Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

RECENT DEVELOPMENTS ON A SINGULAR PREDATOR-PREY MODEL

Yu-Shuo Chen

Department of Mathematics, Tamkang University
Tamsui, New Taipei City 25137, Taiwan

Jong-Shenq Guo*

Department of Mathematics, Tamkang University
Tamsui, New Taipei City 25137, Taiwan

Masahiko Shimojo

Department of Applied Mathematics, Okayama University of Science
Okayama 700-0005, Japan

(Communicated by the associate editor name)

Abstract. This work is concerned with the dynamical behaviors of a singular predator-prey model. We first review some well-known results obtained recently. Then we give some new results on the spreading speed of the predator, the existence vs non-existence of traveling waves connecting the predator-free state to the co-existence state, and the existence vs non-existence of spatially periodic traveling waves to this singular predator-prey system.

1. **Introduction.** For the control of introduced rabbits to protect native birds from introduced cat predation in an island, it is proposed in [4] the model

$$\begin{cases} B' = r_b B\left(1 - \frac{B}{K_b}\right) - \frac{\alpha B}{\alpha B + R}\mu_b C, \\ R' = r_r R\left(1 - \frac{R}{K_r}\right) - \frac{R}{\alpha B + R}\mu_r C - \lambda_r R, \\ C' = r_c C\left(1 - \frac{C}{(B/\mu_b) + (R/\mu_r)}\right) - \lambda_c C, \end{cases} \tag{1}$$

in which B, R, C stand for the population of birds, rabbits, cats, respectively; r_b, r_r, r_c denote their growth rates; K_b, K_r are the carrying capacities and μ_b, μ_r are the predation rates of birds and rabbits; α is the preference rate; $(B/\mu_b) + (R/\mu_r)$ is the carrying capacity of cats; and λ_r, λ_c denote the control rates of rabbits and cats.

By numerical simulations, it is concluded in [4] that control of both introduced species is the best strategy. Without controls, the birds go extinct eventually. Analytically, this claim is equivalent to the stability analysis for equilibria of system (1) with/without controls. For the biological background of system (1), we refer the reader to [4, 5]. However, in reality, species are moving around in the habitat so

2010 *Mathematics Subject Classification.* Primary: 35K55, 35K57; Secondary: 92D25, 92D40.
Key words and phrases. predator-prey system, spreading speed, traveling wave, wave speed, minimal speed.
This work is partially supported by the Ministry of Science and Technology of Taiwan under the grants 106-2811-M-032-008 and 105-2115-M-032-003-MY3.

* Corresponding author: Jong-Shenq Guo.

that the spatial movements have to be considered. Our main concern in this paper is the dynamics of the corresponding system when the spatial dependence is taken into account. Here we consider the classical random movements so that the Laplace operator is modeled.

However, the full 3-species model is too hard to be analyzed. We therefore consider the case without rabbits and the control(s). Then system (1) is reduced to the following two-component ordinary differential system:

$$\begin{cases} B' = r_b\left(1-\dfrac{B}{K}\right)B - \mu C, \\ C' = r_c\left(1-\mu\dfrac{C}{B}\right)C, \end{cases} \tag{2}$$

where we have set $K_b := K$ and $\mu_b := \mu$. Note that the parameter μ is the intake of birds per individual predator (cat) per unit time. When taking into account the spatial dependence, we have the following predator-prey model posed on a smooth domain $\Omega \subset \mathbb{R}^N$:

$$B_t = d_b\Delta B + r_b\left(1-\frac{B}{K}\right)B - \mu C, \quad x\in\Omega,\, t>0, \tag{3}$$

$$C_t = d_c\Delta C + r_c\left(1-\mu\frac{C}{B}\right)C, \quad x\in\Omega,\, t>0, \tag{4}$$

supplemented with the boundary condition

$$\frac{\partial B}{\partial\nu} = \frac{\partial C}{\partial\nu} = 0, \quad x\in\partial\Omega,\, t>0, \tag{5}$$

and the initial condition

$$B(\cdot,0) = B_0 > 0,\ C(\cdot,0) = C_0 \geq 0, \quad x\in\Omega, \tag{6}$$

where $d_b, d_c, r_b, r_c, K, \mu$ are positive constants and ν denotes the outer normal on $\partial\Omega$ (if $\partial\Omega \neq \emptyset$). Here d_b (d_c, resp.) is the diffusion coefficient of birds (cats, resp.). The functions B_0 (C_0, resp.) is the initial distribution of birds (cats, resp.). There are two major difficulties in dealing with system (3)-(6), one is the lack of comparison principle and the other is a singularity occurs when B reaches zero in a finite time. That is why we call this system as a *singular* predator-prey model.

In fact, for a closely related system to (3)-(6) when the functional response of predation is replaced by a linear function of prey, namely, (3) becomes

$$B_t = d_b\Delta B + r_bB(1-B/K) - \mu BC,$$

it follows from the strong maximum principle that $B>0$ for all $t>0$. Hence the rational term C/B in the predator equation never causes troubles in singularity. Same for the case when μBC is replaced by a more general functional response $h(B,C)B$. For the case of linear predation, we refer the reader to, e.g., [13, 15, 16, 6]. Recently, in [3], for $r_b > \mu$ we derive the existence of traveling waves to address the question whether both species can survive eventually, if an alien predator is introduced into the habitat where a prey has been living there. However, the existence of traveling wave solutions is still open when $r_b \leq \mu$.

Introducing the function $P := C/B$, system (2) is reduced to the following system of ordinary differential equations

$$\begin{cases} B' = \left[r_b(1-\frac{B}{K}) - \mu P\right]B, \\ P' = \left[r_c - r_b + r_b\frac{B}{K} - \mu(r_c-1)P\right]P. \end{cases} \tag{7}$$

There are always two nontrivial constant equilibria $(K, 0)$ and $(0, P^{**})$,

$$P^{**} := \frac{r_c - r_b}{\mu(r_c - 1)},$$

when $r_c \neq 1$. Also, when $r_b > 1$, there is the unique co-existence state

$$(B^*, P^*) := (K(1 - 1/r_b), 1/\mu).$$

When taking the spatial dependence into account, system (3)-(4) is reduced to

$$B_t = d_b \Delta B + r_b(1 - B/K)B - \mu PB, \tag{8}$$

$$\begin{aligned} P_t = d_c \Delta P + (d_c - d_b)\frac{P}{B}\Delta B + 2\frac{d_c}{B}\nabla B \cdot \nabla P \\ + \Big[r_c - r_b + \frac{r_b}{K}B - \mu(r_c - 1)P\Big]P. \end{aligned} \tag{9}$$

The dynamical behaviors for solutions to system (7) has been studied extensively in [11]. The global vs non-global existence of solutions to (7) and the asymptotic behaviors of global solutions are derived in [11] for 5 different open domains of the parameter space (r_b, r_c). Also, the corresponding dynamical behaviors of problem (3)-(6) were studied in [11] when $d_b = d_c$ and Ω is a bounded smooth domain. We refer the reader to [11] for the details. Here quenching (i.e., B reaches zero in finite time) occurs for non-global solutions. Note that the cross diffusion term in (9) is disappeared when $d_b = d_c$.

However, some results in [11] are given only based on numerical simulations, in particular, for (r_b, r_c) lies on the boundaries of the 5 open regions. For examples, along the segment

$$r_b + r_c = 2,\ r_b > 1,\ r_c > 0,$$

it is conjectured in [11] that the state (B^*, P^*) is a center of (7) and solutions of (7) starting with initial data in $\mathcal{P}^+$ blow up in finite time, where

$$\mathcal{P}^+ := \left\{(B, P) \mid B > 0, P > 0, \left(\frac{B}{K} + \frac{P}{P^{**}} - 1\right) > 0\right\}.$$

Moreover, solutions of (3)-(6) with $d_b = d_c$ exhibits spatio-temporal oscillations, i.e., solutions become spatially homogeneous and time-periodic asymptotically. In [12], we verify these three numerical observations rigorously.

On the other hand, in a recent work [8], we analyze the associated shadow system (as $d_b \to \infty$ so that B becomes spatially homogeneous) for system (8)-(9). Global existence with asymptotic behaviors and quenching results for the shadow system are derived in [8]. More precisely, all results for the kinetic system and system (3)-(6) with $d_b = d_c$ obtained in [11] are proved in [8] for the following shadow system

$$\begin{cases} \xi_t = \Big\{r_b\left(1 - \frac{\xi}{K}\right) - \frac{\mu}{|\Omega|}\int_\Omega P\,dx\Big\}\xi, \\ P_t = d_c \Delta P + \Big[r_c - r_b + r_b\frac{\xi}{K} - \mu\Big(r_c P - \frac{1}{|\Omega|}\int_\Omega P\,dx\Big)\Big]P, \\ \frac{\partial P}{\partial \nu} = 0, \quad x \in \partial\Omega,\, t > 0, \\ \xi(0) = \xi_0 = B_0 > 0, P(\cdot, 0) = P_0 := C_0/B_0 \geq 0, x \in \overline{\Omega}. \end{cases} \tag{10}$$

Coming back to the (full) reaction-diffusion system (8)-(9), little is known for the case when $d_b \neq d_c$. Up to now, we only have the following global existence result.

Theorem 1.1 ([8]). *Let $N = 1$ and $\Omega = (0, 1)$. Suppose that $r_c \geq 1$, $r_b > 1$, $2\pi^2 d_b + r_b \geq 2$, and $d_b \geq d_c$. Then every solution to system (8)-(9) with zero Neumann boundary condition and initial condition*

$$B(\cdot, 0) = B_0 > 0,\ P(\cdot, 0) = P_0 := C_0/B_0 \geq 0, \quad x \in \Omega$$

exists and is bounded globally in time. Moreover, as $t \to \infty$, $B(\cdot, t) \to B^*$ and $P(\cdot, t) \to P^*$ in $L^\infty(\Omega)$.

In fact, the global existence can be proved under conditions $r_c \geq 1$, $d_b \geq d_c$ and $N = 1$. See [8, Theorem 1.7]. For other related results on system (3)-(4) (or, (8)-(9)), we refer the interested reader to, e.g., [9, 10, 7] for more details.

In this paper, we shall give some new results on system (3)-(4) when $\Omega = \mathbb{R}^N$. In the sequel, for notational simplicity we set

$$d_b = d,\ K = 1,\ r_b = a,\ r_c = b,\ B = u,\ \mu C = v.$$

Also, without loss of generality (by a suitable spatial scaling) we may assume that $d_c = 1$. Therefore, system (3)-(4) for $\Omega = \mathbb{R}^N$ is reduced to

$$u_t = d\Delta u + au(1-u) - v, \quad x \in \mathbb{R}^N,\ t > 0, \tag{11}$$

$$v_t = \Delta v + bv\left(1 - \frac{v}{u}\right), \qquad x \in \mathbb{R}^N,\ t > 0. \tag{12}$$

The rest of this paper is organized as follows. In section 2, we shall describe the main results of this paper. Our first result is to characterize the spreading speed of predator for system (11)-(12). We then study the (planar) traveling wave solutions connecting the predator-free state to the positive co-existence state. It turns out that the minimal wave speed is the same as the spreading speed of the predator under the condition $a \geq 4$. We suspect that the condition $a \geq 4$ should be technical. This question is left for open. The third result is the existence and non-existence of periodic (in space) traveling wave solutions. Finally, the proofs of these results are given in sections 3-5.

2. **Main results.**

In this section, we shall describe the main results obtained in this paper.

The first result is about the spreading speed of the predator. The study of spreading speed is important in ecology, since it tells us how fast the predator can invade the habitat of the existing prey. For the spreading speed in scalar equations and predator-prey systems, we refer the reader to, e.g., [2, 22, 23, 17, 20, 21]. Here the spreading speed is adopted from the notion introduced by Aronson and Weinberger [2] for scalar equations. More precisely, a constant c^* is called the spreading speed (of the predator) of system (11)-(12) if the following two conditions hold:

$$\lim_{t\to\infty} \sup_{|x|>ct} v(x,t) = 0 \quad \text{for } c > c^*, \tag{13}$$

$$\liminf_{t\to\infty} \inf_{|x|<ct} v(x,t) > 0 \quad \text{for } c \in (0, c^*) \tag{14}$$

for any solution (u, v) of the initial value problem for (11)-(12) supplemented with the initial condition

$$u(x, 0) = u_0(x) \equiv 1, \quad v(x, 0) = v_0(x), \quad x \in \mathbb{R}^N, \tag{15}$$

where v_0 is a nonnegative continuous function with nonempty compact support.

Theorem 2.1. *Suppose that $a \geq 4$. Then the spreading speed of system (11)-(12) is given by $c^* = 2\sqrt{b}$.*

It is rather surprising that Theorem 2.1 also implies that solutions exists globally even when $b < 1$. However, a finite time singularity may occur when $b < 1$ for solutions to system (3)-(4) on bounded domain Ω. This is true for the case $d = 1$ (of course for certain initial data) and for the shadow system (10).

The second result is about the planar traveling waves to system (11)-(12). Hence we assume that $N = 1$. A solution of (11)-(12) (with $N = 1$) is called a traveling wave solution with speed s if there exist positive functions ϕ_1 and ϕ_2 defined on $\mathbb{R}$ such that

$$u(x,t) = \phi_1(z), \quad v(x,t) = \phi_2(z), \quad z := x + st.$$

Here ϕ_k, $k = 1, 2$, are called the wave profiles. Then (11)-(12) is reduced to the following system of equations:

$$\begin{cases} d\phi_1''(z) - s\phi_1'(z) + a\phi_1(z)\left[1 - \phi_1(z)\right] - \phi_2(z) = 0, \ z \in \mathbb{R}, \\ \phi_2''(z) - s\phi_2'(z) + b\phi_2(z)\left[1 - \dfrac{\phi_2(z)}{\phi_1(z)}\right] = 0, \ z \in \mathbb{R}, \end{cases} \tag{16}$$

where the prime denotes d/dz.

If $a > 1$, then (11)-(12) has a positive constant state $E^* = (u^*, v^*)$, where $u^* = v^* = 1 - 1/a$. We are interested in whether an alien predator can invade the existing prey in the habitat. Moreover, it is also interesting to see whether both predator and prey can live together. To see this, we shall study the existence of the traveling wave solution connecting the predator-free state $(1, 0)$ and the co-existence state (u^*, v^*). This implies that (ϕ_1, ϕ_2) satisfies the following asymptotic boundary conditions

$$\lim_{z\to-\infty}(\phi_1, \phi_2) = (1, 0), \quad \lim_{z\to\infty}(\phi_1, \phi_2) = (u^*, v^*). \tag{17}$$

Theorem 2.2. *Suppose that $a \geq 4$ and $N = 1$. Then the minimal speed of traveling wave solutions of (11)-(12) connecting $(1, 0)$ and (u^*, v^*) is $2\sqrt{b} := s^*$. In other words, for each $s \geq s^*$, there exists a positive solution of (16) with the condition (17). On the other hand, for $s < s^*$, there exist no nonnegative solutions of (16) with the condition (17).*

This result shows that the minimal wave speed is exactly the same as the minimal wave speed of the Fisher-KPP equation $u_t = u_{xx} + bu(1 - u)$. Indeed, ahead of the invading front of predator, the prey population density is approximately equal to one by (17). Thus we can heuristically reduce the problem to a single equation and obtain the minimal speed. For more background and illustrations of traveling wave solutions and minimal speeds in biology, we refer the reader to [19].

Our third result is the existence of (spatially) periodic traveling wave solutions. The following theorem can be proved by applying a general theory of Hopf bifurcation (cf. [14, 1]).

Theorem 2.3. *Suppose that $N = 1$ and*

$$a > 1, \quad b < 1, \quad a + b < 2. \tag{18}$$

Let

$$d_s := \frac{(2 - a - b)(s^2 + 2b) + \sqrt{\Delta(s)}}{2b(1 - b)} - 1, \tag{19}$$

where

$$\Delta(s) := (2-a-b)^2(s^2+2b)^2 + 4(2-a-b)^2 b(1-b).$$

Case 1: *Assume*

$$a+\sqrt{b} \le 2. \tag{20}$$

Then for any $s>0$ system (11)-(12) has a family of (positive) periodic traveling wave solutions when the diffusion coefficient d is sufficiently close to d_s.

Case 2: *Assume*

$$a+\sqrt{b} > 2. \tag{21}$$

Then there exists a unique minimum speed $s_p>0$ such that for $s>s_p$ system (11)-(12) has a family of periodic traveling wave solutions when the diffusion coefficient d is sufficiently close to d_s, where s_p is the unique positive zero of d_s.

Moreover, we have the following non-existence result for periodic traveling waves.

Theorem 2.4. *Suppose that $N=1$. If $a\ge 2$, $b\ge 1$ and $d \ge 1$, then there exist no non-constant periodic traveling wave solutions.*

3. **Spreading speed.** In this section, we provide a proof of Theorem 2.1 on the spreading speed of (11)-(12). For this, we consider the initial value problem for (11)-(12) with the initial condition (15), in which v_0 is a nonnegative continuous function defined on $\mathbb{R}$ with compact support. Let (u,v) be a solution of (11)-(12) and (15) for $t\in(0,T)$ for some $T\in(0,\infty]$. Note that we have $u>0$ and $v\ge 0$ for all $x\in\mathbb{R}^N$, $t<T$.

Since

$$u_t = d\Delta u + au(1-u) - v \le d\Delta u + au(1-u),\ x\in\mathbb{R}^N,\ 0<t<T,$$

by the comparison principle we have $u(x,t)\le 1$ for all $x\in\mathbb{R}$, $t\in[0,T)$. It follows that

$$v_t = \Delta v + bv\left(1-\frac{v}{u}\right) \le \Delta v + bv(1-v),\ x\in\mathbb{R}^N,\ 0<t<T.$$

Thus, $v(x,t)\le 1$ for all $x\in\mathbb{R}$, $t\in[0,T)$.

For $a>4$, the function $F(u):=au(1-u)-1$ has two distinct roots in $(0,1)$ and they are given by

$$\underline{u} := \frac{1}{2} - \sqrt{\frac{1}{4}-\frac{1}{a}},\quad \overline{u} := \frac{1}{2} + \sqrt{\frac{1}{4}-\frac{1}{a}}.$$

Moreover, for $a=4$, $F(u)$ has the double root $\underline{u}=1/2$.

We now assume that $a\ge 4$. Since

$$u_t = d\Delta u + au(1-u) - v \ge d\Delta u + au(1-u) - 1,\ x\in\mathbb{R}^N,\ 0<t<T,$$

and $w\equiv \underline{u}$ is a sub-solution of

$$\begin{cases} w_t = dw_{xx} + aw(1-w) - 1,\ x\in\mathbb{R}^N,\ 0<t<T,\\ w(x,0)=\underline{u},\ x\in\mathbb{R}^N,\end{cases}$$

we have

$$\underline{u}\le u\le 1,\quad 0\le v\le 1,\ x\in\mathbb{R}^N,\ t\in[0,T). \tag{22}$$

We conclude that (u,v) exists globally in time (so that $T=\infty$) such that estimate (22) holds for all $t\ge 0$.

Since $v\le V$, where V satisfies $V(x,0)=v_0(x)$, $x\in\mathbb{R}^N$, and

$$V_t = \Delta v + bV(1-V),\ x\in\mathbb{R}^N,\ t>0,$$

it follows from the classical result of spreading (cf. [2]) that

$$0 \le \lim_{t\to\infty} \sup_{|x|>(c^*+\varepsilon)t} v(x,t) \le \lim_{t\to\infty} \sup_{|x|>(c^*+\varepsilon)t} V(x,t) = 0 \tag{23}$$

for any $\varepsilon > 0$, where $c^* := 2\sqrt{b}$.

On the other hand, by (22), we have

$$v_t = \Delta v + bv\left(1 - \frac{v}{u}\right) \ge \Delta v + bv(1 - \frac{v}{\underline{u}}),\ x \in \mathbb{R}^N,\ t > 0.$$

Hence $v \ge W$, where $W(x,0) = v_0(x)$, $x \in \mathbb{R}^N$, and

$$W_t = W_{xx} + bW(1 - \frac{W}{\underline{u}}),\ x \in \mathbb{R}^N,\ t > 0.$$

Again, it follows from the classical result of spreading (cf. [2]) that

$$\liminf_{t\to\infty} \inf_{|x|<(c^*-\varepsilon)t} v(x,t) \ge \liminf_{t\to\infty} \inf_{|x|<(c^*-\varepsilon)t} W(x,t) = \underline{u} > 0 \tag{24}$$

for any $\varepsilon \in (0, c^*)$. Therefore, Theorem 2.1 is proved by combining (23) and (24). □

4. **Traveling wave solutions.** This section is devoted to the existence and non-existence of traveling wave solutions of (11)-(12) (with $N = 1$) connecting $(1,0)$ and (u^*, v^*). We shall always assume that $a \ge 4$. Since the method here is very standard, we shall only provide some outline of the proof.

First, we introduce the function space

$$\mathbf{X} = \left\{\mathbf{\Phi} = (\phi_1, \phi_2) \mid \mathbf{\Phi} \text{ is continuous function from } \mathbb{R} \text{ to } \mathbb{R}^2\right\},$$
$$\mathbf{X}_0 = \left\{(\phi_1, \phi_2) \in \mathbf{X} \mid \frac{1}{2} \le \phi_1 \le 1 \text{ and } 0 \le \phi_2 \le 1 \text{ for all } z \in \mathbb{R}\right\}.$$

Define the functions

$$F_1(y_1, y_2) := \tau y_1 + a y_1(1 - y_1) - y_2,$$
$$F_2(y_1, y_2) := \tau y_2 + b y_2\left(1 - \frac{y_2}{y_1}\right).$$

for a constat τ such that $\tau > \max\{a, 3b\}$. Then it is easy to see that $\frac{\partial F_1}{\partial y_1} \ge 0$, $\frac{\partial F_1}{\partial y_2} \le 0$, $\frac{\partial F_2}{\partial y_1} \ge 0$ and $\frac{\partial F_2}{\partial y_2} \ge 0$ for $\frac{1}{2} \le y_1 \le 1$ and $0 \le y_2 \le 1$. Then (16) can be re-written as

$$d_k \phi_k''(z) - s\phi_k'(z) - \tau\phi_k(z) + F_k(\phi_1, \phi_2)(z) = 0,\ z \in \mathbb{R},\ k = 1, 2,$$

where $d_1 = d$ and $d_2 = 1$. Now we define

$$\lambda_k^{\pm}(s) = \frac{s \pm \sqrt{s^2 + 4\tau d_k}}{s d_k},\ k = 1, 2.$$

For convenience, we write $\lambda_k^{\pm}$ instead of $\lambda_k^{\pm}(s)$. Also, notice that $\lambda_k^- < 0 < \lambda_k^+$ and

$$d_k(\lambda_k^{\pm})^2 - s\lambda_k^{\pm} - \tau = 0,\ k = 1, 2.$$

For $(\phi_1, \phi_2) \in \mathbf{X}_0$, we consider the operator $P = (P_1, P_2) : \mathbf{X}_0 \to \mathbf{X}$ defined as follows

$$P_k(\phi_1, \phi_2)(z) = \frac{1}{d_k(\lambda_k^+ - \lambda_k^-)}\left[\int_{-\infty}^{z} e^{\lambda_k^-(z-\xi)} + \int_{z}^{\infty} e^{\lambda_k^+(z-\xi)}\right] F_k(\phi_1, \phi_2)(\xi) d\xi$$

for $z \in \mathbb{R}$, $k = 1, 2$. It is easy to check that the operator P_k satisfies

$$d_k(P_k(\phi_1, \phi_2))'' - s(P_k(\phi_1, \phi_2))' - \tau P_k(\phi_1, \phi_2) + F_k(\phi_1, \phi_2) = 0 \text{ in } \mathbb{R},\ k = 1, 2.$$

Next, we give the definition of upper and lower solutions of (16) as follows.

Definition 4.1. Positive functions $(\overline{\phi}_1, \overline{\phi}_2)$ and $(\underline{\phi}_1, \underline{\phi}_2)$ are called a pair of upper and lower solutions of (16) if $\overline{\phi}_k''$, $\underline{\phi}_k''$, $\overline{\phi}_k'$, $\underline{\phi}_k'$, $k = 1, 2$, are bounded functions and satisfy the following inequalities

$$d\overline{\phi}_1''(z) - s\overline{\phi}_1'(z) + a\overline{\phi}_1(z)\left[1 - \overline{\phi}_1(z)\right] - \underline{\phi}_2(z) \leq 0, \tag{25}$$

$$d\underline{\phi_1}''(z) - s\underline{\phi}_1'(z) + a\underline{\phi}_1(z)\left[1 - \underline{\phi}_1(z)\right] - \overline{\phi}_2(z) \geq 0, \tag{26}$$

$$\overline{\phi}_2''(z) - s\overline{\phi}_2'(z) + b\overline{\phi}_2(z)\left[1 - \overline{\phi}_2(z)/\overline{\phi}_1(z)\right] \leq 0, \tag{27}$$

$$\underline{\phi}_2''(z) - s\underline{\phi}_2'(z) + b\underline{\phi}_2(z)\left[1 - \underline{\phi}_2(z)/\underline{\phi}_1(z)\right] \geq 0. \tag{28}$$

for $z \in \mathbb{R}\backslash D$ with some finite set $D = \{z_1, z_2, \dots, z_m\}$.

Following [18, 3], we have the following existence result for (16).

Lemma 4.2. *Let $s > 0$. Suppose that (16) has a pair of upper and lower solutions $(\overline{\phi}_1, \overline{\phi}_2)$ and $(\underline{\phi}_1, \underline{\phi}_2)$ in $\boldsymbol{X}_0$ satisfying $\overline{\phi}_k(z) \geq \underline{\phi}_k(z)$, $z \in \mathbb{R}$, and $\overline{\phi}_k'(z-) \geq \overline{\phi}_k'(z+)$ and $\underline{\phi}_k'(z-) \leq \underline{\phi}_k'(z+)$, $z \in D$, $k = 1, 2$, where*

$$\overline{\phi}_k'(z\pm) := \lim_{\xi \to z^\pm} \overline{\phi}_k'(\xi), \quad \underline{\phi}_k'(z\pm) := \lim_{\xi \to z^\pm} \underline{\phi}_k'(\xi).$$

Then (16) has a positive solution (ϕ_1, ϕ_2) such that $\overline{\phi}_k(z) \geq \phi_k(z) \geq \underline{\phi}_k(z)$ for all $z \in \mathbb{R}$ for $k = 1, 2$.

Proof. Since the proof is rather standard by now, we only give the outline of the proof here.

Take a constant α with $0 < \alpha < \min\{-\lambda_{1,-}, -\lambda_{2,-}\}$ and denote $\|\cdot\|$ the supremum norm in $\mathbb{R}^2$. Define

$$\mathbf{B}_\alpha(\mathbb{R}, \mathbb{R}^2) := \left\{\mathbf{\Phi} \in \mathbf{X}_0 \mid \sup_{z\in\mathbb{R}} \|\mathbf{\Phi}\| e^{-\alpha|z|} < \infty\right\}, |\mathbf{\Phi}|_\alpha := \sup_{z\in\mathbb{R}} \|\mathbf{\Phi}\| e^{-\alpha|z|}.$$

Then $\left(\mathbf{B}_\alpha(\mathbb{R}, \mathbb{R}^2), |\cdot|_\alpha\right)$ is a Banach space. Also, we let

$$\Gamma := \left\{(\phi_1, \phi_2) \in \mathbf{X}_0 \ : \ \underline{\phi}_k(z) \leq \phi_k(z) \leq \overline{\phi}_k(z) \text{ for all } z \in \mathbb{R},\ k = 1, 2\right\}.$$

First, we claim that P maps Γ into itself. This is equivalent to the following inequalities:

$$\begin{cases} \underline{\phi}_1(z) \leq P_1(\underline{\phi}_1, \overline{\phi}_2)(z) \leq P_1(\overline{\phi}_1, \underline{\phi}_2)(z) \leq \overline{\phi}_1(z),\ z \in \mathbb{R}, \\ \underline{\phi}_2(z) \leq P_2(\underline{\phi}_1, \underline{\phi}_2)(z) \leq P_2(\overline{\phi}_1, \overline{\phi}_2)(z) \leq \overline{\phi}_2(z),\ z \in \mathbb{R}, \end{cases} \tag{29}$$

by the choice of τ. Next, since $P : \Gamma \to \Gamma$ is completely continuous in the sense of the weighted norm $|\cdot|_\alpha$, the lemma follows by applying Schauder's fixed point theorem. □

4.1. **Upper and lower solutions.** To derive the existence of traveling waves, we need to find some suitable pairs of upper and lower solutions of (16). For this, we divide it into two cases: $s > s^*$ and $s = s^*$, where $s^* := 2\sqrt{b}$. The main idea of the following construction is from [3] with some modifications.

4.1.1. *The case $s > s^*$.* For a given wave speed $s > s^* = 2\sqrt{b}$, we define the following positive constants:

$$\lambda_1 = \frac{s + \sqrt{s^2 + 4ad}}{2d},\ \lambda_2 = \frac{s - \sqrt{s^2 - 4b}}{2},\ \lambda_3 = \frac{s + \sqrt{s^2 - 4b}}{2}. \tag{30}$$

Notice that

$$d\lambda_1^2 - s\lambda_1 - a = 0, \quad \lambda_k^2 - s\lambda_k + b = 0,\ k = 2, 3. \tag{31}$$

For given constants $\mu > 1$ and $q > 1$ we define $f(z) := e^{\lambda_2 z} - qe^{\mu\lambda_2 z}$. Then $f(z)$ has exactly one zero $z_0 < 0$ and exactly one maximum point $z_M < z_0$. Note that

$$f(z) \le f(z_M) = \left(1 - \frac{1}{\mu}\right)\left(\frac{1}{q\mu}\right)^{1/(\mu-1)}, \quad z_M = -\frac{\ln(q\mu)}{(\mu - 1)\lambda_2}.$$

For a given positive $\delta < \min\{f(z_M), 1/2\}$, since $f(z)$ is continuous and positive on $(-\infty, z_0)$, we can choose a point $z_2 \in (z_M, z_0)$ such that $f(z_2) = \delta$.

With this choice of δ, we further take the constants μ, ν, η, p, q and ε (*in sequence*) satisfying the following assumptions (A1)-(A3).

(A1) $1 < \mu < \min\{\lambda_3/\lambda_2, 2\},\ \nu > \max\{1, \lambda_2/\lambda_1\}$ and $0 < \eta < \min\{1/\sqrt{2}, \lambda_2/\lambda_1\}$.

(A2) $p > \frac{(a/2)+2}{-[d(\eta\lambda_1)^2 - s(\eta\lambda_1) - a/2]}$ and $q > \max\left\{1, \frac{2b}{-[d(\mu\lambda_2)^2 - s(\mu\lambda_2) + b]}\right\}$.

(A3) $0 < \varepsilon < \min\left\{\frac{\delta}{d(\nu\lambda_1)^2 - s(\nu\lambda_1) - a}, \frac{1 - qe^{(\mu-1)\lambda_2 z_2}}{d(\nu\lambda_1)^2 - s(\nu\lambda_1) - a}\right\}$.

Note that, by using (31) and (A1),

$$d(\eta\lambda_1)^2 - s(\eta\lambda_1) - a/2 < 0, \quad d(\mu\lambda_2)^2 - s(\mu\lambda_2) + b < 0,$$
$$d(\nu\lambda_1)^2 - s(\nu\lambda_1) - a > 0, \quad 1 - qe^{(\mu-1)\lambda_2 z_2} > 1 - qe^{(\mu-1)\lambda_2 z_0} = 0.$$

Hence the constants p, q and ε in (A2)-(A3) are well-defined.

Then we introduce the following functions

$$\begin{aligned}
\overline{\phi}_1(z) &= \begin{cases} 1, & z \ge 0, \\ 1 - \varepsilon(e^{\lambda_1 z} - e^{\nu\lambda_1 z}), & z < 0, \end{cases} \\
\underline{\phi}_1(z) &= \begin{cases} \frac{1}{2}, & z \ge z_1, \\ 1 - \frac{1}{2}(e^{\lambda_1 z} + pe^{\eta\lambda_1 z}), & z < z_1, \end{cases} \\
\overline{\phi}_2(z) &= \begin{cases} 1, & z \ge 0, \\ e^{\lambda_2 z}, & z < 0, \end{cases} \\
\underline{\phi}_2(z) &= \begin{cases} \delta, & z \ge z_2, \\ e^{\lambda_2 z} - qe^{\mu\lambda_2 z}, & z < z_2, \end{cases}
\end{aligned} \tag{32}$$

where $z_1 < 0$ is defined by $e^{\lambda_1 z_1} + pe^{\eta\lambda_1 z_1} = 1$.

Lemma 4.3. *For $s > s^*$, the functions $(\overline{\phi}_1, \overline{\phi}_2)$ and $(\underline{\phi}_1, \underline{\phi}_2)$ defined in (32) are a pair of upper and lower solutions of (16).*

4.1.2. *The case $s = s^*$.* In this case, we have (30) with $\lambda_2 = \lambda_3 = \sqrt{b}$. First, for given positive constants q and h, recall from [3] that the function

$$g(z) := \left[-hz - q(-z)^{1/2}\right] e^{\lambda_2 z}, \quad z \le 0,$$

has exactly two critical points in $(-\infty, 0)$ such that $g > 0$ in $(-\infty, z_0)$, where $z_0 := -(q/h)^2$, $g(z_0) = 0$, and there is a unique maximal point $\tilde{z}$ in $(-\infty, z_0)$. We choose z_2 in $(\tilde{z}, z_0)$ such that $g(z_2) = \delta$ for a fixed $\delta < 1/2$. Set $h = \lambda_2 e^2/2$.

With these constants δ and h, we then choose the constants η, ν, p, q and ε (*in sequence*) satisfying the following assumptions (B1)-(B4).

(B1) $0 < \eta \ll 1$ such that $d(\eta\lambda_1)^2 - s(\eta\lambda_1) - a/2 < 0$ and $\lambda_2 > 2\eta\lambda_1$.
(B2) $\nu > \max\{1, \lambda_2/\lambda_1\}$.
(B3) $p > \max\left\{e, \frac{(a/2)+2he^{-1}/(\eta\lambda_1)}{-[d(\eta\lambda_1)^2-s(\eta\lambda_1)-a/2]}\right\}$ and $q > \max\left\{1, \sqrt{\frac{2}{\lambda_2}}h, 8bh^2\left(\frac{7}{2\lambda_2 e}\right)^{7/2}\right\}$.
(B4) $0 < \varepsilon < \min\left\{\frac{\delta}{d(\nu\lambda_1)^2-s(\nu\lambda_1)-a}, \frac{-hz_2-q(-z_2)^{1/2}}{d(\nu\lambda_1)^2-s(\nu\lambda_1)-a}\right\}$.

Note that $d(\nu\lambda_1)^2 - s(\nu\lambda_1) - a > 0$ and $-hz_2 - q(-z_2)^{1/2} > 0$ so that ε is well-defined.

Then we define the functions

$$\begin{aligned}
\overline{\phi}_1(z) &= \begin{cases} 1, & z \ge 0, \\ 1 - \varepsilon(e^{\lambda_1 z} - e^{\nu\lambda_1 z}), & z < 0, \end{cases} \\
\underline{\phi}_1(z) &= \begin{cases} \frac{1}{2}, & z \ge z_1, \\ 1 - \frac{1}{2}(e^{\lambda_1 z} + pe^{\eta\lambda_1 z}), & z < z_1, \end{cases} \\
\overline{\phi}_2(z) &= \begin{cases} 1, & z \ge -2/\lambda_2, \\ -hze^{\lambda_2 z}, & z < -2/\lambda_2, \end{cases} \\
\underline{\phi}_2(z) &= \begin{cases} \delta, & z \ge z_2, \\ \left[-hz - q(-z)^{1/2}\right] e^{\lambda_2 z}, & z < z_2, \end{cases}
\end{aligned} \tag{33}$$

where $z_1 < 0$ is defined by $e^{\lambda_1 z_1} + pe^{\eta\lambda_1 z_1} = 1$. Note that $z_2 < z_0 < -2/\lambda_2$ and $z_1 < -2/\lambda_2$.

Lemma 4.4. *For $s = s^*$, the functions $(\overline{\phi}_1, \overline{\phi}_2)$ and $(\underline{\phi}_1, \underline{\phi}_2)$ define in (33) are pair of upper and lower solutions of (16).*

The proofs of Lemmas 4.3 and 4.4 are by straightforward calculations, we safely omit it here. Of course, to find a suitable pair of upper-lower-solutions is not always simple. We were unable to find a suitable pair of upper-lower-solutions for $a \in (1, 4)$. Again, as for the spreading speed, the restriction of $a \ge 4$ should be only due to technical reasons. Note that the above choices of upper-lower-solutions are well-defined for $a \in (1, 4)$, but they do not satisfy (25)-(28). In particular, to verify (26) the condition $a \ge 4$ is needed.

4.2. **Existence of traveling wave solutions.** Having the upper and lower solutions of (16), the following theorem follows immediately from Lemma 4.2.

Theorem 4.5. *Suppose that $s \ge s^*$. Then there exists a positive solution (ϕ_1, ϕ_2) of (16) such that $(\phi_1, \phi_2)(-\infty) = (1, 0)$.*

To derive the existence of traveling wave connecting $(1, 0)$ and (u^*, v^*), it remains to verify the tail behavior of solutions at $z = +\infty$. Following [3], we define the following functions

$$m(\theta) := \theta\left(1 - \frac{1}{a}\right), \quad M(\theta) := \theta\left(1 - \frac{1}{a}\right) + (1-\theta)(1+\varepsilon)$$

for some $\varepsilon \in (1/8, 1/4)$. Here $\varepsilon > 1/8$ is imposed so that

$$-\varepsilon a^2 + 2\varepsilon a + 1 < 0 \ \text{ for all } a \ge 4. \tag{34}$$

For $0 < \theta_1 < \theta_2 < 1$, it is easy to see that

$$\begin{aligned}
0 = m(0) < m(\theta_1) < m(\theta_2) < m(1) &= 1 - \frac{1}{a} \\
&= M(1) < M(\theta_2) < M(\theta_1) < M(0) = 1 + \varepsilon.
\end{aligned}$$

Theorem 4.6. *Let* (ϕ_1, ϕ_2) *be a positive solution obtained in Theorem 4.5. Then*

$$\lim_{z\to+\infty}(\phi_1,\phi_2)(z) = \left(1-\frac{1}{a}, 1-\frac{1}{a}\right).$$

Proof. Since the proof is completely similar to that of [3], we only point out the difference due to the nonlinearity here is different from there.

Since $1/2 = \underline{\phi}_1(z) \le \phi_1(z) \le \overline{\phi}_1(z) = 1$ and $\delta = \underline{\phi}_2(z) \le \phi_2(z) \le \overline{\phi}_2(z) = 1$ for all $z > 0$, we have

$$\limsup_{z\to+\infty}\phi_1(z) \le 1, \quad \limsup_{z\to+\infty}\phi_2(z) \le 1, \tag{35}$$

$$\liminf_{z\to+\infty}\phi_1(z) \ge \frac{1}{2}, \quad \liminf_{z\to+\infty}\phi_2(z) \ge \delta > 0. \tag{36}$$

Denote

$$\phi_k^- = \liminf_{z\to+\infty}\phi_k(z), \quad \phi_k^+ = \limsup_{z\to+\infty}\phi_k(z),\ k = 1,2,$$

and define

$$\theta_0 = \sup\left\{\theta \in [0,1)\ :\ m(\theta) < \phi_k^- \le \phi_k^+ < M(\theta),\ k=1,2\right\}.$$

Then it suffices to show that $\theta_0 = 1$.

By contradiction, we assume that $\theta_0 < 1$. By taking a sequence $\{\theta_j\}$ with $\theta_j \uparrow \theta_0$ and passing to the limit, we obtain

$$m(\theta_0) \le \phi_1^- \le \phi_1^+ \le M(\theta_0), \quad m(\theta_0) \le \phi_2^- \le \phi_2^+ \le M(\theta_0).$$

We next show that

$$m(\theta_0) < \phi_1^- \le \phi_1^+ < M(\theta_0), \quad m(\theta_0) < \phi_2^- \le \phi_2^+ < M(\theta_0). \tag{37}$$

Suppose for contradiction that (37) does not hold. Then one of the following four cases must happen.

Case 1: $\phi_1^- = m(\theta_0)$. Note that $m(\theta_0) \ge 1/2$, by (36). Hence

$$\begin{aligned} &\liminf_{z\to\infty}\{a\phi_1(z)[1-\phi_1(z)] - \phi_2(z)\} \\ \ge\ & am(\theta_0)[1-m(\theta_0)] - M(\theta_0) \\ =\ & am(\theta_0)[1-m(\theta_0)] - [m(\theta_0) + (1-\theta_0)(1+\varepsilon)] \\ =\ & (1-\theta_0)[m(\theta_0)(a-1) - 1 - \varepsilon] \ge (1-\theta_0)[(a-1)/2 - 1 - \varepsilon] > 0, \end{aligned}$$

due to $a \ge 4$ and $\varepsilon < 1/4$. Then, following the same argument as that in [3], it leads to a contradiction.

Case 2: $\phi_1^+ = M(\theta_0)$. Note that $M(\theta_0) \ge 1 - 1/a$. Then we have

$$\begin{aligned} &\limsup_{z\to\infty}\{a\phi_1(z)[1-\phi_1(z)] - \phi_2(z)\} \\ \le\ & aM(\theta_0)\,[1-M(\theta_0)] - m(\theta_0) \\ =\ & aM(\theta_0)\,[1-M(\theta_0)] - [M(\theta_0) - (1-\theta_0)(1+\varepsilon)] \\ =\ & (1-\theta_0)\,[(1+\varepsilon) - M(\theta_0)(1+a\varepsilon)] \le (1-\theta_0)\left[(1+\varepsilon) - (1-\frac{1}{a})(1+a\varepsilon)\right] \\ =\ & (1-\theta_0)\left(-\varepsilon a^2 + 2\varepsilon a + 1\right)/a < 0, \end{aligned}$$

due to (34) and $a \ge 4$. With this fact, similar to case 1, this case is also impossible.

Case 3: $\phi_2^- = m(\theta_0)$. The same argument as that in [3] leads to a contradiction.

Case 4: $\phi_2^- = M(\theta_0)$. Similar to case 3, this case is also impossible.

We conclude that (37) holds. Therefore, by the continuity of $m(\theta)$ and $M(\theta)$, due to $\theta_0 < 1$, there exists a small positive constant $\tau \in (0, 1-\theta_0)$ such that

$$m(\theta_0+\tau) < \phi_1^- \le \phi_1^+ < M(\theta_0+\tau), \quad m(\theta_0+\tau) < \phi_2^- \le \phi_2^+ < M(\theta_0+\tau),$$

which gives a contradiction to the definition of θ_0. Consequently, we must have $\theta_0 = 1$ and we finish the proof. □

4.3. **Nonexistence of traveling wave solutions.** In this subsection, we complete the proof of Theorem 2.2 by showing that there is no positive traveling wave solution of (11)-(12) with $N = 1$ connecting $(1, 0)$ and (u^*, v^*) for $s < s^*$.

First we recall the following spreading phenomenon [2]. Consider the following Cauchy problem for Fisher's equation

$$\begin{cases} z_t(x,t) = dz_{xx}(x,t) + rz(x,t)[1 - kz(x,t)], & x \in \mathbb{R},\ t > 0, \\ z(x,0) = z_0(x), & x \in \mathbb{R}, \end{cases}$$

where d, r, k are positive constants and $z_0(x)$ is a positive bounded continuous function. Then we have

$$\liminf_{t\to\infty} \inf_{|x|<ct} z(x,t) = \limsup_{t\to\infty} \sup_{|x|<ct} z(x,t) = \frac{1}{k}.$$

for any $c \in (0, 2\sqrt{dr})$. Then the same proof as that of [3, Theorem 2.6] leads

Theorem 4.7. *For $s < s^*$, there is no positive solution of (16) and (17).*

5. Periodic traveling waves.

5.1. **Proof of Theorem 2.3.** The proof is almost the same as that of [24, Theorem 1.2] except the problem here is different from there. We give some details here for the reader's convenience.

First, we set $\psi_1 := \phi_1'$ and $\psi_2 := \phi_2'$. Then system (16) can be re-written as

$$\begin{cases} \phi_1' = \psi_1, \\ \psi_1' = \{s\psi_1 - a\phi_1(1-\phi_1) + \phi_2\}/d, \\ \phi_2' = \psi_2, \\ \psi_2' = s\psi_2 - b\phi_2(1 - \phi_2/\phi_1), \end{cases} \tag{38}$$

which has two fixed points $(1, 0, 0, 0)$ and $(u^*, 0, u^*, 0)$.

Linearizing (38) at $(u^*, 0, u^*, 0)$, we obtain the following characteristic equation for eigenvalues of the corresponding Jacobian matrix

$$\lambda^4 - s\Big(1 + \frac{1}{d}\Big)\lambda^3 + \Big(\frac{s^2 + 2 - a}{d} - b\Big)\lambda^2 - \frac{s}{d}(2 - a - b)\lambda + \frac{b}{d}(a-1) = 0. \tag{39}$$

For a given $s > 0$, it is easy to see that equation (39) has a pair of pure imaginary solutions, $\lambda = \pm i\omega$, $\omega > 0$, if and only if there is a unique positive $d = d_s$ (given by (19)) such that

$$g(d) := b(1-b)(d+1)^2 - (2-a-b)(s^2+2b)(d+1) - (2-a-b)^2 = 0. \tag{40}$$

Here we have used the fact that $0 < b < 1$ and $a + b < 2$.

To determine the existence of admissible d_s, we observe that $d_0 \ge 0$ if and only if $a + \sqrt{b} \le 2$, since for $s = 0$

$$d_0 = \frac{(2-a-b)/\sqrt{b} - (a-1)}{1-b}.$$

Moreover, it is easy to see from (19) that d_s, as a function of s, is strictly increasing in s for $s \geq 0$. Hence $d_s > 0$ for any $s > 0$, if (20) holds. On the other hand, if (21) holds, then there exists a unique $s_p > 0$ such that $d_s > 0$ if and only if $s > s_p$.

In order to apply the standard theory of Hopf bifurcation, we need to compute the derivative of $\lambda(d)$ at $d = d_s$ for an admissible s. More precisely, the existence of periodic traveling wave solutions follows if we have

$$\mathrm{Re}(\lambda'(d_s)) < 0, \tag{41}$$

where $\mathrm{Re}(z)$ denotes the real part of z. Hereafter the prime denotes the derivative respect to d. In fact, by differentiating (39) with respect to d, the expression of $\lambda'(d)$ can be easily derived.

Since the coefficient, $-s(2-a-b)/d$, of λ in (39) is negative, by [24, Lemma 3.2], condition (41) holds if

$$\left\{\frac{s^2}{d^3} g(d)\right\}'(d_s) > 0.$$

However, we have

$$\begin{aligned}\left\{\frac{s^2}{d^3} g(d)\right\}'(d_s) &= -\frac{s^2}{d_s^4} g(d_s) + \frac{s^2}{d_s^3} g'(d_s) = \frac{s^2}{d_s^3} g'(d_s) \\ &= \frac{s^2}{d_s^3}[2b(1-b)(d_s+1) - (2-a-b)(s^2+2b)] = \frac{s^2}{d_s^3}\sqrt{\Delta(s)} > 0,\end{aligned}$$

by using (19) and (40). Hence (41) is satisfied. Therefore, Theorem 2.3 follows by applying the general theory of Hopf bifurcation. □

5.2. **Proof of Theorem 2.4.** For $a > 1$, the positive co-existence state $(u^*, u^*) = (1 - 1/a, 1 - 1/a)$ exists and, as in [8], we introduce the functional

$$\begin{aligned} E[u,P](t) &:= d\int_0^L \Big(\frac{u_x}{u}\Big)^2 dx + \frac{d-1}{2b}\int_0^L \Big(\frac{v_x}{v}\Big)^2 dx \\ &\quad + a\int_0^L \Big(u - u^* - u^* \ln\frac{u}{u^*}\Big) dx + \int_0^L \Big(P - 1 - \ln P\Big) dx \end{aligned}$$

for a given (global in time) periodic solution (u, v) of period $L > 0$ to (11)-(12). Here $P := v/u$ and P satisfies

$$P_t = P_{xx} - (d-1)\frac{P}{u}u_{xx} + \frac{2}{u}u_x P_x + \{b - a + au - (b-1)P\}P, \ x \in \mathbb{R}, \ t > 0. \tag{42}$$

Then we have the following lemma.

Lemma 5.1. *Suppose that $a > 1$, $8\pi^2 d/L^2 + a \geq 2$, $b \geq 1$ and $d \geq 1$. Let (u, v) be a periodic solution of (11)-(12) with period L and $P = v/u$. Then the functional $E[u, P](t)$ is decreasing in t.*

Proof. The proof is almost the same as that of [8, Lemma 6.1], except that here we have the periodic boundary condition. Recall that the optimal constant of the Poincaré-Wirtinger's inequality for periodic functions of period L is $(2\pi/L)^2$. Hence the same calculations as that in the proof of [8, Lemma 6.1] gives

$$\begin{aligned}
&\frac{d}{dt}E[u,P](t)\\
\leq\ & -[2(2\pi/L)^2 d + a - 2]d\int_0^L \left(\frac{u_x}{u}\right)^2 dx - \frac{(2\pi/L)^2(d-1)}{b}\int_0^L \left(\frac{v_x}{v}\right)^2 dx\\
& -\int_0^L \left\{a^2(u-u^*)^2 + (b-1)(P-1)^2\right\} dx \leq 0,
\end{aligned}$$

if $a > 1$, $8\pi^2 d/L^2 + a \geq 2$, $b \geq 1$ and $d \geq 1$. This proves the lemma. □

Proof of Theorem 2.4. We use a contradiction argument. Suppose that there exists a nontrivial periodic traveling wave solution (u, v) to (11)-(12) of period $L > 0$. Since $a \geq 2$, the assumptions made in Lemma 5.1 hold. With Lemma 5.1 and following exactly the same argument as that in the proof of [8, Theorem 1.6], we can prove that, as $t \to \infty$, $u(\cdot, t) \to u^*$ and $v(\cdot, t) \to u^*$ uniformly in $\mathbb{R}$.

However, due to the periodicity of wave profile, we have

$$(u, v)(x, t + L/s) = (\phi_1, \phi_2)(x + st + L) = (\phi_1, \phi_2)(x + st) = (u, v)(x, t)$$

for all $x \in \mathbb{R}$, $t > 0$. With this time periodicity, we deduce that $(u, v) \equiv (u^*, v^*)$, a contradiction. Hence the theorem is proved. □

Acknowledgments. We would like to dedicate this work to Professor Sze-Bi Hsu on the occasion of his 70th birthday.

REFERENCES

[1] A. Ambrosetti and G. Prodi, *A Primer of Nonlinear Analysis*, Cambridge Studies in Advanced Mathematics, **34**, 1995.

[2] D.G. Aronson and H.F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, in: J.A. Goldstein(Ed.),Partial Differential Equations and Related Topics, in: Lecture Notes in Math., vol. 446, Springer, Berlin, 1975, pp. 5–49.

[3] Y.-Y. Chen, J.-S. Guo and C.-H. Yao Traveling wave solutions for a continuous and discrete diffusive predator-prey model, *J. Math. Anal. Appl.*, **445** (2017), 212–239.

[4] F. Courchamp, M. Langlais and G. Sugihara, Controls of rabbits to protect birds from cat predation, *Biological Conservations*, **89** (1999), 219–225.

[5] F. Courchamp and G. Sugihara, Modelling the biological control of an alien predator to protect island species from extinction, *Ecological Applications*, **9** (1999), 112–123.

[6] Y. Du and S.-B. Hsu, A diffusive predator-prey model in heterogeneous environment, *J. Differential Equations* **203** (2004), 331–364.

[7] A. Ducrot and J.-S. Guo, Quenching behavior for a singular predator-prey model, *Nonlinearity*, **25** (2012), 2059–2073.

[8] A. Ducrot, J.-S. Guo and M. Shimojo, Behaviors of solutions for a singular prey-predator model and its shadow system, *J. Dynam. Differential Equations*, **30** (2018), 1063–1079.

[9] A. Ducrot and M. Langlais, A singular reaction-diffusion system modelling prey-predator interactions : Invasion and co-extinction waves, *J. Differential Equations*, **253** (2012), 502–532.

[10] A. Ducrot and M. Langlais, Global weak solution for a singular two component reaction-diffusion system, *Bull. London Math. Soc.*, **46** (2014), 1–13.

[11] S. Gaucel and M. Langlais, Some remarks on a singular reaction-diffusion arising in predator-prey modelling, *Discrete Contin. Dyn. Syst. Ser. B*, **8** (2007), 61–72.

[12] J.-S. Guo and M. Shimojo, Spatio-temporal oscillation for a singular predator-prey model, *J. Math. Anal. Appl.*, **459** (2018), 1–9.

[13] J. Hainzl, Multiparameter bifurcation of a predator-prey system, *SIAM J. Math. Anal.*, **23** (1992), 150–180.

[14] B.D. Hassard, N.D. Kazarinoff and Y.-H. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge Univ. Press, Cambridge, New York, 1981.

[15] S.-B. Hsu and T.-W. Hwang, Global stability for a class of predator-prey systems, *SIAM J. Appl. Math.*, **55** (1995), 763–783.

[16] S.-B. Hsu and T.-W. Hwang, Uniqueness of limit cycles for a predator-prey system of Holling and Leslie type, *Canad. Appl. Math. Quart.*, **6** (1998), 91–117.

[17] G. Lin, Spreading speeds of a Lotka-Volterra predator-prey system: the role of the predator, *Nonlinear Anal.*, **74** (2011), 2448–2461.

[18] S. Ma, Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem, *J. Differential Equations* **171** (2001), 294–314.

[19] J.D. Murray, *Mathematical Biology. II. Spatial Models and Biomedical Applications*, 3^{rd} edition, Interdisciplinary Applied Mathematics, vol. 18, Springer, New York, 2003.

[20] S. Pan, Asymptotic spreading in a Lotka-Volterra predator-prey system, *J. Math. Anal. Appl.*, **407** (2013), 230–236.

[21] S. Pan, Invasion speed of a predator-prey system, *Appl. Math. Lett.*, **74** (2017), 46–51.

[22] H.F. Weinberger, Long-time behavior of a class of biological model, *SIAM J. Math. Anal.*, **13** (1982), 353–396.

[23] H.F. Weinberger, M.A. Lewis and B. Li, Analysis of linear determinacy for spread in cooperative models, *J. Math. Biol.*, **45** (2002), 183–218.

[24] W. Zuo and J. Shi, Traveling wave solutions of a diffusive ratio-dependent Holling-Tanner system with distributed delay, *Comm. Pure. Appl. Anal.*, **17** (2018), 1179–1200.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: formosa1502@gmail.com
E-mail address: jsguo@mail.tku.edu.tw
E-mail address: shimojo@xmath.ous.ac.jp