TRAVELING WAVE SOLUTIONS FOR A THREE-SPECIES PREDATOR-PREY MODEL WITH TWO ABORIGINE PREYS

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Abstract. In this paper, we study the invading phenomenon of an alien predator to the habitat of two aborigine preys by traveling waves connecting the predator-free state to the co-existence state. Based on an application of Schauder's fixed point theorem with the help of (generalized) upper-lower-solutions, we characterize the minimal wave speed of this invading process. New form of upper-lower-solutions are constructed to derive the existence of traveling waves for all admissible speeds.

1. Introduction

In primary succession, plants such as grass and trees are the pioneer species and then some animals that feed primarily on grass and trees are the second. The animals (predator) is attracted to invade the habitat of the preys (grass and trees). This ecological system can be modeled by the following three species predator-prey system:

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - u - kv - bw),\ x \in \mathbb{R},\ t > 0, \\ v_t = d_2 v_{xx} + r_2 v(1 - hu - v - bw),\ x \in \mathbb{R},\ t > 0, \\ w_t = d_3 w_{xx} + r_3 w(-1 + au + av - w),\ x \in \mathbb{R},\ t > 0, \end{cases} \tag{1.1}$$

where the unknowns u, v and w as functions of (x,t) stand for the population densities of preys u, v and predator w at position x and time t. The parameters d_i, r_i, $i = 1,2,3$, a, b, h and k are positive constants in which d_i, $i = 1,2,3$, are the diffusion rates of u, v and w; r_1 and r_2 are the intrinsic growth rates of u and v, respectively, and r_3 is the death rate of the predator w; $r_i b$, $i = 1,2$, are the predation rates and $r_3 a$ is the conversion rate of u (and v); h and k are the competition coefficients between two preys u and v.

Throughout this paper, we always assume that

$$h, k \in (0,1). \tag{1.2}$$

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In other words, we consider both preys are weak competitors. Also, to make sure the predator can survive without other food resources than these two preys we assume

$$a > \gamma,\ \gamma = \gamma(h,k) := \frac{1-hk}{2-h-k}. \tag{1.3}$$

It is easy to check that system (1.1) has two constant states $E_p = (u_p, v_p, 0)$ (the predator-free state) and $E_c = (u_c, v_c, w_c)$ (the co-existence state), where

$$u_p := \frac{1-k}{1-hk}, \quad v_p := \frac{1-h}{1-hk}, \tag{1.4}$$

$$u_c := \frac{(1+b)(1-k)}{(1-hk)+ab(2-h-k)}, \tag{1.5}$$

$$v_c := \frac{(1+b)(1-h)}{(1-hk)+ab(2-h-k)}, \tag{1.6}$$

$$w_c := \frac{a(2-h-k)-(1-hk)}{(1-hk)+ab(2-h-k)}. \tag{1.7}$$

Note that $\gamma(h,k) \in (1/2, 1)$. Also, under conditions (1.2) and (1.3) we have

$$u_p > u_c > 0\ , v_p > v_c > 0 \ \text{ and } \ w_c > 0. \tag{1.8}$$

Ecologically, it is interesting to see whether these three species can live together in the habitat of two aborigine preys after the invading of an alien species. One of the approaches to study this problem is to study the so-called traveling wave solutions of system (1.1) connecting the predator-free state and the co-existence state. From the view point of invading, the population densities of preys should be decreasing after the predator invades. So it is natural to require (1.8).

Another approach of studying the invading phenomenon is the so-called (asymptotic) spreading speed of the predator, by studying the Cauchy problem for (1.1) with initial condition

$$u(x,0) = u_p, \quad v(x,0) = v_p, \quad w(x,0) = w_0(x),\ x \in \mathbb{R},$$

where w_0 is a nonnegative continuous function with nonempty compact support. For the spreading speed for system (1.1), we refer the reader to the work by Wu [17]. In [17], under the conditions : $b = a$, $d_1 = d_2 = d$ and

$$r_1\beta_1 + r_2h\beta_2 = r_1k\beta_1 + r_2\beta_2,\ \beta_1 := 1-k-a\beta > 0,\ \beta_2 := 1-h-a\beta > 0,$$

for some positive constants h, k, a such that $0 < h, k < 1$ and $0 < a - 1 \ll 1$, It is shown in [17] that the spreading speed of the predator w is $2\sqrt{d_3r_3\beta}$ with $\beta := a(u_p + v_p) - 1$.

A solution of (1.1) is called a traveling wave solution with speed s if there exist positive functions $\{\phi_1, \phi_2, \phi_3\}$ defined on $\mathbb{R}$ such that $u(x,t) = \phi_1(x+st)$, $v(x,t) = \phi_2(x+st)$ and $w(x,t) = \phi_3(x+st)$. Here ϕ_j, $j = 1,2,3$, are the wave profiles. Let $z := x+st$ and substitute

$(u,v,w)(x,t) = (\phi_1,\phi_2,\phi_3)(z)$ into (1.1). Then $\{s,\phi_1,\phi_2,\phi_3\}$ satisfy the following system of equations:

$$\begin{cases} d_1\phi_1''(z) - s\phi_1'(z) + r_1\phi_1(z)[1-\phi_1(z)-k\phi_2(z)-b\phi_3(z)] = 0, \ z\in\mathbb{R}, \\ d_2\phi_2''(z) - s\phi_2'(z) + r_2\phi_2(z)[1-h\phi_1(z)-\phi_2(z)-b\phi_3(z)] = 0, \ z\in\mathbb{R}, \\ d_3\phi_3''(z) - s\phi_3'(z) + r_3\phi_3(z)[-1+a\phi_1(z)+a\phi_2(z)-\phi_3(z)] = 0, \ z\in\mathbb{R}, \end{cases} \tag{1.9}$$

where the prime denotes the derivative with respect to z. As we mentioned above, we are interested in the wave connecting the predator-free state and the co-existence state. Hence problem (1.9) is supplemented with the following asymptotic boundary conditions

$$\lim_{z\to-\infty}(\phi_1,\phi_2,\phi_3)(z) = (u_p,v_p,0) \quad\text{and}\quad \lim_{z\to\infty}(\phi_1,\phi_2,\phi_3)(z) = (u_c,v_c,w_c). \tag{1.10}$$

We now state the main theorem of this paper as follows.

Theorem 1.1. *Given h,k,a such that* (1.2) *and* (1.3) *hold. Let $s^* := 2\sqrt{d_3r_3\beta}$. Assume that*

$$a > \frac{2}{2-h-k}, \tag{1.11}$$

$$0 < b < \min\left\{\frac{1-k}{2a-1}, \frac{1-h}{2a-1}, \frac{a(2-h-k)-2}{2a(2a-1)}\right\}. \tag{1.12}$$

For $s > s^*$, under the condition

$$d_3 \geq \max\left\{\frac{d_1}{2}, \frac{d_2}{2}\right\}, \tag{1.13}$$

system (1.9) *has a solution (ϕ_1,ϕ_2,ϕ_3) such that* (1.10) *holds. For $s = s^*$, under the condition*

$$\max\left\{\frac{d_1}{2}, \frac{d_2}{2}\right\} \leq d_3 \leq \min\{d_1,d_2\}, \tag{1.14}$$

system (1.9) *has a solution (ϕ_1,ϕ_2,ϕ_3) such that* (1.10) *holds. Moreover, there is no positive solution for* (1.9)-(1.10) *if $s < s^*$.*

Due to the nonlinearity of our predator-prey model, system (1.1) does not have the comparison principle. The proof of Theorem 1.1 is based on an application of Schauder's fixed point theorem with the help of (generalized) upper-lower-solutions. This method has been proved to be very successful in the derivation of traveling waves for non-monotone systems since the pioneer works [14, 16], if a suitable upper-lower-solutions can be constructed. We refer the reader to [7, 8, 10, 9, 11, 21, 3, 19] for 2-component systems, [4, 15, 6, 20, 12, 18, 13, 2, 5] for 3-species cases and the references cited therein.

In particular, in [6], they constructed the traveling waves for a predator-prey system with one predator and two preys such that all of these 3 species are alien species and the predator can survive without the predation of the preys. The case when the predator has a negative growth rate, namely, system (1.1), is left open in [6]. In [2], they considered the same predator-prey model as (1.1) to investigate how one alien prey and one alien predator invade the habitat of an aborigine prey. The case of two alien predators and one aborigine prey is studied in [5]. However, the construction of suitable upper-lower-solutions is not always available.

In this paper, we study the predator-prey system (1.1) with *two aborigine preys* and one alien predator. To our knowledge, all existing results on the traveling waves for 3-species predator-prey systems are connecting an unstable state with at most one nonzero component to the co-existence state. In other words, it is always assumed that there is at most one aborigine species living in a habitat. This paper is the first work to construct traveling waves describing one alien predator invades two aborigine preys. In fact, the main difficulty in the construction of generalized upper-lower-solution is the two nonzero components in the unstable state.

By a linearization at the unstable state $(u_p, v_p, 0)$ for each equation in (1.9), the decay rate of $\phi_1 - u_p$ ($\phi_2 - v_p$, resp.) at $z = -\infty$ should be $e^{\lambda_1 z}$ ($e^{\lambda_2 z}$, resp.), where $\lambda_1 > 0$, $\lambda_2 > 0$ and they satisfy

$$d_1\lambda_1^2 - s\lambda_1 - r_1 u_p = 0, \quad d_2\lambda_2^2 - s\lambda_2 - r_2 v_p = 0.$$

However, these are not the correct asymptotic behaviors of ϕ_1 and ϕ_2 at the unstable tail. Surprisingly, under certain conditions on the parameters (as stated in Theorem 1.1), it turns out that the correct decay rate of $\phi_1 - u_p$ (and $\phi_2 - v_p$) is the same as that of ϕ_3, namely, $e^{\lambda_3 z}$ where λ_3 is the smaller positive root to

$$H(\lambda) := d_3\lambda^2 - s\lambda + r_3\beta = 0, \quad \beta := a(u_p + v_p) - 1,$$

for $s \geq s^*$. This counterintuitive behavior actually causes the major difficulty in the construction of upper-lower-solutions.

We find in this work that the minimal wave speed connecting the predator-free state to the co-existence state is the same as the spreading speed of the predator. However, the conditions on the parameters in these two works are very different. In [17], no restrictions are imposed on the diffusion rate d_3 of the predator. But, there are some restrictions on the growth rates of the preys in [17]. On the other hand, in this paper, we do not impose any restrictions on the growth rates of preys. Also, we consider different predation rates and conversion rates. However, we need to restrict ourselves on the diffusion rates d_i, $i = 1, 2, 3$. This may shed light on the limitations of these two different approaches to the invading phenomenon.

The rest of this paper is organized as follows. In §2, we provide some details of the method of generalized upper-lower-solutions originated from [10, 11]. The main task is to construct suitable upper-lower-solutions for each admissible wave speed. Then we give a proof of our main result, Theorem 1.1 in §3. For the existence part, based on results of §2, we only need to verify the wave profiles satisfy the required asymptotic boundary condition at the right-hand tail. The proof is based on constructing a sequence of shrinking rectangles (cf. [6, 3, 5]) with some modifications. The proof for the non-existence part of Theorem 1.1 is standard by using a contradiction argument with the help of the spreading phenomenon of the Cauchy problem for Fisher's equation ([1]). Finally, we provide the details of verification of upper-lower-solutions constructed in §2.

2. Method of generalized upper-lower-solutions

In this section, we shall provide some details of the method of generalized upper-lower-solutions with the help of Schauder's fixed point theorem (cf. e.g., [14, 10, 11]). First, we define the following function spaces

$$\mathbf{X} = \left\{\mathbf{\Phi} = (\phi_1, \phi_2, \phi_3) \mid \mathbf{\Phi} \text{ is continuous function from } \mathbb{R} \text{ to } \mathbb{R}^3\right\},$$
$$\mathbf{X}_0 = \{(\phi_1, \phi_2, \phi_3) \in \mathbf{X} \mid 0 \le \phi_1 \le 1,\ 0 \le \phi_2 \le 1,\ 0 \le \phi_3 \le B \ \text{ for all } z \in \mathbb{R}\},$$

where $B := 2a - 1$.

Define the functions F_k, $k = 1, 2, 3$,

$$F_1(y_1, y_2, y_3) = \tau y_1 + r_1 y_1(1 - y_1 - k y_2 - b y_3),$$
$$F_2(y_1, y_2, y_3) = \tau y_2 + r_2 y_2(1 - h y_1 - y_2 - b y_3),$$
$$F_3(y_1, y_2, y_3) = \tau y_3 + r_3 y_3(-1 + a y_1 + a y_2 - y_3),$$

for some large enough constant τ such that

$$\tau > \max\{r_1(1 + k + bB), r_2(1 + h + bB), r_3(2B + 1)\}.$$

Note that we have

$$\frac{\partial F_1}{\partial y_1} \ge 0,\ \frac{\partial F_1}{\partial y_2} \le 0,\ \frac{\partial F_1}{\partial y_3} \le 0;$$
$$\frac{\partial F_2}{\partial y_1} \le 0,\ \frac{\partial F_2}{\partial y_2} \ge 0,\ \frac{\partial F_1}{\partial y_3} \le 0;$$
$$\frac{\partial F_3}{\partial y_1} \ge 0,\ \frac{\partial F_3}{\partial y_2} \ge 0,\ \frac{\partial F_3}{\partial y_3} \ge 0.$$

Also, system(1.9) can be re-written as

$$d_k \phi_k''(z) - s\phi_k'(z) - \tau\phi_k(z) + F_k(\phi_1, \phi_2, \phi_3)(z) = 0,\ k = 1, 2, 3.$$

Next, we define

$$\nu_k^{\pm}(s) = \frac{s \pm \sqrt{s^2 + 4d_k\tau}}{2d_k},\ k = 1, 2, 3.$$

Clearly $\nu_k^- < 0 < \nu_k^+$ and

$$d_k(\nu_k^\pm)^2 - s\nu_k^\pm - \tau = 0,\ k = 1,2,3.$$

For $(\phi_1,\phi_2,\phi_3) \in \mathbf{X}_0$, we consider the operator $P = (P_1,P_2,P_3) : \mathbf{X}_0 \to \mathbf{X}$ defined by

$$P_k(\phi_1,\phi_2,\phi_3)(z) := \frac{1}{d_k(\nu_k^+ - \nu_k^-)}\left[\int_{-\infty}^{z} e^{\nu_k^-(z-\xi)} + \int_{z}^{\infty} e^{\nu_k^+(z-\xi)}\right] F_k(\phi_1,\phi_2,\phi_3)(\xi)d\xi,$$

for $k = 1,2,3,\ z \in \mathbb{R}$. It is easy to check that the operator P satisfies

$$d_k(P_k(\phi_1,\phi_2,\phi_3))''(z) - s(P_k(\phi_1,\phi_2,\phi_3))'(z) - \tau P_k(\phi_1,\phi_2,\phi_3)(z) + F_k(\phi_1,\phi_2,\phi_3) = 0$$

for $k = 1,2,3,\ z \in \mathbb{R}$. Therefore, to find a solution of (1.9) is equivalent to finding a fixed point of the operator P.

Now, we introduce the definition of (generalized) upper-lower-solutions of (1.9) as follows.

Definition 2.1. *Positive continuous functions* $(\overline{\phi}_1,\overline{\phi}_2,\overline{\phi}_3)$ *and* $(\underline{\phi}_1,\underline{\phi}_2,\underline{\phi}_3)$ *are called a pair of upper-lower-solutions of* (1.9) *if* $\overline{\phi}_i''$, $\underline{\phi}_i''$, $\overline{\phi}_i'$, $\underline{\phi}_i'$, $i = 1,2,3$, *are bounded functions and satisfy the following inequalities*

$$\mathcal{U}_1(z) := d_1\overline{\phi}_1''(z) - s\overline{\phi}_1'(z) + r_1\overline{\phi}_1(z)[1 - \overline{\phi}_1(z) - k\underline{\phi}_2(z) - b\underline{\phi}_3(z)] \le 0, \tag{2.1}$$

$$\mathcal{U}_2(z) := d_2\overline{\phi}_2''(z) - s\overline{\phi}_2'(z) + r_2\overline{\phi}_2(z)[1 - h\underline{\phi}_1(z) - \overline{\phi}_2(z) - b\underline{\phi}_3(z)] \le 0, \tag{2.2}$$

$$\mathcal{U}_3(z) := d_3\overline{\phi}_3''(z) - s\overline{\phi}_3'(z) + r_3\overline{\phi}_3(z)[-1 + a\overline{\phi}_1(z) + a\overline{\phi}_2(z) - \overline{\phi}_3(z)] \le 0, \tag{2.3}$$

$$\mathcal{L}_1(z) := d_1\underline{\phi}_1''(z) - s\underline{\phi}_1'(z) + r_1\underline{\phi}_1(z)[1 - \underline{\phi}_1(z) - k\overline{\phi}_2(z) - b\overline{\phi}_3(z)] \ge 0, \tag{2.4}$$

$$\mathcal{L}_2(z) := d_2\underline{\phi}_2''(z) - s\underline{\phi}_2'(z) + r_1\underline{\phi}_2(z)[1 - h\overline{\phi}_1(z) - \underline{\phi}_2(z) - b\overline{\phi}_3(z)] \ge 0, \tag{2.5}$$

$$\mathcal{L}_3(z) := d_3\underline{\phi}_3''(z) - s\underline{\phi}_3'(z) + r_3\underline{\phi}_3(z)[-1 + a\underline{\phi}_1(z) + a\underline{\phi}_2(z) - \underline{\phi}_3(z)] \ge 0, \tag{2.6}$$

for $z \in \mathbb{R}\backslash E$ *with some finite set* $E = \{z_1, z_2, \ldots, z_m\}$.

Then the following lemma gives the existence of positive solutions of (1.9). Since its proof is standard by now, we safely omit it (cf. [14, 10, 3]).

Lemma 2.2. *Given* $s > 0$, *suppose that* (1.9) *has a pair of upper-lower-solutions* $(\overline{\phi}_1,\overline{\phi}_2,\overline{\phi}_3)$ *and* $(\underline{\phi}_1,\underline{\phi}_2,\underline{\phi}_3)$ *in* $\boldsymbol{X}_0$ *satisfying*

(1) $\overline{\phi}_k(z) \ge \underline{\phi}_k(z),\ z \in \mathbb{R},\ k = 1,2,3$;

(2) $\overline{\phi}_k'(z^-) \ge \overline{\phi}_k'(z^+)$ *and* $\underline{\phi}_k'(z^-) \le \underline{\phi}_k'(z^+)$, $z \in E,\ k = 1,2,3,$

where

$$\overline{\phi}_k'(z^\pm) := \lim_{z\to z^\pm} \overline{\phi}_k'(z), \quad \underline{\phi}_k'(z^\pm) := \lim_{z\to z^\pm} \underline{\phi}_k'(z).$$

Then (1.9) *has a positive solution* (ϕ_1,ϕ_2,ϕ_3) *such that* $\underline{\phi}_k(z) \le \phi_k(z) \le \overline{\phi}_k(z)$ *for all* $z \in \mathbb{R}$ *for* $k = 1,2,3$.

Based on this lemma, it remains to construct a suitable pair of upper-lower-solutions for all admissible wave speeds. For the construction of upper-lower-solutions, we divide our discussion into two cases: $s > s^*$ and $s = s^*$.

Case 1. $s > s^*$**.** Given $s > s^* = 2\sqrt{d_3 r_3 \beta}$. Recall $\beta = au_p + av_p - 1$ and $B = 2a - 1$. Note that $B > 0$, since $a > \gamma > 1/2$.

Let $H(\lambda) := d_3\lambda^2 - s\lambda + r_3\beta$ and $0 < \lambda_3 < \lambda_4$ be two roots of H. We introduce the following upper-lower-solutions

$$\overline{\phi}_1(z) = \begin{cases} u_p + kv_p e^{\lambda_3 z}, & z < 0, \\ 1, & z > 0, \end{cases} \tag{2.7}$$

$$\underline{\phi}_1(z) = \begin{cases} u_p - (u_p - \delta_1) e^{\lambda_3 z}, & z < 0, \\ \delta_1, & z > 0, \end{cases} \tag{2.8}$$

$$\overline{\phi}_2(z) = \begin{cases} v_p + hu_p e^{\lambda_3 z}, & z < 0, \\ 1, & z > 0, \end{cases} \tag{2.9}$$

$$\underline{\phi}_2(z) = \begin{cases} v_p - (v_p - \delta_2) e^{\lambda_3 z}, & z < 0, \\ \delta_2, & z > 0, \end{cases} \tag{2.10}$$

$$\overline{\phi}_3(z) = \begin{cases} Be^{\lambda_3 z}, & z < 0, \\ B, & z > 0, \end{cases} \tag{2.11}$$

$$\underline{\phi}_3(z) = \begin{cases} Be^{\lambda_3 z} - qe^{\mu\lambda_3 z}, & z < z_3, \\ \delta_3, & z > z_3, \end{cases} \tag{2.12}$$

where constants μ, q and δ_i, $i = 1, 2, 3$, are chosen in the following order:

$$1 < \mu < \min\{2, \lambda_4/\lambda_3\}, \tag{2.13}$$

$$q \geq \max\left\{B, \frac{r_3 B(\beta + B)}{-H(\mu\lambda_3)}\right\}, \tag{2.14}$$

$$\delta_1 := (1 - k - bB)/2,\ \delta_2 := (1 - h - bB)/2,\ \delta_3 := \min\{a(\delta_1 + \delta_2) - 1, M/2\}, \tag{2.15}$$

in which $M := B(B/q\mu)^{1/(\mu-1)}(1 - 1/\mu) = f(z_M) = \max_{z\in\mathbb{R}} f(z)$, $f(z) := Be^{\lambda_3 z} - qe^{\mu\lambda_3 z}$. The number $z_3 \in (z_M, z_0)$, where $f(z_0) = 0$ with $z_0 := \frac{\ln(B/q)}{\lambda_3(\mu-1)} \leq 0$, is defined by

$$Be^{\lambda_3 z_3} - qe^{\mu\lambda_3 z_3} = \delta_3.$$

Note that the conditions (1.11) and (1.12) are assumed to ensure $\delta_i > 0$ for $i = 1, 2, 3$. Also, by the choice of μ in (2.13), $H(\mu\lambda_3) < 0$. Furthermore, the condition $q \geq B$ is assumed to ensure that $z_3 < 0$; while $\delta_3 \leq M/2$ is for the continuity of $\underline{\phi}_3$ and that condition (2) in Lemma 2.2 holds. Then we have

Lemma 2.3. *Suppose that* $s > s^*$. *Then the functions* $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ *and* $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ *defined in* (2.7)-(2.12) *are a pair of upper and lower solutions of* (1.9).

Case 2. $s = s^*$**.** Note that $s = 2d_3\lambda_3$ when $s = s^* = 2\sqrt{d_3 r_3 \beta}$. We follow an idea from [3] and consider the following upper-lower-solutions

$$\overline{\phi}_1(z) = \begin{cases} u_p + L_1(-z)e^{\lambda_3 z}, & z < -1/\lambda_3, \\ 1, & z > -1/\lambda_3, \end{cases} \tag{2.16}$$

$$\underline{\phi}_1(z) = \begin{cases} u_p - (u_p - \delta_1)\lambda_3 e(-z)e^{\lambda_3 z}, & z < -1/\lambda_3, \\ \delta_1, & z > -1/\lambda_3, \end{cases} \tag{2.17}$$

$$\overline{\phi}_2(z) = \begin{cases} v_p + L_2(-z)e^{\lambda_3 z}, & z < -1/\lambda_3, \\ 1, & z > -1/\lambda_3, \end{cases} \tag{2.18}$$

$$\underline{\phi}_2(z) = \begin{cases} v_p - (v_p - \delta_2)\lambda_3 e(-z)e^{\lambda_3 z}, & z < -1/\lambda_3, \\ \delta_2, & z > -1/\lambda_3, \end{cases} \tag{2.19}$$

$$\overline{\phi}_3(z) = \begin{cases} L_3(-z)e^{\lambda_3 z}, & z < -1/\lambda_3, \\ B, & z > -1/\lambda_3, \end{cases} \tag{2.20}$$

$$\underline{\phi}_3(z) = \begin{cases} \left[L_3(-z) - q(-z)^{1/2}\right] e^{\lambda_3 z}, & z < z_3^*, \\ \delta_3, & z > z_3^*, \end{cases} \tag{2.21}$$

where

$$L_1 = k v_p \lambda_3 e,\ L_2 = h u_p \lambda_3 e,\ L_3 = B\lambda_3 e,$$

the parameter q is chosen to satisfy

$$q \geq \max\left\{ Be\lambda_3^{1/2}, \frac{4C}{d_3} r_3 B(\beta + B) \right\},\ C := (\lambda_3 e)^2 \left(\frac{7}{2\lambda_3 e} \right)^{7/2}, \tag{2.22}$$

and the constants δ_i, $i = 1, 2, 3$, are defined by (2.15) in which $M := g(z_M^*)$ is the maximum of $g(z) := \left[L_3(-z) - q(-z)^{1/2}\right] e^{\lambda_3 z}$ for $z < 0$. The number $z_3^* \in (z_M^*, z_0^*)$, where $z_0^* := -\left(q/L_3\right)^2 \leq -1/\lambda_3$, is defined by

$$\left[L_3(-z_3^*) - q(-z_3^*)^{1/2}\right] e^{\lambda_3 z_3^*} = \delta_3.$$

Then we obtain

Lemma 2.4. *Suppose that* $s = s^*$. *Then the functions* $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ *and* $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ *defined by* (2.16)-(2.21) *are a pair of upper-lower-solutions of* (1.9).

With the constructed upper-lower-solutions, we deduce the following existence theorem by applying Lemma 2.2. Since the proof is just to check the conditions (1) and (2) in Lemma 2.2, we safely omit it.

Theorem 2.5. *For all* $s \geq s^*$, *there exists a positive solution* (ϕ_1, ϕ_2, ϕ_3) *of* (1.9) *such that*

$$\lim_{z \to -\infty} (\phi_1, \phi_2, \phi_3) = (u_p, v_p, 0) \text{ and } \underline{\phi}_j(z) \leq \phi_j(z) \leq \overline{\phi}_j(z),\ z \in \mathbb{R},\ j = 1, 2, 3.$$

3. Proof of Theorem 1.1

This section is devoted to the proof of our main theorem, Theorem 1.1. For the existence part of Theorem 1.1, it remains to show the convergence of the wave profile to the co-existence state E_c as $z \to \infty$.

Theorem 3.1. *Let (ϕ_1, ϕ_2, ϕ_3) be a solution obtained in Theorem 2.5 for a given $s \geq s^*$. Then $(\phi_1, \phi_2, \phi_3)(z) \to (u_c, v_c, w_c)$ as $z \to \infty$.*

Proof. First, we consider the rectangle $Q := I_1 \times I_2 \times I_3$ with $I_k(\theta) := [m_k(\theta), M_k(\theta)]$, $\theta \in [0, 1]$, where

$$\begin{cases} m_1(\theta) := (1-\theta)(\delta_1 - \varepsilon^2) + \theta u_c, & M_1(\theta) := (1-\theta)(1+\varepsilon^2) + \theta u_c, \\ m_2(\theta) := (1-\theta)(\delta_2 - \varepsilon^2) + \theta v_c, & M_2(\theta) := (1-\theta)(1+\varepsilon^2) + \theta v_c, \\ m_3(\theta) := (1-\theta)(\delta_3 - \varepsilon) + \theta w_c, & M_3(\theta) := (1-\theta)(B+\varepsilon) + \theta w_c \end{cases}$$

for some positive constant ε satisfying

$$0 < \varepsilon < \min\left\{\frac{1}{2a}, \sqrt{\delta_1}, \sqrt{\delta_2}, \delta_3, \frac{\delta_1}{b}, \frac{\delta_2}{b}\right\}. \tag{3.1}$$

From (1.5)-(1.7), it is easy to check that $\delta_1 < u_c < 1$, $\delta_2 < v_c < 1$ and $\delta_3 < w_c < B$, by using condition (1.12). Hence $m_i(\theta)$ is strictly increasing and $M_i(\theta)$ is strictly decreasing in θ such that $(m_1, m_2, m_3)(1) = (M_1, M_2, M_3)(1) = (u_c, v_c, w_c)$.

For convenience, we let

$$\phi_j^+ := \limsup_{z\to\infty} \phi_j(z),\ \phi_j^- := \liminf_{z\to\infty} \phi_j(z),\ j = 1, 2, 3.$$

Obviously, we have

$$m_1(0) = \delta_1 - \varepsilon^2 < \delta_1 \leq \phi_1^- \leq \phi_1^+ \leq 1 < 1 + \varepsilon^2 = M_1(0);$$
$$m_2(0) = \delta_2 - \varepsilon^2 < \delta_2 \leq \phi_2^- \leq \phi_2^+ \leq 1 < 1 + \varepsilon^2 = M_2(0);$$
$$m_1(0) = \delta_3 - \varepsilon < \delta_3 \leq \phi_3^- \leq \phi_3^+ \leq B < B + \varepsilon = M_3(0).$$

Hence the quantity

$$\theta_0 := \sup\left\{\theta \in [0, 1) \mid m_k(\theta) < \phi_k^- \leq \phi_k^+ < M_k(\theta),\ k = 1, 2, 3\right\}$$

is well-defined and $\theta_0 > 0$. Then the theorem follows if we can prove that $\theta_0 = 1$.

Upon choosing the sequence of rectangles, the proof of $\theta_0 = 1$ can be carried out by a method used in [3, 5] as follows. For contradiction, we assume that $\theta_0 \in (0, 1)$.

First, we check that

$$\begin{aligned} &\alpha_1 := 1 - m_1(\theta_0) - kM_2(\theta_0) - bM_3(\theta_0) \\ =\ & 1 - (1-\theta_0)(\delta_1 - \varepsilon^2) - \theta_0 u_c - k(1-\theta_0)(1+\varepsilon^2) - k\theta_0 v_c - b(1-\theta_0)(B+\varepsilon) - b\theta_0 w_c \\ =\ & (1-\theta_0)[1 - k - bB - \delta_1 + \varepsilon^2(1-k) - b\varepsilon] \geq (1-\theta_0)(\delta_1 - b\varepsilon) > 0, \end{aligned}$$

by (2.15) and (3.1). Similarly, we have

$$\begin{aligned}
\omega_1 &:= 1 - M_1(\theta_0) - km_2(\theta_0) - bm_3(\theta_0) = -(1-\theta_0)[\varepsilon^2 + k(\delta_2 - \varepsilon^2) + b(\delta_3 - \varepsilon)] < 0,\\
\alpha_2 &:= 1 - hM_1(\theta_0) - m_2(\theta_0) - bM_3(\theta_0) \geq (1-\theta_0)(\delta_2 - b\varepsilon) > 0,\\
\omega_2 &:= 1 - hm_1(\theta_0) - M_2(\theta_0) - bm_3(\theta_0) = -(1-\theta_0)[\varepsilon^2 + h(\delta_1 - \varepsilon^2) + b(\delta_3 - \varepsilon)] < 0,\\
\alpha_3 &:= -1 + am_1(\theta_0) + am_2(\theta_0) - m_3(\theta_0) = (1-\theta_0)[\varepsilon(1-2a\varepsilon)] > 0,\\
\omega_3 &:= -1 + aM_1(\theta_0) + aM_2(\theta_0) - M_3(\theta_0) = -(1-\theta_0)[\varepsilon(1-2a\varepsilon)] < 0,
\end{aligned}$$

by using (2.15) and (3.1).

Next, by passing to the limit, $m_k(\theta_0) \leq \phi_k^- \leq \phi_k^+ \leq M_k(\theta_0)$, $k = 1,2,3$, and at least one of the following equalities

$$\phi_k^- = m_k(\theta_0),\ \phi_k^+ = M_k(\theta_0),\ k = 1,2,3,$$

must hold. Then, with the help of the positivity of $\{\alpha_i, -\omega_i,\ i = 1,2,3\}$, the method of [3, 5] can be applied to get a contradiction.

For the reader's convenience, we give the idea as follows. Suppose, for example, that $\phi_3^+ = M_3(\theta_0)$. Then there are the following two possibilities. Suppose that $\phi_3(z)$ is monotone for z in a neighborhood of ∞. Then $\phi_3(\infty) = M_3(\theta_0)$. By integrating the third equation in (1.9) from 0 to n for any $n \in \mathbb{N}$, we obtain

$$-d_3\phi_3'(n) + d_3\phi_3'(0) + s\phi_3(n) - s\phi_3(0) = r_3 \int_0^n \phi_3(z)(-1 + a\phi_1 + a\phi_2 - \phi_3)(z)dz. \tag{3.2}$$

Since

$$\limsup_{z\to\infty}\{\phi_3(z)(-1 + a\phi_1 + a\phi_2 - \phi_3)(z)\} \leq M_3(\theta_0)\omega_3 < 0,$$

the right-hand side of (3.2) tends to $-\infty$ as $n \to \infty$. But, the left-hand side of (3.2) is bounded uniformly for all $n \in \mathbb{N}$, a contradiction.

Now, suppose that $\phi_3(z)$ is oscillatory near $z = \infty$. Then there is a sequence of local maximal points $\{z_n\}$ of ϕ_3 such that $z_n \to \infty$ and $\phi_3(z_n) \to M_3(\theta_0)$ as $n \to \infty$. Since $d_3\phi_3''(z_n) - s\phi_3'(z_n) \leq 0$ for all n and

$$\limsup_{n\to\infty}\{\phi_3(z_n)(-1 + a\phi_1 + a\phi_2 - \phi_3)(z_n)\} \leq M_3(\theta_0)\omega_3 < 0,$$

we reach a contradiction. Hence the case $\phi_3^+ = M_3(\theta_0)$ is impossible. The other cases are similar. This proves the theorem. □

Finally, the proof of the non-existence part of Theorem 1.1 is almost the same as that of [5, Proposition 5.1], we only give an outline of the proof here.

Theorem 3.1. *For $s < s^*$, there is no positive solution (ϕ_1, ϕ_2, ϕ_3) of* (1.9)-(1.10).

Proof. First, we claim that $s > 0$ by a contradiction argument. Suppose that there exists a positive solution (ϕ_1, ϕ_2, ϕ_3) of (1.9)-(1.10) for some $s \le 0$. Then, due to $(\phi_1, \phi_2, \phi_3)(-\infty) = (u_p, v_p, 0)$, there is a large $N > 0$ such that

$$-1 + a\phi_1(y) + a\phi_2(y) - \phi_3(y) \ge \frac{\beta}{2},\ \forall\, y \le -N.$$

Integrating the third equation in (1.9) in y from $-\infty$ to $z \le -N$ and then in z from $-\infty$ to $-N$, we obtain

$$0 < \frac{r_3\beta}{2}\int_{-\infty}^{-N}\int_{-\infty}^{z}\phi_3(y)dydz \le -\int_{-\infty}^{-N} d_3\phi_3'(z)dz = -d_3\phi_3(-N) < 0,$$

a contradiction. Hence $s > 0$.

Now suppose that there is a positive solution (ϕ_1, ϕ_2, ϕ_3) of (1.9)-(1.10) for some $s \in (0, s^*)$. Since $0 < s < s^*$, there is a small positive ε such that $0 < s < 2\sqrt{d_3 r_3(\beta - 2a\varepsilon)}$. For this ε, using (1.10) and positivity of (ϕ_1, ϕ_2, ϕ_3), there are constants c_1 and c_2 such that

$$\phi_1(z) + c_1\phi_3(z) > u_p - \varepsilon,\ \forall z \in \mathbb{R}, \tag{3.3}$$

$$\phi_2(z) + c_2\phi_3(z) > v_p - \varepsilon,\ \forall z \in \mathbb{R}. \tag{3.4}$$

Note that $(u, v, w)(x, t) = (\phi_1, \phi_2, \phi_3)(x + st)$ satisfies

$$w_t = d_3 w_{xx} + r_3 w(-1 + au + av - w),\ x \in \mathbb{R},\ t > 0. \tag{3.5}$$

Plugging (3.3) and (3.4) into (3.5), we obtain

$$w_t \ge d_3 w_{xx} + r_3 w[(\beta - 2a\varepsilon) - (ac_1 + ac_2 + 1)w],\ x \in \mathbb{R},\ t > 0.$$

Then the spreading theory of [1] gives

$$\liminf_{t\to\infty} w(y(t), t) \ge \frac{\beta - 2a\varepsilon}{ac_1 + ac_2 + 1} > 0,$$

where $y(t) := -(s + 2\sqrt{d_3 r_3(\beta - 2a\varepsilon)})t/2$, since $|y(t)| < 2\sqrt{d_3 r_3(\beta - 2a\varepsilon)}t$ for all $t > 0$. But, $y(t) + st \to -\infty$ as $t \to \infty$. This implies that $w(y(t), t) = \phi_3(y(t) + st) \to 0$ as $t \to \infty$. Thus we finish our proof by contradiction. □

4. Verification of upper-lower-solutions

Proof of Lemma 2.3. For $z > 0$, $\overline{\phi}_1(z) = 1$, $\underline{\phi}_2(z) = \delta_2$ and $\underline{\phi}_3(z) = \delta_3$. Then

$$\mathcal{U}_1(z) = -r_1(k\delta_2 + b\delta_3) \le 0,\ z > 0.$$

For $z < 0$, $\overline{\phi}_1(z) = u_p + kv_p e^{\lambda_3 z}$ and $\underline{\phi}_2(z) = v_p - (v_p - \delta_2)e^{\lambda_3 z}$. Then

$$\mathcal{U}_1(z) = (d_1\lambda_3^2 - s\lambda_3)kv_p e^{\lambda_3 z} + r_1(u_p + kv_p e^{\lambda_3 z})(-\delta_2 k e^{\lambda_3 z} - b\underline{\phi}_3).$$

Note that $d_1\lambda_3^2 - s\lambda_3 \le 0$ if $\lambda_3 \in [0, s/d_1]$. By (1.13), $d_3 \ge d_1/2$,

$$\lambda_3 = \frac{s - \sqrt{s^2 - 4d_3r_3\beta}}{2d_3} \le \frac{s - \sqrt{s^2 - 4d_3r_3\beta}}{d_1} < \frac{s}{d_1}.$$

It follows that

$$\mathcal{U}_1(z) \;=\; (d_1\lambda_3^2 - s\lambda_3)kv_pe^{\lambda_3 z} + r_1(u_p + kv_pe^{\lambda_3 z})(-\delta_2 ke^{\lambda_3 z} - b\underline{\phi}_3) \le 0$$

for $z < 0$. Hence $\mathcal{U}_1(z) \le 0$ for all $z \ne 0$.

Similarly, $\mathcal{U}_2(z) \le 0$ for all $z \ne 0$, by (1.13).

Next, we show that $\mathcal{U}_3(z) \le 0$ for $z \ne 0$. For $z > 0$, $\overline{\phi}_1(z) = \overline{\phi}_2(z) = 1$ and $\overline{\phi}_3(z) = B$. It follows that

$$\mathcal{U}_3(z) = r_3B(-1 + 2a - B) = 0,\ z > 0,$$

since $B = 2a - 1$. For $z < 0$, $\overline{\phi}_1(z) = u_p + kv_pe^{\lambda_3 z}$, $\overline{\phi}_2(z) = v_p + hu_pe^{\lambda_3 z}$ and $\overline{\phi}_3(z) = Be^{\lambda_3 z}$. Then

$$\begin{aligned}
\mathcal{U}_3(z) &= B(d_3\lambda_3^2 - s\lambda_3)e^{\lambda_3 z} + r_3Be^{\lambda_3 z}\left[\beta + a(kv_p + hu_p)e^{\lambda_3 z} - Be^{\lambda_3 z}\right] \\
&= BH(\lambda_3)e^{\lambda_3 z} + r_3Be^{2\lambda_3 z}\left[a(kv_p + hu_p - 2) + 1\right] \\
&= -r_3Be^{2\lambda_3 z}(a/\gamma - 1) \le 0,
\end{aligned}$$

by the fact that $H(\lambda_3) = 0$ and (1.3). Hence $\mathcal{U}_3(z) \le 0$ for all $z \ne 0$.

Thirdly, we show that $\mathcal{L}_1(z) \ge 0$ for $z \ne 0$. For $z > 0$, $\underline{\phi}_1(z) = \delta_1$, $\overline{\phi}_2(z) = 1$ and $\overline{\phi}_3(z) = B$. It follows that

$$\mathcal{L}_1(z) = r_1\delta_1(1 - \delta_1 - k - bB) \ge 0,\ z > 0.$$

For $z < 0$, $\underline{\phi}_1(z) = u_p - (u_p - \delta_1)e^{\lambda_3 z}$, $\overline{\phi}_2(z) = v_p + hu_pe^{\lambda_3 z}$ and $\overline{\phi}_3(z) = Be^{\lambda_3 z}$. Then, setting

$$p_1 := 1 - \delta_1/u_p,$$

we compute

$$\begin{aligned}
\mathcal{L}_1(z) &= -p_1u_p(d_1\lambda_3^2 - s\lambda_3)e^{\lambda_3 z} + r_1u_p(1 - p_1e^{\lambda_3 z})(p_1u_pe^{\lambda_3 z} - khu_pe^{\lambda_3 z} - bBe^{\lambda_3 z}) \\
&\ge r_1u_p(1 - p_1e^{\lambda_3 z})(p_1u_pe^{\lambda_3 z} - khu_pe^{\lambda_3 z} - bBe^{\lambda_3 z}) \\
&= r_1u_p(1 - p_1e^{\lambda_3 z})(1 - k - \delta_1 - bB)e^{\lambda_3 z} \ge 0,\ z < 0,
\end{aligned}$$

using $d_1\lambda_3^2 - s\lambda_3 \le 0$. Hence $\mathcal{L}_1(z) \ge 0$ for all $z \ne 0$.

Similarly, $\mathcal{L}_2(z) \ge 0$ for all $z \ne 0$.

Finally, we show that $\mathcal{L}_3(z) \ge 0$ for $z \ne z_3$. For $z > z_3$, $\underline{\phi}_3(z) = \delta_3$, $\underline{\phi}_1 \ge \delta_1$ and $\underline{\phi}_2 \ge \delta_2$. Hence, by (2.15),

$$\mathcal{L}_3(z) \ge r_3\delta_3(-1 + a\delta_1 + a\delta_2 - \delta_3) \ge 0,\ \forall\, z > z_3.$$

For $z < z_3 < 0$,

$$\underline{\phi}_3(z) = Be^{\lambda_3 z} - qe^{\mu\lambda_3 z},\ \underline{\phi}_1(z) = u_p - (u_p - \delta_1)e^{\lambda_3 z},\ \underline{\phi}_2(z) = v_p - (v_p - \delta_2)e^{\lambda_3 z}.$$

Then, setting

$$p_1 := 1 - \delta_1/u_p, \quad p_2 := 1 - \delta_2/v_p, \tag{4.1}$$

we compute

$$\begin{aligned} \mathcal{L}_3(z) &= B(d_3\lambda_3^2 - s\lambda_3)e^{\lambda_3 z} - q[d_3(\mu\lambda_3)^2 - s(\mu\lambda_3)]e^{\mu\lambda_3 z} \\ &\quad + r_3(Be^{\lambda_3 z} - qe^{\mu\lambda_3 z})\left[\beta - ap_1u_pe^{\lambda_3 z} - ap_2v_pe^{\lambda_3 z} - Be^{\lambda_3 z} + qe^{\mu\lambda_3 z}\right]. \end{aligned}$$

Due to the fact $H(\lambda_3) = 0$, we have

$$\begin{aligned} \mathcal{L}_3(z) &= -qH(\mu\lambda_3)e^{\mu\lambda_3 z} + r_3(Be^{\lambda_3 z} - qe^{\mu\lambda_3 z})\left(-ap_1u_pe^{\lambda_3 z} - ap_2v_pe^{\lambda_3 z} - Be^{\lambda_3 z} + qe^{\mu\lambda_3 z}\right) \\ &\geq -qH(\mu\lambda_3)e^{\mu\lambda_3 z} + r_3Be^{2\lambda_3 z}(-ap_1u_p - ap_2v_p - B) \\ &= e^{\mu\lambda_3 z}[-qH(\mu\lambda_3) - r_3Be^{(2-\mu)\lambda_3 z}(ap_1u_p + ap_2v_p + B)] \\ &\geq e^{\mu\lambda_3 z}[-qH(\mu\lambda_3) - r_3B(\beta + B - \delta_3)] \geq 0, \end{aligned}$$

for $z < z_3$, by the choices of μ in (2.13) and q in (2.14). This completes the proof of this lemma. □

Proof of Lemma 2.4. Note that $2d_3\lambda_3 - s = 0 \leq 2d_k\lambda_3 - s$, $k = 1, 2$, when $s = s^*$.

First, we show $\mathcal{U}_1(z) \leq 0$ for all $z \neq -1/\lambda_3$. For $z > -1/\lambda_3$, $\overline{\phi}_1(z) = 1$, $\underline{\phi}_2(z) = \delta_2$ and $\underline{\phi}_3(z) = \delta_3$. Then

$$\mathcal{U}_1(z) = -r_1(k\delta_2 + b\delta_3) \leq 0$$

For $z < -1/\lambda_3$, $\overline{\phi}_1(z) = u_p + L_1(-z)e^{\lambda_3 z}$, $\underline{\phi}_2(z) = v_p - (v_p - \delta_2)\lambda_3 e(-z)e^{\lambda_3 z}$, $\underline{\phi}_3(z) \geq 0$, and so

$$\begin{aligned} \mathcal{U}_1(z) &\leq L_1(-2d_1\lambda_3 + s)e^{\lambda_3 z} + L_1(d_1\lambda_3^2 - s\lambda_3)(-z)e^{\lambda_3 z} \\ &\quad + r_1\overline{\phi}_1\{-L_1(-z)e^{\lambda_3 z} + k(v_p - \delta_2)\lambda_3 e(-z)e^{\lambda_3 z}\} \\ &= L_1(-2d_1\lambda_3 + s)e^{\lambda_3 z} + L_1(d_1\lambda_3^2 - s\lambda_3)(-z)e^{\lambda_3 z} - r_1\overline{\phi}_1\lambda_3 e(-z)k\delta_2e^{\lambda_3 z} \\ &\leq 2\lambda_3L_1(d_3 - d_1)e^{\lambda_3 z} + \lambda_3^2L_1(d_1 - 2d_3)(-z)e^{\lambda_3 z} \leq 0, \end{aligned}$$

by (1.14). Hence $\mathcal{U}_1(z) \leq 0$ for all $z \neq -1/\lambda_3$. Similarly, we also have $\mathcal{U}_2(z) \leq 0$ for all $z \neq -1/\lambda_3$.

Secondly, we show that $\mathcal{U}_3(z) \leq 0$ for all $z \neq -1/\lambda_3$. For $z > -1/\lambda_3$,

$$\mathcal{U}_3(z) = r_3B(-1 + 2a - B) = 0.$$

For $z < -1/\lambda_3$, we have

$$\overline{\phi}_1(z) = u_p + L_1(-z)e^{\lambda_3 z},\ \overline{\phi}_2(z) = v_p + L_2(-z)e^{\lambda_3 z},\ \overline{\phi}_3(z) = L_3(-z)e^{\lambda_3 z}.$$

Hence, using $au_p + av_p - 1 = \beta$, $s = 2d_3\lambda_3$ and $H(\lambda_3) = 0$,

$$\begin{aligned}\mathcal{U}_3(z) &= L_3(d_3\lambda_3^2 - s\lambda_3 + r_3\beta)(-z)e^{\lambda_3 z} + L_3(-2d_3\lambda_3 + s)e^{\lambda_3 z} \\ &\quad + r_3 L_3(-z)^2 e^{2\lambda_3 z}(aL_1 + aL_2 - L_3) \\ &= r_3\lambda_3 e L_3(-z)^2 e^{2\lambda_3 z}(akv_p + ahu_p - 2a + 1) \\ &= r_3\lambda_3 e L_3(-z)^2 e^{2\lambda_3 z}(1 - a/\gamma) \le 0,\end{aligned}$$

by (1.3). Hence $\mathcal{U}_3(z) \le 0$ for all $z \ne -1/\lambda_3$.

Thirdly, we show that $\mathcal{L}_1(z) \ge 0$ for all $z \ne -1/\lambda_3$. For $z > -1/\lambda_3$, $\underline{\phi}_1(z) = \delta_1$, $\overline{\phi}_2(z) = 1$ and $\overline{\phi}_3(z) = 1$, it follows that

$$\mathcal{L}_1(z) = r_1\delta_1(1 - \delta_1 - k - bB) \ge 0.$$

For $z < -1/\lambda_3$, we have, recalling $p_1 = 1 - \delta_1/u_p \in (0,1)$,

$$\begin{aligned}\mathcal{L}_1(z) &= -p_1u_p\lambda_3 e(-2\lambda_3 d_1 + s)e^{\lambda_3 z} - p_1u_p\lambda_3 e(d_1\lambda_3^2 - s\lambda_3)(-z)e^{\lambda_3 z} \\ &\quad + r_1\underline{\phi}_1(z)\{(1 - u_p - kv_p) + (p_1u_p\lambda_3 e - kL_2 - bL_3)(-z)e^{\lambda_3 z}\} \\ &= -p_1u_p\lambda_3 e(-2\lambda_3 d_1 + s)e^{\lambda_3 z} - p_1u_p\lambda_3 e(d_1\lambda_3^2 - s\lambda_3)(-z)e^{\lambda_3 z} \\ &\quad + r_1\underline{\phi}_1(z)\{[(1 - kh)u_p - bB - \delta_1]\lambda_3 e(-z)e^{\lambda_3 z}\} \\ &\ge -p_1u_p\lambda_3 e(-2\lambda_3 d_1 + s)e^{\lambda_3 z} - p_1u_p\lambda_3 e(d_1\lambda_3^2 - s\lambda_3)(-z)e^{\lambda_3 z} \\ &= 2p_1u_p\lambda_3^2 e\,(d_1 - d_3)\,e^{\lambda_3 z} + p_1u_p\lambda_3^3 e\,(2d_3 - d_1)\,(-z)e^{\lambda_3 z} \ge 0,\end{aligned}$$

by using $1 - u_p - kv_p = 0$, (2.15) and (1.14).

Similarly, we also have $\mathcal{L}_2(z) \ge 0$ for all $z \ne -1/\lambda_3$.

Finally, we show that $\mathcal{L}_3(z) \ge 0$ for all $z \ne z_3^*$. By the choice of q, $z_3^* \le -1/\lambda_3$. For $z > z_3^*$, $\underline{\phi}_3(z) = \delta_3$, $\underline{\phi}_1 \ge \delta_1$ and $\underline{\phi}_2 \ge \delta_2$. Hence, as before,

$$\mathcal{L}_3(z) \ge r_3\delta_3(-1 + a\delta_1 + a\delta_2 - \delta_3) \ge 0,\ \forall\, z > z_3^*.$$

For $z < z_3^*$, by using the facts that $d_3\lambda_3^2 - s\lambda_3 + r_3\beta = 2d_3\lambda_3 - s = 0$,

$$\begin{aligned}\mathcal{L}_3(z) &= \frac{d_3}{4}q(-z)^{-3/2}e^{\lambda_3 z} \\ &\quad + r_3[L_3(-z) - q(-z)^{1/2}]e^{\lambda_3 z}\left[-(\beta - \delta_3)\lambda_3 e(-z)e^{\lambda_3 z} - L_3(-z)e^{\lambda_3 z} + q(-z)^{1/2}e^{\lambda_3 z}\right] \\ &\ge \frac{d_3}{4}q(-z)^{-3/2}e^{\lambda_3 z} - r_3L_3(-z)^2e^{2\lambda_3 z}\lambda_3 e\,[\beta - \delta_3 + B] \\ &= \frac{d_3}{4}(-z)^{-3/2}e^{\lambda_3 z}\left[q - 4r_3B(\lambda_3 e)^2(-z)^{7/2}e^{\lambda_3 z}(\beta - \delta_3 + B)/d_3\right] \\ &\ge \frac{d_3}{4}(-z)^{-3/2}e^{\lambda_3 z}\left[q - 4r_3B(\lambda_3 e)^2\left(\frac{7}{2e\lambda_3}\right)^{7/2}(\beta - \delta_3 + B)/d_3\right] \ge 0,\end{aligned}$$

by the choice of q in (2.22) and the fact that

$$(-z)^{7/2}e^{\lambda_3 z} \le \left(\frac{7}{2e\lambda_3}\right)^{7/2} \quad \text{for all} \quad z \le 0.$$

Hence we obtain $\mathcal{L}_3(z) \ge 0$ for all $z < z_3^*$. The proof of the lemma is thus completed. □

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