

THE EXACT NUMBER OF STATIONARY SOLUTIONS TO A MICRO-ELECTRO MECHANICAL SYSTEM WITH FRINGING FIELD

HUNG-JUI GUO, JONG-SHENQ GUO, MASAHIKO SHIMOJO, AND CHI-JEN WANG

ABSTRACT. In this paper, we derive the exact multiplicity of a boundary value problem for a quasilinear differential equation arising in the study of MEMS model with fringing field effect. This extends a result obtained in the work by Guo et al. (J. Diff. Eqns., 269 (2020), 7676-7704) to answer an open question left in the paper by Wei and Ye (Proc. Amer. Math. Soc., 138 (2010), 1693-1699).

1. INTRODUCTION

The purpose of this paper is to investigate the exact number of solutions to the following boundary value problem (P) for $u = u(x)$:

$$(1.1) \quad -u_{xx} = \lambda \frac{1 + \delta u_x^2}{(1 - u)^2}, \quad x \in (-1, 1),$$

$$(1.2) \quad u(\pm 1) = 0,$$

where λ and δ are positive constants. Recall from [15] that problem (1.1)-(1.2) has no solution for $\lambda > \lambda^*$, a unique solution for $\lambda = \lambda^*$, and at least 2 solutions for $\lambda \in (0, \lambda^*)$ for a finite positive constant $\lambda^* = \lambda^*(\delta)$. However, the exact number of solutions for $\lambda \in (0, \lambda^*)$ remains open.

Problem (1.1)-(1.2) describes the stationary solutions of the following parabolic equation

$$(1.3) \quad u_t = \Delta u + \lambda \frac{1 + \delta |\nabla u|^2}{(1 - u)^2}, \quad x \in \Omega \subset \mathbb{R}^2, \quad t > 0,$$

with the zero Dirichlet boundary condition. Equation (1.3) describes the dynamic deflection $u(x, t)$ of an elastic membrane inside a micro-electro mechanical system (MEMS), in the case when the distance between the plate and the membrane is relative small compared to the length of the device, when we ignore the inertia, the device embedded in an electric circuit without a capacitor, and the system is with the fringing field effect $\delta |\nabla u|^2$. In (1.3), λ is

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proportional to the square of the applied voltage. The zero Dirichlet boundary condition corresponds to the case when the edge of the membrane is clamped. For more physical background of MEMS, we refer the reader to, e.g., [11, 12, 13].

There were extensive works on the analysis of MEMS model, we refer the reader to [5, 3, 6] for model without fringing field, and [15, 1, 9, 14, 2, 8, 10] for model with fringing field. For the study of stationary solutions, we refer to the earlier works [7, 3]. An open question about the exact multiplicity of solutions raised in [15] was solved in [4] when $\delta = 1$. However, it is not clear whether the process carried out in [4] for $\delta = 1$ can be extended to any $\delta > 0$. The purpose of this paper is to clarify this question for $\delta \in (0, 1)$.

Our main theorem reads

Theorem 1.1. *Let $\delta \in (0, 1)$ and let $\lambda^* = \lambda^*(\delta)$ be the positive critical value obtained in [15]. Then problem (1.1)-(1.2) has exactly two solutions for each $\lambda \in (0, \lambda^*)$.*

The rest of this paper is organized as follows. In §2, for a given $\delta > 0$ we first derive an implicit formula for $\mu := \lambda\delta$ and $\eta := u(0; \mu)$, where $u = u(x; \mu)$ is a solution of (1.1)-(1.2). Then we give some useful estimates for $\mu(\eta)$. Finally, we give a proof of Theorem 1.1 in §3. Although most of the processes of the proof of Theorem 1.1 are originated from [4], there are certain nontrivial modifications due to the dependence on the parameter δ . To be self-contained, we provide as much details as it needs for the reader's convenience.

2. PRELIMINARIES

In the sequel, we fix the constant $\delta > 0$ and set $\mu := \lambda\delta$. Also, we let $u = u(x; \mu)$ be a (classical) solution of (1.1)-(1.2) for a given (λ, δ) . Note that any solution u of problem (1.1)-(1.2) is strictly concave and symmetric (with respect to $x = 0$). In this section, we first derive a relation between μ and the maximum of u , i.e., $u(0; \mu) := \eta$. Then we derive some useful estimates to be used later.

First, motivated by [15, 4], we rewrite (1.1) as

$$(2.1) \quad -(e^{\frac{\mu}{1-u}} u')' = \frac{\lambda e^{\frac{\mu}{1-u}}}{(1-u)^2}, \quad u' := u_x.$$

Multiplying both sides of (2.1) by $e^{\frac{\mu}{1-u}} u'$, we obtain

$$-(e^{\frac{\mu}{1-u}} u')'(e^{\frac{\mu}{1-u}} u') = \frac{\lambda e^{\frac{\mu}{1-u}}}{(1-u)^2} (e^{\frac{\mu}{1-u}} u').$$

By integrating it from 0 to $x > 0$, using $u'(0) = 0$ and $u(0) = \eta$, we get

$$-(e^{\frac{\mu}{1-u}} u')^2 = \frac{1}{\delta} \left(e^{\frac{2\mu}{1-u}} - e^{\frac{2\mu}{1-\eta}} \right).$$

It follows that

$$\left(e^{\frac{\mu}{1-u}} u'\right)(x) = -\frac{1}{\sqrt{\delta}} \sqrt{e^{\frac{2\mu}{1-\eta}} - e^{\frac{2\mu}{1-u(x)}}}, \quad x > 0,$$

and so

$$(2.2) \quad \left(e^{\frac{\mu}{1-u}} (e^{\frac{2\mu}{1-\eta}} - e^{\frac{2\mu}{1-u}})^{-1/2} u'\right)(x) = -\frac{1}{\sqrt{\delta}}, \quad x > 0.$$

By integrating (2.2) from 1 to $x \in [0, 1]$, using $u(1) = 0$ we obtain

$$(2.3) \quad \frac{1-x}{\sqrt{\delta}} = \int_0^{u(x)} e^{\frac{\mu}{1-s}} \left(e^{\frac{2\mu}{1-\eta}} - e^{\frac{2\mu}{1-s}}\right)^{-1/2} ds = \int_0^{u(x)} \left(e^{2\mu(\frac{1}{1-\eta} - \frac{1}{1-s})} - 1\right)^{-1/2} ds.$$

It follows from $u(0) = \eta$ that $I(\eta, \mu) = 1/\sqrt{\delta}$, where

$$(2.4) \quad I(\eta, a) := \int_0^\eta \left(e^{2a(\frac{1}{1-\eta} - \frac{1}{1-s})} - 1\right)^{-1/2} ds, \quad a > 0, \quad \eta \in (0, 1).$$

The following lemma plays an important role for the estimation of μ .

Lemma 2.1. *Given $\delta > 0$. For each $\eta \in (0, 1)$, $I(\eta, a)$ is continuous and decreasing in $a \in (0, \infty)$, and there is a unique positive $\mu = \mu(\eta)$ such that $I(\eta, \mu) = 1/\sqrt{\delta}$.*

Proof. Given $\eta \in (0, 1)$. It is clearly that $I(\eta, a)$ is continuous and decreasing in a . Since $I(\eta, a) \rightarrow \infty$ as $a \rightarrow 0^+$, and $I(\eta, a) \rightarrow 0$ as $a \rightarrow \infty$, the lemma follows. $\square$

From Lemma 2.1 it follows that the corresponding solution $u(x)$ of (1.1)-(1.2) can be determined from (2.3) for a given $\eta \in (0, 1)$ with $u(0) = \eta$ and $\mu = \mu(\eta)$.

Here are some numerical results of $\mu(\eta)$ for $\delta = 1, 0.5, 0.1$. The case of $\delta = 1$ is the same as Figure 1 in [4].

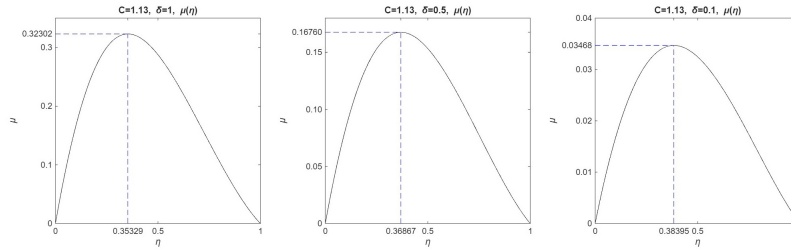


FIGURE 1. Curves of $\mu(\eta)$ derived from $I(\eta, \mu) = 1/\sqrt{\delta}$ for $\delta = 1, 0.5, 0.1$.

Moreover, we have the following upper and lower bounds of $\mu(\eta)$ in terms of the quadratic polynomial $\eta(1 - \eta)$.

Lemma 2.2. *Given $\delta > 0$. It holds $c_1\eta(1 - \eta) \leq \mu(\eta) \leq c_2\eta(1 - \eta)$ for all $\eta \in (0, 1)$, where $c_1 = \ln(\sqrt{1 + \delta})$ and $c_2 = 2\delta$.*

Proof. First, for a positive constant C , we compute

$$\begin{aligned} I(\eta, C\eta(1-\eta)) &:= \int_0^\eta \frac{1}{\sqrt{e^{2C\eta(1-\eta)[1/(1-\eta)-1/(1-s)]} - 1}} ds \\ &\leq \int_0^\eta \frac{1}{\sqrt{2C\eta(\eta-s)/(1-s)}} ds \\ &\leq \frac{1}{\sqrt{2C\eta}} \int_0^\eta \frac{1}{\sqrt{\eta-s}} ds = \sqrt{\frac{2}{C}}, \end{aligned}$$

using $e^x - 1 \geq x$ for all $x > 0$ and $1-s \leq 1$ for all $s \in (0, \eta)$. Therefore, with $C = 2\delta$, by Lemma 2.1, we deduce that $\mu(\eta) \leq 2\delta\eta(1-\eta)$ for any $\eta \in (0, 1)$.

For the lower bound estimate, we observe that

$$\begin{aligned} I(\eta, (\ln \sqrt{1+\delta})\eta(1-\eta)) &= \int_0^\eta \frac{1}{\sqrt{e^{2(\ln \sqrt{1+\delta})\eta(1-\eta)[1/(1-\eta)-1/(1-s)]} - 1}} ds \\ &= \int_0^\eta \frac{1}{\sqrt{(1+\delta)^{\eta(\frac{\eta-s}{1-s})} - 1}} ds \\ &\geq \int_0^\eta \frac{1}{\sqrt{(1+\delta)^{\eta^2} - 1}} ds = \frac{\eta}{\sqrt{(1+\delta)^{\eta^2} - 1}} > \frac{1}{\sqrt{\delta}}. \end{aligned}$$

Here we have used $0 < (1+\delta)^y < 1+\delta y$ for all $y \in (0, 1)$. This can be checked by observing that the function $g(y) := (1+\delta)^y$ is convex for $y \in (0, 1)$ and $g(y) = 1+\delta y$ for $y = 0, 1$. Hence the lower bound estimate of $\mu(\eta)$ is derived and the proof is complete. $\square$

Next, we establish a better upper bound of μ in terms of the following cubic polynomial

$$(2.5) \quad H(\eta) := [c_1\eta + c_2(1-\eta)]\eta(1-\eta),$$

which is motivated by Lemma 2.2 and the Hermite interpolation. Note that $c_4 = 1.1$ was chosen in [4] for the case $\delta = 1$, but here we choose a larger constant $c_4 = 1.13$ that works for all $\delta \in (0, 1]$.

Lemma 2.3. *Let $\delta \in (0, 1]$. It holds $\mu(\eta) \leq c_4 H(\eta)$ for all $\eta \in (0, 1)$, where $c_4 = 1.13$.*

Proof. Given $\delta \in (0, 1]$. Let

$$b(\eta, s) := 2CH(\eta) \left[\frac{1}{1-\eta} - \frac{1}{1-s} \right] = 2D \left(\frac{\eta-s}{1-s} \right) > 0, \quad 0 < s < \eta < 1,$$

where

$$D := D(\eta) = C[c_1\eta + c_2(1-\eta)]\eta > 0, \quad \forall \eta \in (0, 1),$$

and $C \in (1, 2)$ is a constant to be determined later. Using $e^z - 1 > z + z^2/2$ for all $z > 0$, we obtain

$$\int_0^\eta \frac{1}{\sqrt{e^b - 1}} ds \leq \int_0^\eta \frac{1}{\sqrt{b + \frac{1}{2}b^2}} ds = \frac{1}{\sqrt{2D}} \int_0^\eta \frac{1}{\sqrt{\left(\frac{\eta-s}{1-s}\right) + D \left(\frac{\eta-s}{1-s}\right)^2}} ds := \frac{1}{\sqrt{2D}} J(\eta, D).$$

By [4, Property 5.4], we have

$$J(\eta, D) = \frac{\{-\ln(1-\eta) + 2\ln[h_1(\eta) + h_2(\eta)]\} (D + \frac{1}{2})(1-\eta) + h_1(\eta)h_2(\eta)}{(D+1)^{3/2}},$$

where $h_1(\eta) := \sqrt{D(\eta)\eta + 1}$ and $h_2(\eta) := \sqrt{(D(\eta) + 1)\eta}$.

We consider the function

$$G(\eta) = G(\eta; \delta) := \frac{(D+1)^{3/2}}{(D + \frac{1}{2})(1-\eta)} [\sqrt{2D} - \sqrt{\delta} J(\eta, D)], \quad \eta \in (0, 1).$$

Then we have

$$G(\eta; \delta) = \frac{(D+1)^{3/2}\sqrt{2D}}{(D + \frac{1}{2})(1-\eta)} - \sqrt{\delta} \frac{h_1 h_2}{(D + \frac{1}{2})(1-\eta)} - \sqrt{\delta} \ln \left(\frac{(h_1 + h_2)^2}{1-\eta} \right).$$

Note that $G(0; \delta) = 0$.

Now, we show that $\frac{dG}{d\eta}(\eta; \delta) > 0$ for $\eta \in (0, 1)$ for $\delta \in (0, 1]$. To see this, we compute the following three derivatives w.r.t. η .

First, we compute

$$\begin{aligned} & \left\{ \frac{\sqrt{2D(D+1)^3}}{(D + \frac{1}{2})(1-\eta)} \right\}' \\ &= \frac{\frac{(4D+1)(D+1)^2 D'}{\sqrt{(D+1)^3 D}}}{\sqrt{2}(D + \frac{1}{2})(1-\eta)} - \frac{\sqrt{2}\sqrt{(D+1)^3 D}(-\frac{1}{2} - D + (1-\eta)D')}{(D + \frac{1}{2})^2(1-\eta)^2} \\ &= \frac{\sqrt{D+1}\{2D(D + \frac{1}{2})(D+1) + (1-\eta)(\frac{1}{2} + D + 2D^2)D'\}}{\sqrt{2}\sqrt{D}(D + \frac{1}{2})^2(1-\eta)^2} \\ &:= \frac{\sqrt{D+1}P_6(\eta; \delta)}{\sqrt{2}\sqrt{D}(D + \frac{1}{2})^2(1-\eta)^2}, \end{aligned}$$

where $P_6(\eta; \delta)$ is a polynomial of degree 6 in η with δ dependent coefficients expressed by

$$\begin{aligned} P_6(\eta; \delta) &:= \frac{1}{2}c_2C + (c_1 - \frac{c_2}{2} + c_2^2C)C\eta + (-c_2^2 + 3c_1c_2 + 2c_2^3C)C^2\eta^2 \\ &\quad + [(-c_2^2 - c_1c_2 + 2c_1^2) + 8c_2^2(c_1 - c_2)C]C^2\eta^3 \\ &\quad + [(c_1 - c_2)^2 + 2c_2(c_1 - c_2)(5c_1 - 6c_2)C]C^2\eta^4 \\ &\quad + 4(c_1 - 2c_2)(c_1 - c_2)^2C^3\eta^5 - 2(c_1 - c_2)^3C^3\eta^6. \end{aligned}$$

Secondly, we have

$$\begin{aligned}
& \left\{ \frac{h_1 h_2}{(D + \frac{1}{2})(1 - \eta)} \right\}' \\
&= \frac{1 + D + 2\eta(D + D^2) + [\eta + \eta^2(2D + 1)]D'}{2h_1 h_2 (D + \frac{1}{2})(1 - \eta)} - \frac{h_1 h_2 [-(D + \frac{1}{2}) + (1 - \eta)D']}{(D + \frac{1}{2})^2 (1 - \eta)^2} \\
&= \frac{(1 + \eta + 2\eta D)(1 + D)(\frac{1}{2} + D) + \eta(1 - \eta)(-\frac{3}{2} + \frac{1}{2}\eta - D)D'}{2h_1 h_2 (D + \frac{1}{2})^2 (1 - \eta)^2} \\
&:= \frac{P_7(\eta; \delta)}{2h_1 h_2 (D + \frac{1}{2})^2 (1 - \eta)^2},
\end{aligned}$$

where P_7 is a polynomial of degree 7 in η expressed by

$$\begin{aligned}
P_7(\eta; \delta) &:= \frac{1}{2} + \frac{1}{2}\eta + \left(6c_2 - \frac{3}{2}c_1\right)C\eta^2 + \left[-7c_2 + \frac{13}{2}c_1 + c_2(6c_2 - c_1)C\right]C\eta^3 \\
&\quad + [(c_2 - c_1) + (-12c_2^2 + 13c_1c_2 - c_1^2)C + 2c_2^3C^2]C\eta^4 \\
&\quad + [6(c_2 - c_1)^2 - 6c_2^2(c_2 - c_1)C]C^2\eta^5 + [6c_2(c_2 - c_1)^2]C^3\eta^6 \\
&\quad + [-2(c_2 - c_1)^3]C^3\eta^7.
\end{aligned}$$

Thirdly, we compute

$$\begin{aligned}
\left\{ \ln \left(\frac{(h_1 + h_2)^2}{1 - \eta} \right) \right\}' &= \frac{\frac{D + \eta D'}{h_1} + \frac{1 + D + \eta D'}{h_2}}{h_1 + h_2} + \frac{1}{1 - \eta} \\
&= \frac{(\frac{D + \eta D'}{h_1} + \frac{1 + D + \eta D'}{h_2})(h_1 - h_2)}{1 - \eta} + \frac{1}{1 - \eta} \\
&= \frac{-(D\eta + \eta)(D + \eta D') + (D\eta + 1)(1 + D + \eta D')}{(1 - \eta)h_1 h_2} \\
&= \frac{(1 + D) + \eta(1 - \eta)D'}{(1 - \eta)h_1 h_2} := \frac{P_3(\eta; \delta)}{(1 - \eta)h_1 h_2},
\end{aligned}$$

where P_3 is a polynomial of degree 3 in η expressed by

$$P_3(\eta; \delta) := 1 + 2c_2C\eta + (-4c_2 + 3c_1)C\eta^2 + 2(c_2 - c_1)C\eta^3.$$

We conclude that

$$\frac{dG}{d\eta}(\eta; \delta) = \frac{\sqrt{2}h_1\sqrt{\eta}(D + 1)P_6 - \sqrt{\delta}\sqrt{D}[P_7 + 2(D + \frac{1}{2})^2(1 - \eta)P_3]}{2\sqrt{D}h_1h_2(D + \frac{1}{2})^2(1 - \eta)^2}.$$

To proceed further, we consider the following real polynomial of degree 19 in variable η with δ dependent coefficients:

$$P(\eta; \delta) := 4(D\eta + 1)(D + 1)^2(P_6)^2 - c_2C[c_1\eta + c_2(1 - \eta)] \left(P_7 + 2(D + \frac{1}{2})^2(1 - \eta)P_3 \right)^2.$$

Then $\frac{dG}{d\eta}(\eta; \delta) > 0$ is equivalent to $P(\eta; \delta) > 0$ for $\eta \in (0, 1)$ for $\delta \in (0, 1]$.

Note that $P(\eta; 0) \equiv 0$ for all $\eta \in (0, 1)$. Moreover, as $\delta \downarrow 0^+$, we have $c_1(\delta) \approx \delta/2$. Combining this with $c_2(\delta) = 2\delta$, we compute the Taylor's expansion of $P(\eta; \delta)$ in δ as

$$\begin{aligned} P(\eta; \delta) &= P(\eta; 0) + \frac{\partial P}{\partial \delta}(\eta; 0)\delta + \frac{\frac{\partial^2 P}{\partial \delta^2}(\eta; 0)}{2}\delta^2 + \frac{\frac{\partial^3 P}{\partial \delta^3}(\eta; \xi)}{6}\delta^3 \\ &= [(-4 + 4C) + (3 - 4C)\eta + C\eta^2]C\delta^2 + \frac{\frac{\partial^3 P}{\partial \delta^3}(\eta; \xi)}{6}\delta^3 \text{ for some } \xi \in (0, \delta). \end{aligned}$$

Set $\Lambda(\eta) := (-4 + 4C) + (3 - 4C)\eta + C\eta^2$, $0 < \eta < 1$. Note that the discriminant of the quadratic polynomial $\Lambda(\eta)$ is given by $9 - 8C$, which is negative if and only $C > 9/8 = 1.125$, and $\Lambda(0) > 0$ if $C > 1$. We conclude that $P(\eta; \delta) > 0$ for all $\eta \in (0, 1)$ for $\delta > 0$ small enough, if we choose $C > 9/8$. In particular, this holds if we choose $C = 1.13$.

For general $\delta > 0$, we show numerically using Python in Figure 2 the 2D graphs of $P(\eta; \delta)$, $\eta \in (0, 1)$, for $\delta \in (0, 1)$ with $C = 1.13$. One can observe that these graphs are all above the η -axis. There is also an “Interval Min” in each mini graph of Figure 2, indicating that all values of $P(\eta; \delta)$ within these ranges are positive. Therefore, this shows (numerically) that $\frac{dG}{d\eta}(\eta; \delta) > 0$ for $\eta \in (0, 1)$ for $\delta \in (0, 1]$.

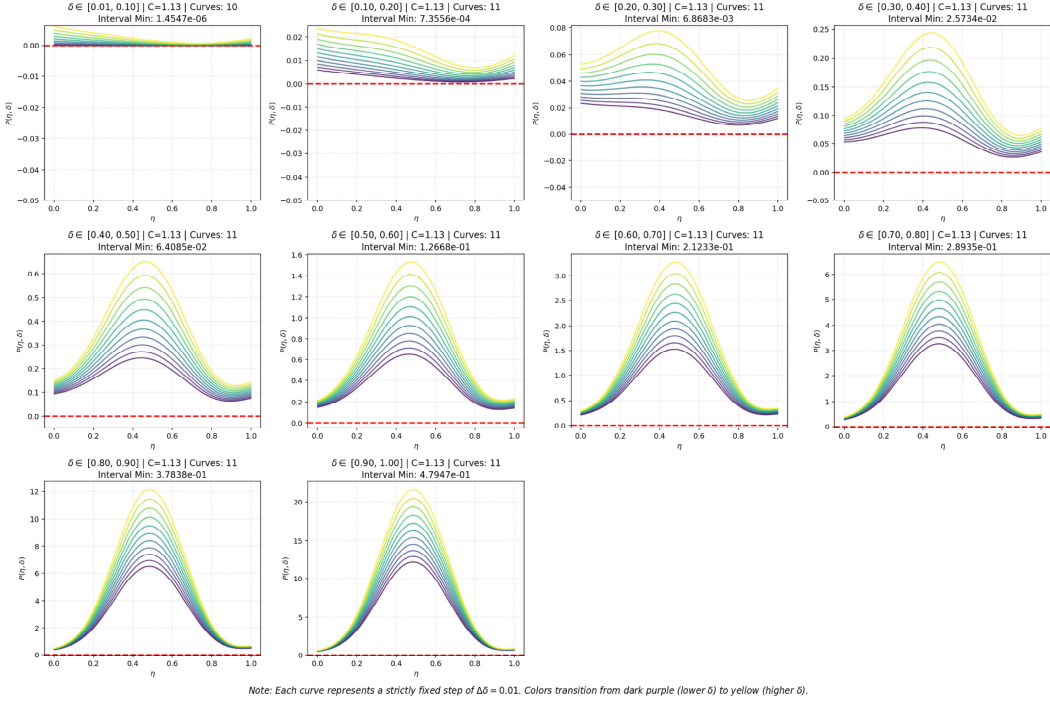


FIGURE 2. 2D graphs of $P(\eta; \delta)$, $0 < \eta < 1$, for $\delta \in (0, 1)$

Since G is increasing in $\eta \in (0, 1)$ and $G(0; \delta) = 0$, we have $G(\eta; \delta) \geq 0$ for $\eta \in (0, 1)$ for $\delta \in (0, 1]$. It follows that $\sqrt{2D} \geq \sqrt{\delta}J(\eta, D)$ for all $\eta \in (0, 1)$. This implies that

$$I(\eta, c_4 H(\eta)) = \int_0^\eta \frac{1}{\sqrt{e^b - 1}} ds \leq \frac{1}{\sqrt{2D}} J(\eta, D) \leq \frac{1}{\sqrt{\delta}}, \forall \eta \in (0, 1).$$

Hence the lemma is proved rigorously for small $\delta > 0$ and numerically for $\delta \in (0, 1]$. $\square$

3. PROOF OF THEOREM 1.1

Our aim is to derive the exact number of solutions of problem (1.1)-(1.2) for any given $\mu_0 > 0$. This is equivalent to finding the number of η 's in $(0, 1)$ such that $\mu(\eta) = \mu_0$. To this end, we shall show that $\mu(\eta)$ has only one critical point in $(0, 1)$.

First, we recall from (2.4) and $I(\eta; \mu) = 1/\sqrt{\delta}$ that

$$(3.1) \quad \int_0^\eta \frac{e^{\mu[r(s)-r(\eta)]}}{\sqrt{1 - e^{2\mu[r(s)-r(\eta)]}}} ds = \frac{1}{\sqrt{\delta}}.$$

Hereafter $r(y) := 1/(1 - y)$. As in [4], we introduce the following functions:

$$\begin{cases} Q(\eta) := \frac{e^{-\mu\eta r(\eta)}}{\sqrt{1 - e^{-2\mu\eta r(\eta)}}} = \frac{1}{\sqrt{e^{2\mu\eta r(\eta)} - 1}}, \\ I_n(\eta) := \int_0^\eta \frac{e^{\mu[r(s)-r(\eta)]}}{\sqrt{1 - e^{2\mu[r(s)-r(\eta)]}}} (1 - s)^n ds, \quad n \in \mathbb{Z}, \\ J_n(\eta) := \int_0^\eta \arcsin(e^{\mu[r(s)-r(\eta)]}) (1 - s)^n ds, \quad n \in \mathbb{Z}. \end{cases}$$

It is easy to check that

$$0 < J_n(\eta) \leq \frac{\pi}{2} \frac{1 - (1 - \eta)^{n+1}}{n + 1}, \quad \text{if } n \neq -1; \quad -\frac{\pi}{2} \ln(1 - \eta), \quad \text{if } n = -1.$$

For any integer n , using the identity

$$\begin{aligned} & \frac{d}{ds} \{ \arcsin(e^{\mu[r(s)-r(\eta)]}) (1 - s)^{n+2} \} \\ &= \mu \frac{e^{\mu[r(s)-r(\eta)]}}{\sqrt{1 - e^{2\mu[r(s)-r(\eta)]}}} (1 - s)^n - (n + 2) \arcsin(e^{\mu[r(s)-r(\eta)]}) (1 - s)^{n+1}, \end{aligned}$$

an integration by parts gives

$$(3.2) \quad \mu I_n(\eta) = \frac{\pi}{2} (1 - \eta)^{n+2} - \arcsin(e^{-\mu\eta r(\eta)}) + (n + 2) J_{n+1}(\eta), \quad \forall \eta \in (0, 1), \quad n \in \mathbb{Z}.$$

In particular, taking $n = 0$ and using (3.1), we obtain from (3.2) that

$$(3.3) \quad \frac{\mu}{\sqrt{\delta}} = \frac{\pi}{2} (1 - \eta)^2 - \arcsin(e^{-\mu\eta r(\eta)}) + 2J_1(\eta).$$

Differentiating (3.3) with respect to η and using (3.1), we get

$$\frac{\mu'}{\sqrt{\delta}} = [\mu(1 - \eta)^{-2} + \mu'\eta(1 - \eta)^{-1}]Q(\eta) + 2\frac{\mu'}{\sqrt{\delta}} - 2[\mu(1 - \eta)^{-2} + \mu'(1 - \eta)^{-1}]I_1(\eta).$$

Hence we obtain

$$(3.4) \quad [2I_1(\eta) - \eta Q(\eta) - (1 - \eta)/\sqrt{\delta}] \mu'(\eta) = \frac{\mu(\eta)}{1 - \eta} [Q(\eta) - 2I_1(\eta)].$$

Moreover, we have the following formula for $\mu'(\eta)$:

$$(3.5) \quad \mu'(\eta) = \frac{\mu(\eta)}{1 - \eta} \cdot \frac{-2I_1(\eta) + Q(\eta)}{2I_1(\eta) - \eta Q(\eta) - (1 - \eta)/\sqrt{\delta}},$$

if $2I_1(\eta) - \eta Q(\eta) - (1 - \eta)/\sqrt{\delta} \neq 0$.

Next, we prove the following important lemma. Although the idea of the proof is originated from [4], due to the δ -dependence, the proof here is much more complex.

Lemma 3.1. *Let $\delta \in (0, 1]$. It holds $2I_1(\eta) - \eta Q(\eta) - \frac{(1-\eta)}{\sqrt{\delta}} > 0$ for all $\eta \in (0, 1)$.*

Proof. Following [4], we consider the quartic function (polynomial of degree 4)

$$P_4(\eta) = [(1 + \frac{\eta}{3})(1 - \eta) + 1]^2 - \frac{4c_4}{c_2} [c_1\eta + c_2(1 - \eta)], \quad \eta \in (0, 1).$$

We compute

$$P'_4(\eta) = \frac{4}{9} \left\{ (\eta + 1)^3 - 7(\eta + 1) + 9c_4 \left(1 - \frac{c_1}{c_2} \right) \right\}.$$

Note that the function $f(\delta) := \ln(1 + \delta)/\delta$ is decreasing in $\delta > 0$ such that $f(0^+) = 1$ and $f(+\infty) = 0$. Let $p = -7$ and $q = 9c_4(1 - c_1/c_2) = 9c_4[1 - f(\delta)/4]$. We compute

$$\frac{q^2}{4} + \frac{p^3}{27} = \frac{81}{4} \left[1 - \frac{f(\delta)}{4} \right]^2 c_4^2 - \frac{343}{27} > \frac{729}{64} c_4^2 - \frac{343}{27} > 0, \text{ due to } c_4 > 1.056.$$

Then, using Cardano's formula, the cubic polynomial $P'_4(\eta)$ has only one (real) root at

$$\eta = -1 + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} < -1.$$

Hence $P_4(\eta)$ is increasing for $\eta \in [0, 1]$. Since $P_4(0) = 0$, we have $P_4(\eta) > 0$ for $\eta \in (0, 1)$.

Therefore, using $c_2 = 2\delta$, we obtain

$$(3.6) \quad \frac{\sqrt{2[c_1\eta + c_2(1 - \eta)]} \sqrt{\frac{c_4}{\delta}} - 1}{1 - \eta} < 1 + \frac{\eta}{3}, \quad \eta \in (0, 1).$$

On the other hand, using

$$\frac{1}{\sqrt{\eta}} \ln \left(\frac{1 + \sqrt{\eta}}{\sqrt{1 - \eta}} \right) > 1 + \frac{\eta}{3}, \quad \eta \in (0, 1),$$

we obtain from (3.6) that

$$(3.7) \quad \frac{1}{\sqrt{\eta}} \ln \left(\frac{1 + \sqrt{\eta}}{\sqrt{1 - \eta}} \right) > \frac{\sqrt{2[c_1\eta + c_2(1 - \eta)]} \sqrt{\frac{c_4}{\delta}} - 1}{1 - \eta}, \quad \eta \in (0, 1).$$

Next, since

$$\max_{\eta \in [0,1]} \{2[c_1\eta + c_2(1-\eta)]\eta^2\} = \frac{8c_2^3}{27(c_2 - c_1)^2},$$

we have

$$(3.8) \quad y(\eta) := 2c_4[c_1\eta + c_2(1-\eta)]\eta^2 \leq c_4 \cdot \frac{8c_2^3}{27(c_2 - c_1)^2} < 1, \forall \eta \in [0, 1].$$

Indeed, define

$$F(\delta) := \frac{8c_2^3}{27(c_2 - c_1)^2} = \frac{256\delta}{27[4 - \frac{\ln(1+\delta)}{\delta}]^2} =: \frac{256}{27} \frac{\delta}{g(\delta)^2}.$$

Then we claim that $F(\delta)$ is increasing in $\delta \in (0, 1]$. For this, we differentiate the identity

$$\ln \left[\frac{27}{256} F(\delta) \right] = \ln \delta - 2 \ln[g(\delta)]$$

to obtain

$$\frac{F'(\delta)}{F(\delta)} = \frac{1}{\delta} - 2 \frac{g'(\delta)}{g(\delta)} = \frac{h(\delta)}{\delta g(\delta)}, \quad h(\delta) := g(\delta) - 2\delta g'(\delta).$$

Now we compute

$$h(\delta) = 4 - \frac{\ln(1+\delta)}{\delta} - 2\delta \left\{ \frac{\ln(1+\delta)}{\delta^2} - \frac{1}{\delta(1+\delta)} \right\} = 4 - 3 \frac{\ln(1+\delta)}{\delta} + \frac{2}{1+\delta}.$$

Since $\ln(1+\delta)/\delta \in (0, 1)$ for $\delta > 0$, we obtain that $h(\delta) > 0$ for all $\delta > 0$. Hence $F(\delta)$ is increasing in $\delta > 0$. Note that

$$F(0^+) = 0, \quad F(1) \approx 0.867.$$

Hence the last inequality in (3.8) holds for any $\delta \in (0, 1]$, by the choice of c_4 in Lemma 2.3.

With (3.8), it follows from $e^y - 1 < 2y$ for $0 < y \leq 1$ and (3.7) that

$$\begin{aligned} & 2\sqrt{e^{2c_4[c_1\eta + c_2(1-\eta)]\eta^2} - 1} \\ & < 2\sqrt{4c_4[c_1\eta + c_2(1-\eta)]\eta^2} = \sqrt{8c_4}\eta\sqrt{2[c_1\eta + c_2(1-\eta)]} \\ & < 3\sqrt{c_4}\eta\sqrt{2[c_1\eta + c_2(1-\eta)]} < \sqrt{\delta} \left\{ 3\eta + 3\sqrt{\eta}(1-\eta) \ln \left(\frac{1+\sqrt{\eta}}{\sqrt{1-\eta}} \right) \right\}, \quad \eta \in (0, 1). \end{aligned}$$

Now, we recall from [4, (3.10)] the following estimate:

$$(3.9) \quad I_1(\eta) \geq \frac{1}{4} \left[(5 - 3\eta)\eta + 3\sqrt{\eta}(1-\eta)^2 \ln \left(\frac{1+\sqrt{\eta}}{\sqrt{1-\eta}} \right) \right] Q(\eta).$$

Then, from (3.9) and the above estimation, it follows that

$$\begin{aligned}
& 2I_1(\eta) - \eta Q(\eta) - (1 - \eta)/\sqrt{\delta} \\
\geq & Q(\eta) \left(\frac{3}{2} \right) \left[(1 - \eta)\eta + \sqrt{\eta}(1 - \eta)^2 \ln \left(\frac{1 + \sqrt{\eta}}{\sqrt{1 - \eta}} \right) \right] - (1 - \eta)/\sqrt{\delta} \\
= & Q(\eta) \left(\frac{1 - \eta}{2\sqrt{\delta}} \right) \left[3\sqrt{\delta}\eta + 3\sqrt{\delta}\sqrt{\eta}(1 - \eta) \ln \left(\frac{1 + \sqrt{\eta}}{\sqrt{1 - \eta}} \right) - 2\sqrt{e^{2\mu\eta r(\eta)} - 1} \right] \\
> & Q(\eta) \left(\frac{1 - \eta}{\sqrt{\delta}} \right) \left(\sqrt{e^{2c_4[c_1\eta + c_2(1 - \eta)]\eta^2} - 1} - \sqrt{e^{2\mu\eta r(\eta)} - 1} \right) \geq 0,
\end{aligned}$$

using Lemma 2.3. The lemma is thus proved. $\square$

From Lemma 3.1 and (3.5), we see that

$$(3.10) \quad \mu'(\eta) = 0 \text{ if and only if } Q(\eta) = 2I_1(\eta).$$

Recall the quantity $r_2 \in (0.38, 0.39)$ defined in [4, Lemma 3.2] which satisfies

$$(3.11) \quad (5 - 3x)x + 3\sqrt{x}(1 - x)^2 \ln \left(\frac{1 + \sqrt{x}}{\sqrt{1 - x}} \right) < 2 \text{ for } x \in (0, r_2),$$

$$(3.12) \quad (5 - 3x)x + 3\sqrt{x}(1 - x)^2 \ln \left(\frac{1 + \sqrt{x}}{\sqrt{1 - x}} \right) > 2 \text{ for } x \in (r_2, 1).$$

Therefore, we obtain the following important property for the critical points of μ :

$$(3.13) \quad \mu'(\eta) = 0 \text{ only when } \eta \leq r_2,$$

using (3.9), (3.12) and (3.4). As shown in Figure 1, the critical μ^* are consistent with the above derived property.

Our goal is to derive that μ has a unique critical point in $(0, 1)$. This can be done if we can derive that $\mu''(\eta) < 0$ when $\mu'(\eta) = 0$. In the sequel, we let $\bar{\eta} \in (0, r_2]$ be any point such that $\mu'(\bar{\eta}) = 0$ and set $\bar{\mu} = \mu(\bar{\eta})$. Using $\mu'(\bar{\eta}) = 0$, (3.5) and (3.10), we can easily derive that

$$(3.14) \quad \mu''(\bar{\eta}) = \frac{\bar{\mu}}{1 - \bar{\eta}} \cdot \frac{-2I_1'(\bar{\eta}) + Q'(\bar{\eta})}{2I_1(\bar{\eta}) - \bar{\eta}Q(\bar{\eta}) - (1 - \bar{\eta})/\sqrt{\delta}}.$$

Also, we recall from [4, (3.17)]

$$(3.15) \quad -2I_1'(\bar{\eta}) + Q'(\bar{\eta}) = \frac{1}{(1 - \bar{\eta})^2} \left\{ 6I_2(\bar{\eta}) - \left[2 + \frac{\bar{\mu}}{1 - e^{-2\bar{\mu}\bar{\eta}/(1 - \bar{\eta})}} \right] Q(\bar{\eta}) \right\}.$$

Next, exactly the same proof of [4, Lemma 3.3] gives

Lemma 3.2. *It holds*

$$6I_2(\eta) - \left[2 + \frac{\mu}{1 - e^{-2\mu\eta/(1 - \eta)}} \right] Q(\eta) < 0$$

for $\eta \in (0, r_2]$.

Note that here we have $x_0 = x_0(\delta) := 2c_2/[3(c_2 - c_1)] \in (2/3, 8/9)$ for $\delta \in (0, \infty)$ in the proof of [4, Lemma 3.3]. It holds that $x_0(\delta) > r_2$ for any $\delta > 0$. Therefore, we safely omit the details of the proof of Lemma 3.2.

Now we are ready to prove the following uniqueness on the critical points of $\mu(\eta)$.

Proposition 3.3. *There is a unique point η^* on $(0, 1)$ such that $\mu'(\eta) > 0$ for $0 < \eta < \eta^*$, $\mu'(\eta^*) = 0$, and $\mu'(\eta) < 0$ for $\eta^* < \eta < 1$.*

Proof. Since, by Lemma 2.2, $\mu(0^+) = \mu(1^-) = 0$ and $\mu(\eta) > 0$ for $\eta \in (0, 1)$, there exists a point $\eta^* \in (0, 1)$ such that $\mu'(\eta^*) = 0$. By (3.13), $\eta^* \in (0, r_2]$. From (3.14), (3.15) and Lemma 3.2 it follows that $\mu''(\eta^*) < 0$. Hence $\mu(\eta)$ can only have one critical point η^* and the proposition is proved. $\square$

Finally, Theorem 1.1 is just a direct consequence of Proposition 3.3.

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DEPARTMENT OF COMPUTER SCIENCE AND INFORMATION ENGINEERING, TAINAN 70101, NATIONAL CHENG KUNG UNIVERSITY

Email address: `hjguo@gs.ncku.edu.tw`

DEPARTMENT OF APPLIED MATHEMATICS AND DATA SCIENCE, TAMKANG UNIVERSITY, TAMSUI, NEW TAIPEI CITY 251301, TAIWAN

Email address: `jsguo@mail.tku.edu.tw`

DEPARTMENT OF MATHEMATICAL SCIENCES, TOKYO METROPOLITAN UNIVERSITY, HACHIOJI, TOKYO 192-0397, JAPAN

Email address: `shimojo@tmu.ac.jp`

DEPARTMENT OF MATHEMATICS, NATIONAL CHUNG CHENG UNIVERSITY, MIN-HSIUNG, CHIA-YI 62102, TAIWAN

Email address: `cjwang@ccu.edu.tw`