SPREADING DYNAMICS FOR A PREDATOR-PREY SYSTEM WITH TWO PREDATORS AND ONE PREY IN A SHIFTING HABITAT

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ABSTRACT. We study the spreading dynamics for a three-species predator-prey system with two weak competing predators and one prey in a shifting habitat. First, we derive some extinction results for each species. Then we provide some persistence theorems for each species with moving speeds exceed the shifting speed, but less than some certain quantities. Finally, the convergence to a certain constant state is proven in each persistent regime.

1. Introduction

In this paper, we consider the following three-species predator-prey system

$$(1.1) \quad u_t(x,t) = d_1 u_{xx}(x,t) + r_1 u(x,t)[\alpha(x-st) - u(x,t) - av(x,t) - aw(x,t)],\ x \in \mathbb{R}, t > 0,$$

$$(1.2) \quad v_t(x,t) = d_2 v_{xx}(x,t) + r_2 v(x,t)[-1 + bu(x,t) - v(x,t) - hw(x,t)],\ x \in \mathbb{R},\ t > 0,$$

$$(1.3) \quad w_t(x,t) = d_3 w_{xx}(x,t) + r_3 w(x,t)[-1 + bu(x,t) - kv(x,t) - w(x,t)],\ x \in \mathbb{R},\ t > 0,$$

in which u, v, w stand for the population densities of the single prey and two predators, constants d_i, $i = 1, 2, 3$, are their diffusion rates, a is the predation rate, b is the conversion rate, and h, k are competition coefficients between two predators. The intrinsic growth rate for u is $r_1\alpha(x - st)$, where α is a piecewise continuously differentiable and non-decreasing function such that

$$-\infty < \alpha(-\infty) < 0 < \alpha(\infty) < \infty$$

in which we may assume without loss of generality that $\alpha(\infty) = 1$. This assumption implies that the environment is favorable to the prey ahead of the shifting edge, $\alpha(x - st) = 0$, and becomes hostile to the prey behind the shifting edge. Here the positive constant s stands for the speed of shifting habitat due to the climate change effect for example. The intrinsic growth rates for predators are assumed to be $-r_i$, $i = 2, 3$. We assume all parameters are positive. Hence, in particular, either predator cannot survive without the prey.

From the modeling point view, although the shifting term α does not present in each equation of predator, the effect of changing environment actually comes to play for both

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predators. This can be seen from the results we obtained below. In this paper, for simplicity we assume the predation rates for the prey are the same for both predators and also for the conversion rates.

We assume throughout this paper the condition

$$0 < h, k < 1, \quad b > 1. \tag{1.4}$$

Biologically, condition (1.4) means that both predators are weak competitors (in the absence of the prey) and each predator can survive with the feeding of the prey. Then the constant equilibria for the limiting system of (1.1)-(1.3) at $x - st = \infty$, namely,

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - u - av - aw),\ x \in \mathbb{R}, t > 0, \\ v_t = d_2 v_{xx} + r_2 v(-1 + bu - v - hw),\ x \in \mathbb{R},\ t > 0, \\ w_t = d_3 w_{xx} + r_3 w(-1 + bu - kv - w),\ x \in \mathbb{R},\ t > 0, \end{cases} \tag{1.5}$$

are

$$(1, 0, 0),\ (u_p, v_p, 0),\ (u_p, 0, w_p),\ (u_c, v_c, w_c),$$

where

$$u_p := \frac{1+a}{1+ab},\ v_p = w_p := \frac{b-1}{1+ab}, \tag{1.6}$$

$$u_c := \frac{1+a\gamma}{1+ab\gamma},\ \gamma := \frac{2-h-k}{1-hk},\ v_c := \frac{1-h}{1-hk}(bu_c - 1),\ w_c := \frac{1-k}{1-hk}(bu_c - 1). \tag{1.7}$$

Note that, under condition (1.4), the positive co-existence state (u_c, v_c, w_c) exists and it is stable (in the ODE sense). Moreover, the other three constant equilibria are unstable (in the ODE sense).

The effect of shifting heterogeneity on ecological species has been studied recently in a series of works by Berestycki and his coauthors ([5, 6, 3, 2, 4]). We also refer the reader to, for examples, [17, 19, 13, 16, 23, 18, 7, 20, 21, 22, 9, 12, 8] and references cited therein. One of the main concerns of these studies is to understand the large time extinction or survival of all species and their spreading dynamics. Another attention is paid to the existence of so-called *forced waves*, which are traveling wave solutions with wave speed s (the speed of shifting habitat). In the above-mentioned works, either a single species or two species ecological systems were investigated. This includes two species competition systems, cooperative systems and predator-prey systems.

Little is done for three species ecological systems. Motivated by a recent work [8] on a two species predator-prey system, our aim of this paper is to extend the 2-species case to a 3-species system. In particular, we are interested in the spreading dynamics of system (1.1)-(1.3). For system (1.5), we refer the reader to [10] for the study of spreading dynamics to characterize the asymptotic spreading speeds and [14] for the existence of traveling wave solutions connecting the predator-free state $(1, 0, 0)$ and the co-existence state (u_c, v_c, w_c).

Let (u, v, w) be a solution of (1.1)-(1.3) with the initial condition

$$u(x, 0) = u_0(x),\ v(x, 0) = v_0(x),\ w(x, 0) = w_0(x),\ \forall\, x \in \mathbb{R}. \tag{1.8}$$

Throughout this paper, the initial data (u_0, v_0, w_0) are assumed to be continuous and non-trivial in the sense $u_0 \not\equiv 0$, $v_0 \not\equiv 0$ and $w_0 \not\equiv 0$. We also assume that $u_0 \in [0, 1]$ and $v_0, w_0 \in [0, b-1]$ in $\mathbb{R}$. Then, by the standard theory of parabolic equations, system (1.1)-(1.3) with initial condition (1.8) has a unique solution (u, v, w) such that $0 < u \leq 1$, $0 < v \leq b-1$ and $0 < w \leq b-1$ for all $t > 0$.

In the sequel, we denote

$$s_1^* := 2\sqrt{d_1 r_1},\ s_i^* := 2\sqrt{d_i r_i (b-1)},\ i = 2, 3.$$

Here s_1^* stands for the Fisher invasion speed of u with $\alpha \equiv 1$ and without predators. Also, s_2^* (s_3^*, resp.) is the Fisher invasion speed of v (w, resp.) with $u \equiv 1$ and without the other predator.

First, we discuss the extinction phenomenon. If the shifting speed s exceeds the Fisher invasion speed of the prey, then the prey goes extinction, which leads to the extinction of the predators. Moreover, if the shifting speed exceeds the Fisher invasion speed of a predator, then the predator goes extinction. In fact, we have

Theorem 1.1. *Let (u, v, w) be a solution of* (1.1)-(1.3) *with compactly supported initial data (u_0, v_0, w_0). Then we have*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} [u(x,t) + v(x,t) + w(x,t)] = 0, \quad \text{if } s > s_1^*, \tag{1.9}$$

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} v(x,t) = 0, \quad \text{if } s > s_2^*;\ \lim_{t\to\infty} \sup_{x\in\mathbb{R}} w(x,t) = 0, \quad \text{if } s > s_3^*. \tag{1.10}$$

On the other hand, behind the transition to the devastating environment, u is already brought down to a very low density, which leads to the vanishing of each predator. We call this as the *partial extinction* in this paper. More precisely, we provide the following partial extinction result.

Theorem 1.2. *Let (u, v, w) be a solution of* (1.1)-(1.3) *with initial data (u_0, v_0, w_0). Then we have*

$$\lim_{t\to\infty} \sup_{x\leq(s-\varepsilon)t} u(x,t) = 0,\ \text{if } s \leq s_1^*, \tag{1.11}$$

$$\lim_{t\to\infty} \sup_{x\leq(s-\varepsilon)t} v(x,t) = 0,\ \text{if } s \leq s_2^* \text{ and } u_0 \text{ has a compact support}, \tag{1.12}$$

$$\lim_{t\to\infty} \sup_{x\leq(s-\varepsilon)t} w(x,t) = 0,\ \text{if } s \leq s_3^* \text{ and } u_0 \text{ has a compact support}, \tag{1.13}$$

for any small $\varepsilon > 0$

In other words, each species must keep up with the climate change in order to survive. In fact, to survive each species cannot move too fast as shown in a classical result by Aronson and Weinberger [1] (see also Theorem 2.1 in §2). Therefore, the only chance to survive for a species is to move in a speed bigger than the shifting speed s, but less than its Fisher invading speed.

In the sequel, for simplicity we only consider the case when

$$d_2 r_2 > d_3 r_3. \tag{1.14}$$

We shall call v the *fast predator* and w the *slow predator*, since $s_2^* > s_3^*$.

We now state our main spreading results caused by the climate change as follows. First, if the Fisher invasion speed of u is bigger than the shifting speed and the speed of leading edge of the fast predator is smaller than the shifting speed, only the prey can escape from the climate change, and the predators are wiped out because of the lack of food.

Theorem 1.3. *Let (u,v,w) be a solution of (1.1)-(1.3) with compactly supported initial data (u_0,v_0,w_0) such that $u_0 \not\equiv 0$. Suppose that $s_1^* > s_2^*$. If $s \in (s_2^*, s_1^*)$, then we have*

$$\lim_{t\to\infty}\left\{\sup_{(s+\varepsilon)t\le x\le(s_1^*-\varepsilon)t}[|u(x,t)-1|+v(x,t)+w(x,t)]\right\}=0$$

for any $\varepsilon \in (0,(s_1^*-s)/2)$.

Next, if the shifting speed is smaller than the Fisher invasion speeds of u and v and larger than that of w, only the slow predator w dies out by the environmental change.

Theorem 1.4. *Let (u,v,w) be a solution of (1.1)-(1.3) with compactly supported initial data (u_0,v_0,w_0) such that $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$. Suppose that $s_3^* < s < \bar{s}^* := \min\{s_1^*, s_2^*\}$. Then we have*

$$\lim_{t\to\infty}\left\{\sup_{(s+\varepsilon)t\le x\le(\bar{s}^*-\varepsilon)t}[|u(x,t)-u_p|+|v(x,t)-v_p|+w(x,t)]\right\}=0$$

for any $\varepsilon \in (0,(\bar{s}^*-s)/2)$.

The remaining range for the shifting speed is when $s < \hat{s}^* := \min\{s_1^*, s_2^*, s_3^*\}$. From the above theorems, we may expect that all species can escape from the disaster, if $s < \hat{s}^*$. In this aspect, Propositions 3.2 and 3.5 (see below in §3) are the best results we are able to prove in this paper. Note that, since v and w compete each other, on the favorable environment of u, there is a possibility that one of the predators defeats.

In order to describe this sustainable coexistence, we introduce the following two quantities

$$\begin{cases} s_2^{**} := \sqrt{\frac{1-h}{1+ab}}s_2^* = 2\sqrt{d_2r_2(-1+bu_p-hw_p)}, \\ s_3^{**} := \sqrt{\frac{1-k}{1+ab}}s_3^* = 2\sqrt{d_3r_3(-1+bu_p-kv_p)}. \end{cases}$$

Then we have

Theorem 1.5. *Let (u,v,w) be a solution of (1.1)-(1.3) with initial data (u_0,v_0,w_0) such that $u_0 \not\equiv 0$, $v_0 \not\equiv 0$ and $w_0 \not\equiv 0$. Suppose that $s < \hat{s}^{**} := \min\{\hat{s}^*, s_2^{**}, s_3^{**}\}$. Then we have*

$$\lim_{t\to\infty}\left\{\sup_{(s+\varepsilon)t\le x\le(\hat{s}^*-\varepsilon)t}[|u(x,t)-u_c|+|v(x,t)-v_c|+|w(x,t)-w_c|]\right\}=0$$

for any $\varepsilon \in (0,(\hat{s}^{**}-s)/2)$.

It is well-known that some difficulties arise due to the lack of comparison principle for predator-prey systems. We adopt the method used in [11, 10, 8] to prove Theorems 1.3-1.5 on spreading dynamics. This method by now is rather standard, but powerful, to derive the spreading behaviors of certain reaction-diffusion systems. However, here we extend work [10] to system (1.1)-(1.3) which involving the shifting nonlinearity and extend [11, 8] of two species case to three interacting species. The extra interacting species makes the dynamical behavior of (1.1)-(1.3) more complex than two species case. We were unable to prove definitely the convergence to the co-existence state in the habitat with moving speeds between $\hat{s}^{**}$ and $\hat{s}^{*}$. As indicated in [10, Theorem 2.6], even for system (1.5) (without the shifting effect), there is a so-called non-local pulling phenomenon. So we are not sure whether Theorem 1.5 can be improved by removing the restriction $s < \hat{s}^{**}$. We leave this delicate problem to be an open question.

For the reader's convenience, we recall the recent progress of the propagation dynamics for system (1.5) from [10] as follows. Suppose, in addition to (1.4), that $2a(b-1) < 1$. Let (u, v, w) be a solution of (1.5) with initial data (u_0, v_0, w_0) such that

$$1 - 2a(b-1) := \beta \leq u_0 \leq 1,\ 0 \leq v_0, w_0 \leq b-1.$$

Then the following results are derived in [10].

(1) If both v_0 and w_0 are compactly supported, then

$$\lim_{t\to\infty} \sup_{|x|\geq ct} |u(x,t) - 1| = 0,\ \forall\, c > \max\{s_2^*, s_3^*\},$$
$$\lim_{t\to\infty} \sup_{|x|\geq ct} v(x,t) = 0,\ \forall\, c > s_2^*; \quad \lim_{t\to\infty} \sup_{|x|\geq ct} w(x,t) = 0,\ \forall\, c > s_3^*.$$

(2) Suppose $s_2^* = s_3^*$. If both v_0 and w_0 are nontrivial and compactly supported, then

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} (v+w)(x,t) > 0,\ \forall\, c \in (0, s_2^*),$$
$$\liminf_{t\to\infty} \inf_{|x|\leq ct} v(x,t) > 0,\ \forall\, c \in (0, s_2^{**}), \quad \liminf_{t\to\infty} \inf_{|x|\leq ct} w(x,t) > 0,\ \forall\, c \in (0, s_3^{**}),$$
$$\lim_{t\to\infty} \sup_{|x|\leq ct} [|u(x,t) - u_c| + |v(x,t) - v_c| + |w(x,t) - w_c|] = 0,\ \forall\, c \in (0, \min\{s_2^{**}, s_3^{**}\}).$$

(3) Suppose $s_2^* > s_3^*$. If both v_0 and w_0 are nontrivial and compactly supported, then

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} v(x,t) > 0,\ \forall\, c \in (0, s_2^*),$$
$$\lim_{t\to\infty} \sup_{c_1 t\leq|x|\leq c_2 t} [|u(x,t) - u_p| + |v(x,t) - v_p| + w(x,t)] = 0,\ \text{if } s_3^* < c_1 < c_2 < s_2^*,$$
$$\lim_{t\to\infty} \sup_{|x|\leq ct} [|u(x,t) - u_c| + |v(x,t) - v_c| + |w(x,t) - w_c|] = 0,\ \forall\, c \in (0, s_3^{**}).$$

(4) *Nonlocal pulling:* under certain parameter range (see [10, Theorem 2.6] for details), there is $c_0 > s_3^{**}$ such that

$$\liminf_{t\to\infty} \inf_{|x|\leq c_0 t} w(x,t) > 0.$$

The rest of this paper is organized as follows. In the next section, we investigate the extinction or partial extinction of each species and prove Theorems 1.1 and 1.2. Then we study the persistence of each species and give the detailed proofs of Theorems 1.3, 1.4 and 1.5 in section 3.

2. Extinction and partial extinction

In this section, we shall discuss extinction or partial extinction results for each species and prove Theorems 1.1 and 1.2. Although the proofs are similar to those in [8], we provide some details here for the reader's convenience.

First, the Fisher invasion speed of each species yields the following theorem. Note that this theorem is irrelevant to the shifting environment.

Theorem 2.1. *Let (u,v,w) be a solution of (1.1)-(1.3) with initial data (u_0,v_0,w_0). If $u_0(x)=v_0(x)=w_0(x)=0$ for $x\geq K$ for some constant K, then*

$$\lim_{t\to\infty}\sup_{x\geq(s_1^*+\varepsilon)t}u(x,t)=0,\ \lim_{t\to\infty}\sup_{x\geq(s_1^*+\varepsilon)t}v(x,t)=0,\ \lim_{t\to\infty}\sup_{x\geq(s_1^*+\varepsilon)t}w(x,t)=0,\tag{2.1}$$

$$\lim_{t\to\infty}\sup_{x\geq(s_2^*+\varepsilon)t}v(x,t)=0,\ \lim_{t\to\infty}\sup_{x\geq(s_3^*+\varepsilon)t}w(x,t)=0\tag{2.2}$$

for any $\varepsilon>0$.

Proof. The proofs of (2.1) and (2.2) can be done as that in [8, Theorem 5.3 (ii)]. We provide some details here. Given $\varepsilon>0$. Let $U(x,t):=Ae^{-\nu_1[x-(s_1^*+\varepsilon/2)t]}$, where ν_1 is the smaller positive root of

$$d_1\nu^2-(s_1^*+\varepsilon/2)\nu+r_1=0.$$

Then, using $u_0(x)=0$ for $x\geq K$, it holds $u_0(x)\leq U(x,0)$ for all $x\in\mathbb{R}$ for some positive constant A large enough. Since

$$u_t\leq d_1u_{xx}+r_1u[\alpha(x-st)-u],\quad U_t=d_1U_{xx}+r_1U\geq d_1U_{xx}+r_1U[\alpha(x-st)-U],$$

by a comparison principle, $u\leq U$ for all $t>0$. Hence

$$\lim_{t\to\infty}\sup_{x\geq(s_1^*+\varepsilon)t}u(x,t)=0.$$

For (2.2), we compare v with $\bar v$ and w with $\bar w$, where

$$\bar v(x,t):=Be^{-\nu_{21}[x-(s_2^*+\varepsilon/2)t]},\ \bar w(x,t):=Ce^{-\nu_{31}[x-(s_3^*+\varepsilon/2)t]},$$

and ν_{i1} is the smaller positive root of

$$d_i\nu^2-(s_i^*+\varepsilon/2)\nu+r_i(b-1)=0,\ i=2,3,$$

for some positive constants B and C. Then (2.2) follows.

Finally, recall $u\leq 1$ and let V be the solution of

$$V_t(x,t)=d_2V_{xx}(x,t)+r_2V(x,t)[-1+b\min\{1,U(x,t)\}-V(x,t)],\ x\in\mathbb{R},\ t>0,\tag{2.3}$$

with initial condition $V(x,0)=v_0(x)$. Then, by comparison, $v(x,t)\leq V(x,t)$ for all $x\in\mathbb{R}$, $t>0$. Suppose that $s_2^*>s_1^*$. Then, by choosing a positive constant ν_{22} small enough such that

$$d_2\nu_{22}^2-(s_1^*+\varepsilon)\nu_{22}-r_2/2<0$$

and setting

$$\hat{v}(x,t) := \min\left\{b-1, \hat{B}e^{-\nu_{22}[x-(s_1^*+\varepsilon/2)t]}\right\}$$

for some large enough positive constant $\hat{B}$, a simple comparison gives $V \le \hat{v}$. Indeed, this can be done by checking that $\hat{v}$ is a supersolution of (2.3). Hence

$$\lim_{t\to\infty} \sup_{x\ge(s_1^*+\varepsilon)t} v(x,t) = 0.$$

Similarly, we can prove

$$\lim_{t\to\infty} \sup_{x\ge(s_1^*+\varepsilon)t} w(x,t) = 0$$

when $s_3^* > s_1^*$. Since the case when $s_i^* \le s_1^*$ is included in (2.2), this completes the proof of (2.1). Therefore, the theorem is proved. □

Next, we give the proofs of Theorems 1.1 and 1.2 as follows.

Proof of Theorem 1.1. First, since u satisfies $u_t \le d_1 u_{xx} + r_1 u[\alpha(x-st) - u]$, by a simple comparison with the help of [17, Theorem 2.1], we can easily see that u tends to zero uniformly over $\mathbb{R}$ as $t\to\infty$, if $s > s_1^*$.

Secondly, let $s > s_2^*$, Then, by [16, Theorem 1.1], there is a monotone traveling wave solution $U(x,t) = \phi(\xi)$, $\xi := x - st$, to

$$U_t(x,t) = d_1 U_{xx}(x,t) + r_1 U(x,t)[\alpha(x-st) + \delta - U(x,t)],\ x\in\mathbb{R},\ t\in\mathbb{R},$$

such that $\phi(-\infty) = 0$ and $\phi(\infty) = 1+\delta$, where δ is a small enough positive constant such that

$$s > 2\sqrt{d_2 r_2[b(1+\delta)-1]},\ \alpha(-\infty) + \delta < 0.$$

Since u_0 has a compact support and $u_0 \le 1$, we can find a constant x_0 such that $u_0(x) \le \phi(x+x_0)$ for all $x\in\mathbb{R}$. Then, by comparison, we have $u(x,t) \le \phi(x+x_0-st)$ and so $v \le V$, where V is the solution of

$$V_t = d_2 V_{xx} + r_2 V[b\phi(x+x_0-st) - 1 - V],\ x\in\mathbb{R},\ t>0, \quad V(x,0) = v_0(x),\ x\in\mathbb{R}.$$

It follows from [17, Theorem 2.1] again that v tends to zero uniformly in $\mathbb{R}$ as $t\to\infty$. The case for w can be proved similarly and so (1.10) is proved.

Finally, suppose that $s > s_1^*$. Given $\varepsilon \in (0, 1/b)$. Since $s > s_1^*$, there is $T \gg 1$ such that $u(x,t) \le \varepsilon$ for all $x\in\mathbb{R}$, $t \ge T$. Choose $\sigma \in (0, r_2(1-b\varepsilon))$ and set

$$V(x,t) := (b-1)e^{-\sigma(t-T)}.$$

Note that

$$V_t - \{d_2 V_{xx} + r_2 V(-1 + b\varepsilon - V)\} \ge 0.$$

Hence $v \le V$ for $x\in\mathbb{R}$, $t\ge T$, since $v(x,T) \le b-1 = V(x,T)$ for all $x\in\mathbb{R}$ and v satisfies

$$v_t \le d_2 v_{xx} + r_2 v(-1 + b\varepsilon - v),\ x\in\mathbb{R},\ t\ge T.$$

Hence $v(x,t) \to 0$ uniformly in $\mathbb{R}$ as $t \to \infty$. The case for w can be treated similarly. This completes the proof of (1.9) and so the theorem follows. □

Proof of Theorem 1.2. First, we apply [17, Theorem 2.2 (i)] and using a comparison to obtain $u \le \bar{u}$, where $\bar{u}$ is the solution of

$$\begin{cases} \bar{u}_t = d_1 \bar{u}_{xx} + r_1 \bar{u}[\alpha(x-st) - \bar{u}], \ x \in \mathbb{R},\, t > 0, \\ \bar{u}(x,0) = u_0(x), \ x \in \mathbb{R}. \end{cases}$$

Then we conclude

$$\lim_{t\to\infty} \sup_{x \le (s-\varepsilon)t} u(x,t) \le \lim_{t\to\infty} \sup_{x \le (s-\varepsilon)t} \bar{u}(x,t) = 0, \ \text{ if } s \le s_1^*.$$

Hence (1.11) follows. We remark here [17, Theorem 2.2 (i)] does not require the compactness assumption on the initial data.

Next, we recall from the proof of (1.10) that $v \le V$, using the compactness of u_0. Then (1.12) follows from [17, Theorem 2.2 (i)], using $s \le s_2^* < 2\sqrt{d_2 r_2[b(1+\delta)-1]}$. The proof for (1.13) is similar. The proof is complete. □

3. Spreading dynamics

This section is devoted to the proofs of the main theorems on the spreading dynamics.

3.1. **Survival of the prey.** First, we give a proof of Theorem 1.3 as follows.

Proof of Theorem 1.3. Given a fixed $\varepsilon \in (0, (s_1^* - s)/2)$. First, since $s > s_2^* > s_3^*$ (by (1.14)), it follows from Theorem 1.1 that both v and w tend to zero uniformly in $\mathbb{R}$ as $t \to \infty$. Consequently, for a given small positive constant δ such that $s_\delta^* := 2\sqrt{d_1 r_1 (1-\delta)} > s_1^* - \varepsilon/2$, there is $T \gg 1$ such that $u \ge \underline{u}$ for all $t \ge T$, where $\underline{u}$ is the solution of

$$\begin{cases} \underline{u}_t = d_1 \underline{u}_{xx} + r_1 \underline{u}[\alpha(x-st) - \delta - \underline{u}], \ x \in \mathbb{R},\, t > T, \\ \underline{u}(x,T) = \min\{1-\delta, u(x,T)\}. \end{cases}$$

It follows from [17, Theorem 2.2 (iii)] that

$$\lim_{t\to\infty} \left\{ \sup_{(s+\varepsilon/2)t \le x \le (s_\delta^* - \varepsilon/2)t} |\underline{u}(x,t) - 1 + \delta| \right\} = 0. \tag{3.1}$$

On the other hand, since $[s+\varepsilon, s_1^* - \varepsilon] \subset [s + \varepsilon/2, s_\delta^* - \varepsilon/2]$ and $u \le 1$, we deduce from (3.1) that

$$\lim_{t\to\infty} \left\{ \sup_{(s+\varepsilon)t \le x \le (s_1^* - \varepsilon)t} |u(x,t) - 1| \right\} = 0,$$

since δ is arbitrarily. The proof is thus complete. □

Next, we prepare the following lemma for the weak persistence of prey u with speed $c \in (s, s_1^*)$.

Lemma 3.1. *Suppose that $s < s_1^*$. Then for any $c \in (s, s_1^*)$ there exists $\delta_1(c) \in (0,1)$ (independent of initial data (u_0, v_0, w_0) with $u_0 \neq 0$) such that*

$$\limsup_{t\to\infty} u(ct,t) \geq \delta_1(c) \tag{3.2}$$

for any solution (u, v, w) of (1.1)-(1.3).

Proof. We assume for contradiction that there are sequences $\{(u_{0,n}, v_{0,n}, w_{0,n})\}$ and $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$\lim_{n\to\infty} \sup_{t\geq t_n} u_n(ct,t) = 0, \tag{3.3}$$

where (u_n, v_n, w_n) is the solution of (1.1)-(1.3) with the initial datum $(u_{0,n}, v_{0,n}, w_{0,n})$. Then, for any $R > 0$, we claim

$$\lim_{n\to\infty} \{ \sup_{|x-ct|\leq R, t\geq t_n} u_n(x,t) \} = 0. \tag{3.4}$$

Indeed, otherwise there exist sequences $\{x_n\} \subset [-R, R]$ and $\{\tau_n\}$ with $\tau_n \geq t_n$ such that

$$\liminf_{n\to\infty} u_n(x_n + c\tau_n, \tau_n) > 0.$$

Without loss of generality (up to a subsequence) we may assume that $x_n \to x_0$ as $n \to \infty$ for some $x_0 \in [-R, R]$. By the standard parabolic estimates and using $c > s$, up to extraction of a subsequence, we have

$$(u_n, v_n, w_n)(x + c\tau_n, t + \tau_n) \to (u_\infty, v_\infty, w_\infty)(x,t) \text{ as } n \to \infty$$

locally uniformly in $\mathbb{R} \times \mathbb{R}$, where $(u_\infty, v_\infty, w_\infty)$ is an entire solution of

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - u - av - aw), \\ v_t = d_2 v_{xx} + r_2 v(-1 + bu - v - hw), \\ w_t = d_3 w_{xx} + r_3 w(-1 + bu - kv - w). \end{cases} \tag{3.5}$$

Hereafter, an *entire* solution is a solution defined for all $x, t \in \mathbb{R}$. Since $u_\infty(0,0) = 0$ by (3.3), the strong maximum principle implies that $u_\infty \equiv 0$. However, $u_\infty(x_0, 0) > 0$, a contradiction. Hence (3.4) is proved.

From (3.4), we can further derive that

$$\lim_{n\to\infty} \{ \sup_{|x-ct|\leq R, t\geq t_n} v_n(x,t) \} = \lim_{n\to\infty} \{ \sup_{|x-ct|\leq R, t\geq t_n} w_n(x,t) \} = 0. \tag{3.6}$$

Indeed, by the same limiting argument as for (3.4), the limit function (v_∞, w_∞) satisfies (using also $u_\infty \equiv 0$)

$$\begin{cases} v_t = d_2 v_{xx} + r_2 v(-1 - v - hw), \ x \in \mathbb{R}, \ t \in \mathbb{R}, \\ w_t = d_3 w_{xx} + r_3 w(-1 - kv - w), \ x \in \mathbb{R}, \ t \in \mathbb{R}. \end{cases}$$

Then (3.6) follows by using a comparison and the fact that any nonnegative bounded entire solution V of

$$V_t = dV_{xx} + rV(-1 - V), \ d > 0, \ r > 0,$$

must be identically zero.

Now, let

$$\lambda_R := \frac{c^2}{4d_1} + \frac{d_1\pi^2}{4R^2},\ \phi(x) := e^{-cx/(2d_1)} \cos\left(\frac{\pi x}{2R}\right).$$

Then ϕ satisfies

$$-d_1\phi_{xx} - c\phi_x = \lambda_R \phi \text{ in } (-R,R);\ \ \phi(\pm R) = 0.$$

Since $c < s_1^*$, we can find a small positive constant δ such that $c^2/(4d_1) < r_1(1-2\delta)$ and R large enough such that $\lambda_R < r_1(1-2\delta)$. Then, by (3.4) and (3.6), for large enough n the positive function u_n satisfies

$$(u_n)_t \geq d_1(u_n)_{xx} + r_1(1-\delta)u_n \text{ for } x \in (ct-R, ct+R),\ t \geq t_n.$$

Then $\hat{u}_n(x,t) := u_n(x+ct,t)$ satisfies

$$(\hat{u}_n)_t \geq d_1(\hat{u}_n)_{xx} + c(\hat{u}_n)_x + r_1(1-\delta)\hat{u}_n \text{ for } x \in (-R,R),\ t \geq t_n.$$

A comparison principle gives that

$$u_n(x+ct,t) = \hat{u}_n(x,t) \geq Ae^{r_1\delta t}\phi(x),\ |x| \leq R,\ t \geq t_n,$$

provided that the positive constant A is chosen small enough so that $\hat{u}_n(x,t_n) \geq Ae^{r_1\delta t_n}\phi(x)$ for all $x \in [-R,R]$. This implies that $u_n(ct,t) \to \infty$ as $t \to \infty$, a contradiction. Hence (3.2) follows and the lemma is proved. □

Remark 1. It is easy to see that the same argument as above also leads to (3.2) with a constant $\delta_1'(c) \in (0,1)$ for any solution (u,v,w) of system (3.5) for any $c \in (0, s_1^*)$.

Then we show the uniform persistence of prey u for all speeds $c \in (s, \hat{s}^*)$.

Proposition 3.2. *Suppose that $s < \hat{s}^* = \min\{s_1^*, s_2^*, s_3^*\}$. Then for any $\varepsilon \in (0, (\hat{s}^* - s)/2)$ there is a positive constant θ_1 (independent of initial data (u_0,v_0,w_0) with $u_0 \not\equiv 0$) such that*

$$\liminf_{t\to\infty} \left\{ \inf_{(s+\varepsilon)t \leq x \leq (\hat{s}^*-\varepsilon)t} u(x,t) \right\} \geq \theta_1 \tag{3.7}$$

for any solution (u,v,w) of (1.1)-(1.3).

Proof. We divide our proof into two steps.

Step 1. Claim: for any $c \in (s, \hat{s}^*)$ there exists $\delta_2(c) \in (0,1)$ (independent of initial data (u_0,v_0,w_0)) such that any solution (u,v,w) of (1.1)-(1.3) satisfies

$$\liminf_{t\to\infty} u(ct,t) \geq \delta_2(c). \tag{3.8}$$

Again, proceed by a contradiction. Assume that there are sequences $\{(u_{0,n}, v_{0,n}, w_{0,n})\}$ and $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that the corresponding solution $\{(u_n,v_n,w_n)\}$ satisfies

$$\lim_{n\to\infty} u_n(ct_n,t_n) = 0. \tag{3.9}$$

By (3.2), we can choose a sequence $\{t_n'\}$ with $t_n' < t_n$ and $t_n' \to \infty$ such that

$$u_n(ct_n' t_n') \geq \delta_1(c)/2 \text{ for all } n.$$

Recall the lower bound $\varepsilon_1(\kappa, c)$, $\kappa := \max\{1, b-1\}$, derived in [11, Lemma 5.2] for the problem

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - u - av), \\ v_t = d_2 v_{xx} + r_2 v(-1 + bu - v). \end{cases} \tag{3.10}$$

Let $\delta_1'(c)$ be the constant introduced in Remark 1. Then, for

$$\tau_n := \sup\{t_n' \le t \le t_n \mid u_n(ct, t) \ge \gamma_1(c)\},\ \gamma_1(c) := \min\{\delta_1(c), \delta_1'(c), \varepsilon_1(\kappa, c)\}/2,$$

it follows from a limiting argument and a strong maximum principle that $t_n - \tau_n \to \infty$ as $n \to \infty$. Indeed, by taking the limit, we have (up to extraction of a subsequence)

$$(u_n, v_n, w_n)(x + c\tau_n, t + \tau_n) \to (u_\infty, v_\infty, w_\infty)(x, t)$$

locally uniformly in $\mathbb{R} \times \mathbb{R}$, where $(u_\infty, v_\infty, w_\infty)$ is an entire solution of (3.5). If (up to a subsequence) $t_n - \tau_n \to t_0$ as $n \to \infty$ for some $t_0 \in \mathbb{R}$, then

$$u_\infty(ct_0, t_0) = \lim_{n\to\infty} u_n(c(t_n - \tau_n) + c\tau_n, (t_n - \tau_n) + \tau_n) = \lim_{n\to\infty} u_n(ct_n, t_n) = 0,$$

by (3.9). It then follows from the strong maximum principle that $u_\infty \equiv 0$. This is a contradiction to $u_\infty(0,0) = \gamma_1(c)$, since $u_n(c\tau_n, \tau_n) = \gamma_1(c)$ for all n. Hence $t_n - \tau_n \to \infty$ as $n \to \infty$.

Furthermore, since

$$u_n(ct, t) \le \gamma_1(c),\ \forall\, t \in (\tau_n, t_n),$$

we obtain

$$u_\infty(ct, t) \le \gamma_1(c) \ \text{ for all } t \ge 0, \tag{3.11}$$

due to $t_n - \tau_n \to \infty$ as $n \to \infty$.

Now, suppose that $v_\infty = w_\infty \equiv 0$. Then u_∞ satisfies

$$(u_\infty)_t = d_1(u_\infty)_{xx} + r_1 u_\infty(1 - u_\infty).$$

Since $u_\infty(\cdot, 0) \not\equiv 0$ and $c < s_1^*$, we have (cf. [1]) that $u_\infty(ct, t) \to 1$ as $t \to \infty$, a contradiction to (3.11). If $v_\infty \not\equiv 0$ and $w_\infty \equiv 0$, then (u_∞, v_∞) satisfies (3.10). Then (3.11) contradicts [11, Lemma 5.2], since $s < \min\{s_1^*, s_2^*\}$ yields $u_\infty(ct, t) \ge \varepsilon_1(c)$. The case when $v_\infty \equiv 0$ and $w_\infty \not\equiv 0$ can be treated similarly, by using $s < \min\{s_1^*, s_3^*\}$. Lastly, by Remark 1, the case when both v_∞ and w_∞ are nontrivial also leads to a contradiction. This proves (3.8).

Step 2. To reach the conclusion of the proposition, we use a contradiction argument. For a given $\varepsilon \in (0, (\hat{s}^* - s)/2)$, we assume that there are a sequence of initial data $\{(u_{0,n}, v_{0,n}, w_{0,n})\}$ and sequences $\{x_{n,k}\}$, $\{t_{n,k}\}$ such that $t_{n,k} \to \infty$ as $k \to \infty$,

$$x_{n,k} \in [(s + \varepsilon)t_{n,k}, (\hat{s}^* - \varepsilon)t_{n,k}],\ u_n(x_{n,k}, t_{n,k}) \le \frac{1}{n},\ \forall\, n, k \in \mathbb{N}.$$

Note that, by Step 1, we have

$$\liminf_{t\to\infty} u_n((\hat{s}^* - \varepsilon/2)t, t) \ge \delta_2(c_1),\ c_1 := \hat{s}^* - \varepsilon/2.$$

Then, for the sequence $\{t'_{n,k} := x_{n,k}/c_1\}$, we have $t'_{n,k} < t_{n,k}$, $t'_{n,k} \to \infty$ as $k \to \infty$ and

$$u_n(x_{n,k}, t'_{n,k}) = u_n(c_1 t'_{n,k}, t'_{n,k}) \geq \delta_2(c_1)/2 \text{ for all } k \gg 1. \tag{3.12}$$

Now, for each n we choose $k = k(n) \gg 1$ such that (3.12) holds and define

$$\tau_n := \sup\{t'_{n,k} \leq t \leq t_n := t_{n,k} \mid u_n(ct, t) \geq \delta_0\},\ \delta_0 := \min\{\varepsilon_1(\kappa, c), \delta'_1(c), \delta_2(c_1)\}/2,$$

where the constant $\varepsilon_1(\kappa, c)$ is the constant defined in [11, Lemma 5.2]. Then, as in Step 1, the strong maximum principle implies that $t_n - \tau_n \to \infty$ as $n \to \infty$. From this, for (up to extraction of a subsequence) the limit

$$(u_\infty, v_\infty, w_\infty)(x, t) := \lim_{n\to\infty}(u_n, v_n, w_n)(x + c\tau_n, t + \tau_n),\ (x, t) \in \mathbb{R} \times \mathbb{R},$$

an entire solution of (3.5), we have

$$u_\infty(0, 0) = \delta_0,\ u_\infty(0, t) \leq \delta_0,\ \forall\, t \geq 0.$$

Then the same argument as that in Step 1 leads to a contradiction. This completes the proof of the proposition. □

When $s_3^* < \bar{s}^*$, we have the following uniform persistence of prey u for speeds over $(s, \bar{s}^*)$.

Proposition 3.3. *Suppose that $s_3^* < s < \bar{s}^* = \min\{s_1^*, s_2^*\}$. Then for any $\varepsilon \in (0, (\bar{s}^* - s)/2)$ there is a positive constant θ_2 (independent of compactly supported initial data (u_0, v_0, w_0) with $u_0 \not\equiv 0$) such that*

$$\liminf_{t\to\infty}\left\{\inf_{(s+\varepsilon)t \leq x \leq (\bar{s}^*-\varepsilon)t} u(x, t)\right\} \geq \theta_2 \tag{3.13}$$

for any solution (u, v, w) of (1.1)-(1.3).

Proof. Since $s > s_3^*$, w tends to zero uniformly in $\mathbb{R}$ as $t \to \infty$. Hence the case $w_\infty \not\equiv 0$ cannot happen in the proof of Proposition 3.2. The same proof as that for Proposition 3.2 leads to the conclusion of this proposition. □

3.2. **Survival of the fast predator.** In this subsection, we give a proof of Theorem 1.4 and discuss the survival of predators with moving speeds between the shifting speed s and $\hat{s}^*$. First, we prove the following lemma, which states that at least one of the predators persists *weakly* for each speed $c \in (s, \hat{s}^*)$.

Lemma 3.4. *Suppose that $s < \hat{s}^*$. Then for any $c \in (s, \hat{s}^*)$ there exists $\delta_3(c) \in (0, 1)$ (independent of initial data (u_0, v_0, w_0)) such that*

$$\limsup_{t\to\infty}(v + w)(ct, t) \geq \delta_3(c) \tag{3.14}$$

for any solution (u, v, w) of (1.1)-(1.3).

Proof. Assume for contradiction that there are sequences $\{(u_n, v_n, w_n)\}$ and $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty}\{\sup_{t\geq t_n}(v_n + w_n)(ct, t)\} = 0.$$

Then we have

$$\lim_{n\to\infty}\{\sup_{t\geq t_n} v_n(ct,t)\}=0,\quad \lim_{n\to\infty}\{\sup_{t\geq t_n} w_n(ct,t)\}=0. \tag{3.15}$$

As in the proof of (3.4), we can infer from (3.15) and the strong maximum principle that

$$\lim_{n\to\infty}\left\{\sup_{|x-ct|\leq R, t\geq t_n} v_n(x,t)\right\}=0,\quad \lim_{n\to\infty}\left\{\sup_{|x-ct|\leq R, t\geq t_n} w_n(x,t)\right\}=0 \tag{3.16}$$

for any $R>0$. Furthermore, we can deduce from (3.16) that

$$\limsup_{n\to\infty}\left\{\sup_{|x-ct|\leq R, t\geq t_n} u_n(x,t)\right\}=1 \tag{3.17}$$

for any $R>0$, by a contradiction argument similar to that for (3.4). Indeed, otherwise there is a sequence $\{(x_n,t_n')\}$ with $t_n'\geq t_n$ and $x_n\in[ct_n'-R, ct_n'+R]$ such that

$$\limsup_{n\to\infty} u_n(x_n,t_n')<1.$$

Then, up to extraction of a subsequence, $(u_n,v_n,w_n)(x+x_n,t+t_n')$ converges to an entire solution $(u_\infty,v_\infty,w_\infty)$ of (3.5) as $n\to\infty$. It follows from (3.16) and the strong maximum principle that $v_\infty=w_\infty\equiv 0$, since $v_\infty(0,t)=w_\infty(0,t)=0$ for all $t>0$. Hence u_∞ satisfies

$$(u_\infty)_t=d_1(u_\infty)_{xx}+r_1u_\infty(1-u_\infty)\ \text{ in } \mathbb{R}\times\mathbb{R}.$$

However, by Proposition 3.2, $u_\infty\geq\theta_1>0$.Hence $u_\infty\equiv 1$, a contradiction. Thus (3.17) is proved.

Since $v_n(\cdot,0)\not\equiv 0$, we have $v_n>0$ for $t>0$. Then for any small $\delta>0$, $R\gg 1$ and $n\gg 1$ it holds

$$(v_n)_t\geq d_2(v_n)_{xx}+r_2(b-1-2\delta)v_n,\ |x-ct_n|\leq R,\ t\geq t_n.$$

From this, by the same argument as in the proof of Lemma 3.1 and using $s<\hat{s}^*\leq s_2^*$, we reach a contradiction. The lemma is proved. □

Remark 2. The same argument as above also leads to (3.14) with a constant $\delta_3'(c)\in(0,1)$ for any solution (u,v,w) of system (3.5) for any $c\in(0,\hat{s}^*)$.

Next, we show the uniform persistence of predators as follows.

Proposition 3.5. *Suppose that $s<\hat{s}^*$. Then for any $\varepsilon\in(0,(\hat{s}^*-s)/2)$ there is a positive constant θ_3 (independent of initial data (u_0,v_0,w_0)) such that*

$$\liminf_{t\to\infty}\left\{\inf_{(s+\varepsilon)t\leq x\leq(\hat{s}^*-\varepsilon)t}(v+w)(x,t)\right\}\geq\theta_3$$

for any solution (u,v,w) of (1.1)-(1.3).

Proof. The proof is similar to that of Proposition 3.2. We only outline it here.

Claim: for any $c\in(s,\hat{s}^*)$ there exists $\delta_4(c)\in(0,1)$ (independent of initial data (u_0,v_0,w_0)) such that any solution (u,v,w) of (1.1)-(1.3) satisfies

$$\liminf_{t\to\infty}(v+w)(ct,t)\geq\delta_4(c). \tag{3.18}$$

Proceeding as Step 1 in the proof of Proposition 3.2 with u replaced by $v+w$, we end up with the limit solution $(u_\infty, v_\infty, w_\infty)$ satisfies

$$(v_\infty + w_\infty)(0,0) = \gamma_2(c),\ (v_\infty + w_\infty)(ct,t) \le \gamma_2(c),\ \forall\, t \ge 0, \tag{3.19}$$

where

$$\gamma_2(c) := \min\{\delta_3(c), \delta_3'(c), \varepsilon_1(\kappa, c)\}/2.$$

Then the same argument as that in the proof of Proposition 3.2, using Remark 2, leads to a contradiction and so (3.18) is proved.

With (3.18), we can complete the proof by repeating Step 2 in the proof of Proposition 3.2. We safely omit the detail here. □

In particular, we immediately have the following result on the survival of the fast predator.

Proposition 3.6. *Suppose that $s_3^* < s < \bar{s}^*$. Then for any $\varepsilon \in (0, (\bar{s}^* - s)/2)$ there is a positive constant θ_4 (independent of compactly supported initial data (u_0, v_0, w_0)) such that*

$$\liminf_{t\to\infty} \left\{ \inf_{(s+\varepsilon)t \le x \le (\bar{s}^* - \varepsilon)t} v(x,t) \right\} \ge \theta_4 \tag{3.20}$$

for any solution (u,v,w) of (1.1)-(1.3).

Proof. Since w tends to zero uniformly over $\mathbb{R}$ as $t \to \infty$, due to $s > s_3^*$. A similar proof to that of Lemma 3.4 using Proposition 3.3 leads that

$$\limsup_{t\to\infty} v(ct,t) \ge \delta_3(c)$$

for any solution (u,v,w) of (1.1)-(1.3) for any $c \in (s, \bar{s}^*)$. From this, the same proof as that for Proposition 3.2 (with u replaced by v) leads to the desired conclusion. □

Combining Theorem 1.1, Proposition 3.3 and Proposition 3.6, we now give a proof of Theorem 1.4.

Proof of Theorem 1.4. We apply a contradiction argument used in [15]. Suppose that there exist $\delta > 0$ and a sequence of points $\{(x_n, t_n)\}$ with $t_n \to \infty$ and $x_n \in [(s+\varepsilon)t_n, (\bar{s}^* - \varepsilon)t_n]$ such that

$$|u(x_n, t_n) - u_p| + |v(x_n, t_n) - v_p| \ge \delta,\ \forall\, n. \tag{3.21}$$

Then, up to extraction a subsequence, we have

$$(u,v,w)(x+x_n, t+t_n) \to (u_\infty, v_\infty, w_\infty)(x,t) \ \text{ locally uniformly for } (x,t) \in \mathbb{R}^2,$$

where $w_\infty \equiv 0$, by $s > s_3^*$ and (1.10), and (u_∞, v_∞) is an entire solution of

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - u - av), \\ v_2 = d_2 v_{xx} + r_2 v(-1 + bu - v). \end{cases}$$

Now, using Corollaries 3.3 and 3.6, we can find a constant $T \gg 1$ such that

$$\theta_2/2 \le u(x+x_n, t+t_n) \le 1, \quad \theta_4/2 \le v(x+x_n, t+t_n) \le b-1,$$

for $x + x_n \in [(s + \varepsilon/2)(t + t_n), (\bar{s}^* - \varepsilon/2)(t + t_n)]$, if $t + t_n \geq T$, where $\theta_2 = \theta_2(\varepsilon/2)$ and $\theta_4 = \theta_4(\varepsilon/2)$ are constants defined in Corollaries 3.3 and 3.6, respectively. This implies that $\theta_2/2 \leq u_\infty \leq 1$ and $\theta_4/2 \leq v_\infty \leq b - 1$ in $\mathbb{R}^2$. It follows from [10, Lemma 4.1] that $(u_\infty, v_\infty) \equiv (u_p, v_p)$. This contradicts (3.21). Hence the proof is complete. □

Now, we come to the second result on the survival of the fast predator as follows.

Proposition 3.7. *Suppose that* $s < \hat{s}_2^{**} := \min\{\hat{s}^*, s_2^{**}\}$. *Then for any* $\varepsilon \in (0, (\hat{s}^{**} - s)/2)$ *there is a positive constant* θ_5 *(independent of initial data* (u_0, v_0, w_0)*) such that*

$$\liminf_{t\to\infty} \left\{ \inf_{(s+\varepsilon)t \leq x \leq (\hat{s}^{**}-\varepsilon)t} v(x,t) \right\} \geq \theta_5 \tag{3.22}$$

for any solution (u, v, w) *of* (1.1)-(1.3).

Proof. First, we claim: for any $c \in (s, \hat{s}_2^{**})$ there exists $\delta_5(c) \in (0,1)$ (independent of initial data (u_0, v_0, w_0)) such that any solution (u, v, w) of (1.1)-(1.3) satisfies

$$\limsup_{t\to\infty} v(ct, t) \geq \delta_5(c). \tag{3.23}$$

Assume for contradiction that there are sequences $\{t_n\}$ with $t_n \to \infty$ and $\{(u_n, v_n, w_n)\}$ such that

$$\lim_{n\to\infty} \{\sup_{t \geq t_n} v_n(ct, t)\} = 0.$$

Then, as in the proof of (3.4), we have

$$\lim_{n\to\infty} \left\{ \sup_{|x-ct| \leq R, t \geq t_n} v_n(x,t) \right\} = 0 \tag{3.24}$$

for any $R > 0$.

Next, we claim that

$$\limsup_{n\to\infty} \left\{ \sup_{|x-ct| \leq R, t \geq t_n} u_n(x,t) \right\} = u_p, \ \limsup_{n\to\infty} \left\{ \sup_{|x-ct| \leq R, t \geq t_n} w_n(x,t) \right\} = w_p. \tag{3.25}$$

Indeed, (3.25) can be proved by a contradiction argument similar to that for (3.17) as follows. Up to extraction of a subsequence, the limit

$$(u_\infty, v_\infty, w_\infty)(x,t) := \lim_{n\to\infty} (u_n, v_n, w_n)(x + ct_n, t + t_n), \ (x,t) \in \mathbb{R}^2,$$

exists such that $v_\infty \equiv 0$, using (3.24), and (u_∞, w_∞) is an entire solution of

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - u - aw), \\ w_t = d_3 w_{xx} + r_3 w(-1 + bu - w). \end{cases}$$

Recall Propositions 3.2 and 3.5. Then from the same argument as in the proof of Theorem 1.4 it follows that $u_\infty \equiv u_p$ and $w_\infty \equiv w_p$. This leads to a contradiction and so (3.25) is proved.

Now, given $\delta > 0$ small. Then, for n large, v_n satisfies

$$(v_n)_t \geq d_2(v_n)_{xx} + r_2(-1 + bu_p - hw_p - \delta)v_n, \ |x - ct_n| \leq R, \ t \geq t_n.$$

From this we reach a contradiction, by the same argument as in the proof of Lemma 3.1 and using $c < s_2^{**}$. Hence (3.23) is proved.

With (3.23), the proposition can be proved by a similar argument to the proof of Proposition 3.2. We safely omit it here and finish the proof. □

3.3. **Survival of the slow predator.** In this subsection, we give a proof of Theorem 1.5. First, a similar proof to that of Proposition 3.7, we obtain the following uniform persistence of the slow predator w. We shall not repeat the proof here.

Proposition 3.8. *Suppose that* $s < \hat{s}_3^{**} := \min\{\hat{s}^*, s_3^{**}\}$. *Then for any* $\varepsilon \in (0, (\hat{s}_3^{**} - s)/2)$ *there is a positive constant* θ_6 *(independent of initial data* (u_0, v_0, w_0)*) such that*

$$\liminf_{t\to\infty} \left\{ \inf_{(s+\varepsilon)t \le x \le (\hat{s}_3^{**}-\varepsilon)t} w(x,t) \right\} \ge \theta_6 \tag{3.26}$$

for any solution (u, v, w) *of* (1.1)-(1.3).

Finally, we prove Theorem 1.5.

Proof of Theorem 1.5. The proof is similar to that for Theorem 1.4. In fact, here we use Propositions 3.2, 3.7 and 3.8 to ensure the limit entire solution is bounded below by a positive constant and bounded above. Then the theorem follows from [10, Lemma 4.3]. □

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