, dx = 2 \int_{\mathbb{R}} \tilde{U}(x,0) \cos x\xi \, dx \end{aligned}$$

is a real-valued function. Therefore,

$$\begin{aligned}\mathcal{F}[\bar{U}](\xi,\tau) &= \exp\Big[\tau\Big\{(\cos\xi-1)+i\Big(\frac{e^{\lambda_c}-e^{-\lambda_c}}{e^{\lambda_c}+e^{-\lambda_c}}\Big)\sin\xi\Big\}\Big]\mathcal{F}[\bar{U}](\xi,0)\\ &= \exp\Big[\tau(\cos\xi-1)\Big]\mathcal{F}[\bar{U}](\xi,0)\cdot\exp\Big[\tau i\Big(\frac{e^{\lambda_c}-e^{-\lambda_c}}{e^{\lambda_c}+e^{-\lambda_c}}\Big)\sin\xi\Big].\end{aligned}$$

By the inverse Fourier transformation, we have the following inequality

$$\begin{aligned}|\bar{U}(x,\tau)| &= \Big|\frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{F}[\bar{U}](\xi,\tau)e^{ix\xi}\,d\xi\Big| \le \frac{1}{2\pi}\int_{\mathbb{R}}|\mathcal{F}[\bar{U}](\xi,\tau)|\,d\xi\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\exp[\tau(\cos\xi-1)]\cdot|\mathcal{F}[\bar{U}](\xi,0)|\,d\xi.\end{aligned}$$

Thus we conclude

$$\sup_{x\in\mathbb{R}}|\bar{U}(x,\tau)| \le \frac{1}{2\pi}\int_{\mathbb{R}}\exp[\tau(\cos\xi-1)]\cdot|\mathcal{F}[\bar{U}](\xi,0)|\,d\xi.$$

Note that $\lim_{\tau\to\infty}\exp[\tau(\cos\xi-1)]=0$ provided that $2\pi\xi\notin\mathbb{Z}$. Now we use the assumption $\mathcal{F}[\bar{U}(0)]\in L^1(\mathbb{R})$ and the Lebesgue dominated convergence theorem, to conclude that $\lim_{\tau\to\infty}\sup_{x\in\mathbb{R}}|\bar{U}(x,\tau)|=0$.

Finally, by applying the comparison principle, $\tilde{U}(x,\tau)\le\bar{U}(x,\tau)$ for all $x\in\mathbb{R}$ and $\tau\ge 0$. Thus we conclude that

$$0\le\lim_{\tau\to\infty}\sup_{x\in\mathbb{R}}\tilde{U}(x,\tau)\le\lim_{\tau\to\infty}\sup_{x\in\mathbb{R}}\bar{U}(x,\tau)=0,\quad x\in\mathbb{R}.$$

Hence $V(x,t)\to 0$ as $t\to\infty$ uniformly for $x\in\mathbb{R}$. In particular, $W(z,t)\to 0$ locally uniformly for $z\in\mathbb{R}$. Thus, Theorem 1.1 is proved. □

3. NON-EQUAL DIFFUSION CASE

We shall prove Theorem 1.2 in this section. Since the system (1.1) has a bounded invariant set $\mathcal{I}:=\prod_{i=1}^m[\underline{M}_i,\overline{M}_i]$ such that $u^0(z)\in\mathcal{I}$ and $\Phi(z)\in\mathcal{I}$ for all $z\in\mathbb{R}$. Then, we have

$$u(z,t)\in\mathcal{I},\quad z\in\mathbb{R},\ t\ge 0.$$

Now we define

$$\chi(u)=\prod_{i=1}^m\chi_i(u_i),$$

where $0\le\chi_i\le 1$ is a smooth function satisfying

$$\chi_i(u_i)=\begin{cases}1, & \underline{M}_i\le u_i\le\overline{M}_i\\ 0, & u_i\ge\overline{M}_i+\varepsilon \text{ or } u_i\le\underline{M}_i-\varepsilon,\end{cases}$$

where $\varepsilon>0$ is sufficiently small positive constant so that $\prod_{=1}^m\big[\underline{M}_i-\varepsilon,\overline{M}_i+\varepsilon\big]\subset\Omega$. Then we introduce

$$f_i(u)=u_ig_i(u)\chi(u):\mathbb{R}^m\to\mathbb{R},\qquad i=1,\dots,m. \tag{3.1}$$

which is globally Lipschitz continuous on $\mathcal{I}$. Note that the solution of $\partial_t\tilde{u}_i=d_i\mathcal{D}_2[\tilde{u}_i]+f_i(\tilde{u})$ is a solution to (1.1) as long as the solution is included in the invariant set $\mathcal{I}$.

Define $v := (v_1, \dots, v_m)$, where

$$v_i(z,t) = \tilde{u}_i(z+ct,t) - \phi_i(z).$$

By a simple calculation, we obtain

$$\partial_t v_i - c(\partial_z v_i + \phi_i') = d_i \left(\mathcal{D}_2[v_i] + \mathcal{D}_2[\phi_i]\right) + f_i\left(v+\Phi\right),\ z \in \mathbb{R},\ t>0,\ i=1,\dots,m.$$

On the other hand, the traveling wave profile $\Phi = (\phi_1, \dots, \phi_m)$ satisfies

$$d_i \mathcal{D}_2[\phi_i] + c\phi_i' + \phi_i g_i(\Phi) = 0,\ z \in \mathbb{R},\ i=1,\dots,m.$$

Since $\Phi(z) \in \mathcal{I}$ for all $z \in \mathbb{R}$, Φ also satisfies the modified system

$$d_i \mathcal{D}_2[\phi_i] + c\phi_i' + f_i\left(\Phi\right) = 0,\ z \in \mathbb{R},\ i=1,\dots,m.$$

Therefore, the function v satisfies the following system

$$\partial_t v_i = d_i \mathcal{D}_2[v_i] + c\partial_z v_i + f_i(v+\Phi) - f_i(\Phi),\ z \in \mathbb{R},\ t>0,\ i=1,\dots,m,$$

Now write $v_i(z,t) = w_i(z,t)e^{\alpha z}$ for each $i = 1, \dots, m$. Hereafter α is a constant. One can check that the function $w = (w_1, \dots, w_m)$ satisfies

$$\begin{aligned}\partial_t w_i = {} & d_i(e^{\alpha} w_i(z+1,t) - 2w_i(z,t) + e^{-\alpha} w_i(z-1,t)) \\ & + c\partial_z w_i + c\alpha w_i + e^{-\alpha z}\left[f_i(e^{\alpha z} w_i + \Phi) - f_i(\Phi)\right].\end{aligned} \tag{3.2}$$

Then the following proposition holds true.

Proposition 3.1. *The system* (3.2) *generates a strongly continuous semiflow on* $H^1(\mathbb{R})^m$.

Proof. To see this consider the linear operator $A_i : D(A_i) \subset H^1(\mathbb{R}) \to H^1(\mathbb{R})$ given by

$$\begin{aligned}& D(A_i) = H^2(\mathbb{R}) \\ & A_i\,\varphi(z) := d_i\{e^{\alpha}\varphi(z+1) - 2\varphi(z) + e^{-\alpha}\varphi(z-1)\} + c\varphi'(z) + c\alpha\varphi(z),\ \forall \varphi \in H^2(\mathbb{R}).\end{aligned}$$

Observe that $c\partial_x : H^2(\mathbb{R}) \to H^1(\mathbb{R})$ generates a strongly continuous semigroup on $H^1(\mathbb{R})$ given by

$$T_{c\partial_x}(t)\varphi = \varphi(ct + \cdot),$$

while $\varphi \mapsto d_i\{e^{\alpha}\varphi(\cdot+1) - 2\varphi + e^{-\alpha}\varphi(\cdot-1)\} + c\alpha\varphi$ is a linear bounded operator of $H^1(\mathbb{R})$. Hence A_i is the infinitesimal generator of a strongly continuous semigroup $T_{A_i}(t) : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ for each $i \in \{1, \dots, m\}$ and $t \geq 0$.

Now let $\hat{U}_0$ be the Fourier transform of a function $U_0 \in H^1(\mathbb{R})$. Taking the Fourier transform in z of the function $U(z,t) = T_{A_i}(t)U_0$ yields

$$\partial_t \hat{U} = \Big[d_i\big\{\{(e^{\alpha}+e^{-\alpha})\cos\xi - 2\} + i(e^{\alpha}-e^{-\alpha})\sin\xi\big\} + ic\xi + c\alpha\Big]\hat{U} \text{ with } \hat{U}(0,\xi) = \hat{U}_0(\xi). \tag{3.3}$$

Then the solution to (3.3) is given by

$$\hat{U}(t,\cdot) = e^{ic\xi t}e^{c\alpha t}\exp\Big\{t[d_i\big\{\{(e^{\alpha}+e^{-\alpha})\cos\xi - 2\} + i(e^{\alpha}-e^{-\alpha})\sin\xi\big\} + ic\xi + c\alpha]\Big\}\hat{U}(0,\cdot).$$

Next we have

$$\left|e^{ic\xi t}e^{c\alpha t}\exp\left\{t[d_i\{\{(e^{\alpha}+e^{-\alpha})\cos\xi-2\}+i(e^{\alpha}-e^{-\alpha})\sin\xi\}+ic\xi+c\alpha]\right\}\right|$$
$$=\exp\left\{t\Big(d_i\big((e^{\alpha}+e^{-\alpha})\cos\xi-2\big)+c\alpha\Big)\right\}.$$

By taking the inverse Fourier transform yields

$$\|U(t)\|_{H^1(\mathbb{R})}\le e^{c\alpha t}\exp\left\{td_i\Big(e^{\alpha}+e^{-\alpha}-2\Big)\right\}\|U_0\|_{H^1(\mathbb{R})}.$$

Then the proposition is an immediate consequence of the globally Lipschitz continuity of the functions $f_i:\mathbb{R}^m\to\mathbb{R}$. In fact, for any $\varphi=(\varphi_1,..,\varphi_m)\in L^2(\mathbb{R})^m$, by the globally Lipschitz continuity of f_i we have

$$|f_i\left(e^{\alpha z}\varphi(z)+\Phi(z)\right)-f_i(\Phi(z))|\le Ce^{\alpha z}\|\varphi(z)\|_{\mathbb{R}^m}$$

for each $i=1,\dots,m$, and almost every $z\in\mathbb{R}$, for some positive constant C. This also implies that the function $G(\varphi)$ defined by $G:=(G_1,\dots,G_m)$ with

$$G_i(\varphi):z\mapsto e^{-\alpha z}\left[f_i\left(e^{\alpha z}\varphi(z)+\Phi(z)\right)-f_i(\Phi(z))\right],$$

belongs to $L^2(\mathbb{R})^m$, and the mapping G satisfies the Lipschitz estimate

$$\|G(\varphi)-G(\psi)\|_{L^2(\mathbb{R})^m}\le C\|\varphi-\psi\|_{L^2(\mathbb{R})^m}\ \text{ for any }\varphi,\psi\in L^2(\mathbb{R})^m.$$

Moreover, for $\varphi=(\varphi_1,\dots,\varphi_m)\in H^1(\mathbb{R})^m$, we have

$$\begin{aligned}[G_i(\varphi)]' &= -\alpha e^{-\alpha z}\left[f_i\left(e^{\alpha z}\varphi+\Phi\right)-f_i(\Phi)\right]\\ &\quad +e^{-\alpha z}\left[(f_i)'\left(e^{\alpha z}\varphi+\Phi\right)-(f_i)'(\Phi)\right]\Phi'+\left[(f_i)'\left(e^{\alpha z}\varphi+\Phi\right)(\alpha\varphi+\varphi')\right]\end{aligned}$$

in $L^2(\mathbb{R})$, since each term on the right-hand side of the above equality belongs to $L^2(\mathbb{R})$. Hence $G(\varphi)\in H^1(\mathbb{R})^m$ for $\varphi\in H^1(\mathbb{R})^m$. This shows that G maps $H^1(\mathbb{R})^m$ into itself and is Lipschitz continuous.

By formulating (3.2) as an evolution equation of the form $w'(t)=Aw(t)+G(w(t))$, where $Aw:=(A_1w_1,\dots,A_mw_m)$, we see that system (3.2) generates a strongly continuous semiflow on $H^1(\mathbb{R})^m$. This proves the proposition. □

With Proposition 3.1 at hand, the following proposition on the decay estimate of $W(z,t)$ at spatial infinity can be proved in the same manner as that of [9, Proposition 2.2] (see also [10, Proposition 2.1]). We safely omit its proof here.

Proposition 3.2. *Let the assumptions made in Theorem 1.2 be enforced. Let u be the solution of* (1.6) *with a positive initial data $u^0\in BUC(\mathbb{R},\mathcal{I})$ such that $z\mapsto\left(u_i^0(z)-\phi_i(z)\right)e^{-\alpha^+z}\in H^1(\mathbb{R})$ and $z\mapsto\left(u_i^0(z)-\phi_i(z)\right)e^{-\alpha^-z}\in H^1(\mathbb{R})$ for some constants $\alpha^+>\alpha_0$ and $\alpha^-<-\max\{\alpha_0,(\alpha_0+\mu)/2\}$. Then the solution $u=u(z,t)$ to* (1.6) *with initial data u^0 satisfies*

$$\mathcal{K}_i[u_i(\cdot,t)](z)e^{\mu z}\to 0\ \text{ as } z\to\pm\infty,\quad i=1,\dots,m,$$

for all $t>0$, where μ is given in (1.9).

Finally, the assumption (1.9) yields that

$$d_i\mathcal{D}_2[e^{\mu z}] - c(e^{\mu z})_z + Re^{\mu z} \leq -\varepsilon e^{\mu z}, \quad i = 1, 2, \ldots, m, \tag{3.4}$$

for some positive constant ε. Multiplying (2.1) by a function $e^{\mu z}$, we calculate with the help of Fubini's theorem and integration by parts to obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} We^{\mu z}\,dz &\leq \int_{\mathbb{R}} \sum_{i=1}^{m} \sigma_i\{d_i\mathcal{D}_2[W_i] + c\partial_z W_i + RW_i\}e^{\mu z}\,dz \\ &= \int_{\mathbb{R}} \sum_{i=1}^{m} \sigma_i W_i\{d_i\mathcal{D}_2[e^{\mu z}] - c(e^{\mu z})_z + Re^{\mu z}\}\,dz, \end{aligned}$$

where the last equality follows by using Proposition 3.2. It follows from (3.4) that

$$\frac{d}{dt}\int_{\mathbb{R}} W(z,t)e^{\mu z}\,dz \leq -\varepsilon \int_{\mathbb{R}} W(z,t)e^{\mu z}\,dz.$$

By integrating this inequality, we conclude that

$$0 \leq \int_{\mathbb{R}} W(z,t)e^{\mu z}\,dz \leq e^{-\varepsilon t}\int_{\mathbb{R}} W(z,0)e^{\mu z}\,dz \;\to 0 \;\text{ as } t \to \infty, \tag{3.5}$$

since $W(z,t)e^{\mu z}$ is uniformly bounded and integrable over $\mathbb{R}$ for all $t \geq 0$. Hence, we have

$$\lim_{t\to\infty}\int_{\mathbb{R}} \mathcal{K}[u(\cdot,t)](z)e^{\mu z}\,dz = 0.$$

By a similar argument as that of [9], we can get the $L^p_{\text{loc}}(\mathbb{R})$-convergence of the solution $u(z,t)$ to the traveling wave $\Phi(z)$ as $t \to \infty$ for any $p \in [1,\infty)$. This completes the proof of Theorem 1.2. □

4. Applications of Theorem 1.1-1.2

First, we consider the following predator-prey system

$$\begin{cases} \tilde{u}_t = d_1\mathcal{D}_2[\tilde{u}] + a\tilde{u}(1 - \tilde{u} - k\tilde{v}), & x \in \mathbb{R},\, t > 0, \\ \tilde{v}_t = d_2\mathcal{D}_2[\tilde{v}] + b\tilde{v}(1 - \tilde{v}/\tilde{u}), & x \in \mathbb{R},\, t > 0, \end{cases} \tag{4.1}$$

with positive initial data (u^0, v^0) at $t = 0$, where a, b, k are positive constants. According to [6], under the assumption $k \in (0,1)$, there is a positive solution (ϕ_1, ϕ_2) of

$$\begin{cases} d_1\mathcal{D}_2[\phi_1](z) + c\phi_1'(z) + a\phi_1(z)[1 - \phi_1(z) - k\phi_2(z)] = 0, & z \in \mathbb{R}, \\ d_2\mathcal{D}_2[\phi_2](z) + c\phi_2'(z) + b\phi_2(z)\left[1 - \frac{\phi_2(z)}{\phi_1(z)}\right] = 0, & z \in \mathbb{R}, \end{cases} \tag{4.2}$$

satisfying

$$\lim_{z\to+\infty}(\phi_1,\phi_2) = (1,0), \quad \lim_{z\to-\infty}(\phi_1,\phi_2) = \left(\frac{1}{1+k}, \frac{1}{1+k}\right),$$

if and only if

$$c \geq c^* = \inf_{\lambda>0}\frac{d_2(e^{\lambda} + e^{-\lambda} - 2) + b}{\lambda}.$$

The set $\mathcal{I}_k := [1-k, 1] \times [0, 1]$ is an invariant set for system (4.1), and the traveling waves (ϕ_1, ϕ_2) constructed in [6] satisfy $\phi_1 \geq 1-k$. Recall from [14] that (1.14) holds, by choosing $\sigma_1 = b$ and $\sigma_2 = 1$. Also, by setting $R = b$, there exists a constant $\mu > 0$ such that (1.9) holds, provided that $c > c^* = c_R = c_R^2$ and $d_1 \leq d_2$. Moreover, there exists a constant $\alpha_0 > 0$ such that (1.15) is satisfied, by the construction of general upper-lower solutions in [6]. Thus, we obtain the following result:

Theorem 4.1. *Given $c \geq c^*$ and $a > 1/16$. Assume*

$$\frac{1}{a(1-k)^2} - \frac{2}{\sqrt{a}} \leq k \leq \frac{2}{\sqrt{a}},$$

holds for some $k \in (0,1)$ and $b \geq ak$. Let $(u(z,t), v(z,t))$ be a solution of system

$$\begin{cases} u_t = d_1 \mathcal{D}_2[u] + c u_z + a u(1 - u - kv), & z \in \mathbb{R},\ t > 0, \\ v_t = d_2 \mathcal{D}_2[v] + c v_z + b v(1 - v/u), & z \in \mathbb{R},\ t > 0 \end{cases}$$

with initial data $(u^0, v^0)(z) \in \mathcal{I}_k$ for all $z \in \mathbb{R}$. Set $\sigma_1 = b$, $\sigma_2 = 1$, then the following results hold.

1. *Let $d_1 = d_2$ and assume $e^{\lambda_c z}\mathcal{K}[(u^0, v^0)] \in L^1(\mathbb{R})$ and $\mathcal{F}[e^{\lambda_c z}\mathcal{K}[(u^0, v^0)]] \in L^1(\mathbb{R})$. Then $(u, v)(z,t)$ converges to $(\phi_1, \phi_2)(z)$ as $t \to +\infty$ locally uniformly in $\mathbb{R}$, where $\{c, (\phi_1, \phi_2)\}$ is a traveling wave obtained in [6].*
2. *Let $d_1 \leq d_2$. There exist $\alpha^+ > \alpha_0$ and $\alpha^- < -\max\{\alpha_0, (\alpha_0 + \mu)/2\}$ such that*

$$(u^0 - \phi_1)e^{-\alpha^{\pm} z} \in H^1(\mathbb{R}), \qquad (v^0 - \phi_2)e^{-\alpha^{\pm} z} \in H^1(\mathbb{R}),$$

where $\mu > 0$ is a constant satisfying (1.9) and α_0 is the smaller root of the equation $d_2(e^{\lambda} + e^{-\lambda} - 2) - c\lambda + b = 0$ as in [6]. Then $(u, v)(z,t)$ converges to $(\phi_1, \phi_2)(z)$ as $t \to \infty$ in the sense described in Theorem 1.2.

Next, we apply Theorem 1.1 to derive the stability of traveling waves in the following SIR model studied in [11]:

$$\begin{cases} \tilde{S}_t(x,t) = \mathcal{D}_2[\tilde{S}](x,t) - \beta \tilde{S}(x,t)\tilde{I}(x,t), & x \in \mathbb{R},\ t > 0, \\ \tilde{I}_t(x,t) = \mathcal{D}_2[\tilde{I}](x,t) + \beta \tilde{S}(x,t)\tilde{I}(x,t) - \gamma \tilde{I}(x,t), & x \in \mathbb{R},\ t > 0, \end{cases} \tag{4.3}$$

where β, γ are positive constants. There exists a family of equilibrium points $(s^*, 0)$ for $s^* > 0$ to the equations (4.3), which is the fixed points to the corresponding spatial independent ordinary differential systems:

$$\begin{cases} s'(t) = -\beta s(t) i(t), & t > 0, \\ i'(t) = \beta s(t) i(t) - \gamma i(t), & t > 0. \end{cases}$$

The dynamical behavior of the corresponding ordinary differential system is completely determined by the basic reproduction number $\mathcal{R}_0 := \beta s^*/\gamma$. More precisely, if $\beta s(0)/\gamma \leq 1$, disease will always die out, when we consider the corresponding ordinary differential system. On the other hand, if $\beta s^*/\gamma > 1$, the phase plane shows a curve from the point $(s^*, 0)$ to another point $(s_*, 0)$ satisfying $\beta s_*/\gamma < 1$ in the (s, i) phase plane. For this ordinary differential equation, there may not exist a bounded invariant set. Hence we can only apply Theorem 1.1 for this model.

In the following, we assume that $\mathcal{R}_0 > 1$. Then the quantity

$$c^* := \inf_{\lambda>0} \frac{\left[e^{\lambda} + e^{-\lambda} - 2\right] + \beta s^* - \gamma}{\lambda} \tag{4.4}$$

is well-defined and positive. For any $c \geq c^*$, system (4.3) admits a traveling wave $\{c, (\phi_1, \phi_2)\}$ connecting $(s^*, 0)$ at $z = \infty$ to another disease-free state at $z = -\infty$. More precisely, there exists a positive solution to the equations

$$\begin{cases} \mathcal{D}_2[\phi_1](z) + c\phi_1'(z) - \beta\phi_1(z)\phi_2(z) = 0, & z \in \mathbb{R}, \\ \mathcal{D}_2[\phi_2](z) + c\phi_2'(z) + \beta\phi_1(z)\phi_2(z) - \gamma\phi_2(z) = 0, & z \in \mathbb{R}, \end{cases}$$

satisfying $(\phi_1, \phi_2)(+\infty) = (s^*, 0)$ and $\phi_2(-\infty) = 0$.

The proof of existence is based on a pair upper and lower solutions method and the Schauder's fixed point theorem. According to the construction of a pair of upper and lower solutions, $0 \leq \phi_1 \leq s^*$ and $\phi_2 > 0$. For this problem,

$$g_1(S, I) = -\beta I, \qquad g_2(S, I) = \beta S - \gamma.$$

Thus it is possible to choose $R = \beta s^* - \gamma$.

By choosing $\sigma_1 = \sigma_2 = 1$, we can check the condition (1.14), since

$$\begin{aligned} (S - \phi_1)(g_1(S, I) - g_1(\phi_1, \phi_2)) &= (S - \phi_1)(-\beta I + \beta\phi_2), \\ (I - \phi_2)(g_2(S, I) - g_2(\phi_1, \phi_2)) &= \beta(I - \phi_2)(S - \phi_1). \end{aligned}$$

implies $\mathcal{M} = 0$. Therefore, we have the following theorem:

Theorem 4.2. *Let $c \geq c^*$ and let $(S(z,t), I(z,t))$ be a solution of system*

$$\begin{cases} S_t = \mathcal{D}_2[S] + cS_z - \beta SI, & z \in \mathbb{R},\ t > 0, \\ I_t = \mathcal{D}_2[I] + cI_z + \beta SI - \gamma I, & z \in \mathbb{R},\ t > 0. \end{cases}$$

with initial data (S^0, I^0) such that $e^{\lambda_c z}\mathcal{K}[(S^0, I^0)], \mathcal{F}[e^{\lambda_c z}\mathcal{K}[(S^0, I^0)]] \in L^1(\mathbb{R})$ for $\sigma_1 = \sigma_2 = 1$. Then $(S, I)(z, t)$ converges to $(\phi_1, \phi_2)(z)$ as $t \to +\infty$ locally uniformly in $\mathbb{R}$, where $\{c, (\phi_1, \phi_2)\}$ is a traveling wave obtained in [11].

Finally, we mention the limitation of our Theorem 1.1. Consider the stability of traveling waves in some specific epidemic model studied in [5].

$$\begin{cases} \tilde{S}_t(x,t) = \mathcal{D}_2[\tilde{S}](x,t) + \mu - \mu\tilde{S}(x,t) - \beta\tilde{S}(x,t)\tilde{I}(x,t), & x \in \mathbb{R},\ t > 0, \\ \tilde{I}_t(x,t) = \mathcal{D}_2[\tilde{I}](x,t) + \beta\tilde{S}(x,t)\tilde{I}(x,t) - (\mu+\gamma)\tilde{I}(x,t), & x \in \mathbb{R},\ t > 0, \end{cases} \tag{4.5}$$

where μ, β, γ are positive constants. This system has two constant states

$$E_+ = (1, 0), \qquad E_- = \Big(\frac{\mu+\gamma}{\beta}, \frac{\mu(\beta-\mu-\gamma)}{\beta(\mu+\gamma)}\Big)$$

provided that

$$\beta > \mu + \gamma.$$

According to [5], there exists a positive solution to the equations

$$\begin{cases} \mathcal{D}_2[\phi_1](z) + c\phi_1'(z) + \mu - \mu\phi_1 - \beta\phi_1(z)\phi_2(z) = 0, & z \in \mathbb{R}, \\ \mathcal{D}_2[\phi_2](z) + c\phi_2'(z) + \beta\phi_1(z)\phi_2(z) - (\mu+\gamma)\phi_2(z) = 0, & z \in \mathbb{R} \end{cases}$$

such that $(\phi_1, \phi_2)(+\infty) = E_+$ and ϕ_1, ϕ_2 are positive at space minus infinity, if and only if

$$c \geq c_* := \inf_{\lambda>0} \frac{\left[e^{\lambda} + e^{-\lambda} - 2\right] + \beta - \mu - \gamma}{\lambda}.$$

For this problem,

$$g_1(S, I) = \mu\frac{1}{S} - \beta I, \qquad g_2(S, I) = \beta S - (\mu + \gamma)$$

and it is easy to check (1.14). If it is possible to choose $R = \beta - (\mu - \gamma)$, then we can apply Theorem 1.1 to derive the stability of traveling waves. Unfortunately, there is a possibility that ϕ_1 become sufficiently small at some $z \in \mathbb{R}$, and this is the obstacles of applying Theorem 1.1.

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