TRAVELING WAVES FOR A THREE-SPECIES COMPETITION SYSTEM WITH TWO WEAK ABORIGINAL COMPETITORS

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Abstract. In this paper, we study the traveling wave solutions for a three species competition system with two weak aboriginal competitors and one strong alien competitor. We are concerned with the existence of traveling waves such that these two co-existence aboriginal competitors are wiped out by the invading alien strong competitor. First, we derive the existence of wave profiles based on an application of Schauder's fixed point theorem with the help of constructing suitable generalized upper-lower solutions to capture the unstable wave tail limit. Then a new method for deriving the stable wave tail limit is introduced. Finally, the minimal invading speed is characterized.

1. Introduction

In this paper, we study the following diffusive Lotka-Volterra competition system

$$\begin{cases} u_t = du_{xx} + r_1 u(1 - u - a_2 v - a_3 w),\ x \in \mathbb{R},\ t > 0, \\ v_t = dv_{xx} + r_2 v(1 - b_1 u - v - b_3 w),\ x \in \mathbb{R},\ t > 0, \\ w_t = dw_{xx} + r_3 w(1 - c_1 u - c_2 v - w),\ x \in \mathbb{R},\ t > 0, \end{cases} \tag{1.1}$$

where u, v, w are three competitors and all parameters $d, r_1, r_2, r_3, a_2, a_3, b_1, b_3, c_1, c_2$ are positive constants in which d stands for the diffusion coefficient(s) for all species, r_i the intrinsic growth rate, a_i, b_j, c_k are inter-specific competition coefficients and the carrying capacity of each species is normalized to be 1.

We are concerned with the existence of traveling wave solutions, namely, a solution (u, v, w) of (1.1) in the form

$$(u, v, w)(x, t) = (\phi_1, \phi_2, \phi_3)(z),\ z := x - st,$$

for some constant $s \in \mathbb{R}$ (the wave speed) and some functions $\{\phi_i \mid i = 1, 2, 3\}$ (the wave profiles). In particular, we are interested in the traveling wave solution connecting two constant equilibria of (1.1). Hence we are looking for unknown $\{s, \phi_1, \phi_2, \phi_3\}$ that satisfies

$$\begin{cases} d\phi_1'' + s\phi_1' + r_1\phi_1(1 - \phi_1 - a_2\phi_2 - a_3\phi_3) = 0,\ z \in \mathbb{R}, \\ d\phi_2'' + s\phi_2' + r_2\phi_2(1 - b_1\phi_1 - \phi_2 - b_3\phi_3) = 0,\ z \in \mathbb{R}, \\ d\phi_3'' + s\phi_3' + r_3\phi_3(1 - c_1\phi_1 - c_2\phi_2 - \phi_3) = 0,\ z \in \mathbb{R}. \end{cases} \tag{1.2}$$

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The study of traveling waves of biological systems has attracted a lot of attention in past years. Certain difficulties arise for systems without comparison principle. In this case, the classical monotone iteration method is not applicable. To analyze these non-cooperative systems, an application of Schauder's fixed point theorem with the help of generalized upper-lower solutions has been very successful in deriving traveling waves. For two or three species predator-prey systems, we refer the reader to works [11, 8, 14, 13, 6, 7, 4, 5] and references cited therein. For 2-species competition system with delay, we refer the reader to, e.g., [9, 12]. We also refer the reader to the n-species competition system with delay [10].

In the works [9, 12, 10] on competition systems, the constructed waves connect the zero state to the positive co-existence state. Only recently have there been interesting studies about 3-species competition systems which exhibit traveling waves that connect two non-zero states (cf. [2, 3]). These studies employ some very sophisticated methods which are very different from the above-mentioned method of (generalized) upper-lower solutions. The main purpose of this work is to construct another class of traveling waves for 3-species competition systems by using this very fundamental method.

We consider the situation that a strong alien competitor u is introduced to the habitat of two aboriginal weak competing species v and w. Therefore, we assume that

$$b_1, c_1 > 1, \quad a_2, a_3, b_3, c_2 < 1. \tag{1.3}$$

In (1.3), in the absence of u, that $b_3 < 1$ and $c_2 < 1$ is a necessary and sufficient condition for the weak competition between two species v and w. In this case, there is the semi-co-existence state $E_c := (0, v_c, w_c)$, where

$$v_c := \frac{1-b_3}{1-b_3c_2} \in (0,1),\ w_c := \frac{1-c_2}{1-b_3c_2} \in (0,1).$$

By computing the Jacobian matrix of the vector field

$$(r_1u(1-u-a_2v-a_3w), r_2v(1-b_1u-v-b_3w), r_3w(1-c_1u-c_2v-w)),$$

we can easily check that the equilibria $(0,0,0), (0,0,1), (0,1,0)$ are unstable and $(1,0,0)$ is stable (node) for the (diffusion-free) ODE system of (1.1). In this paper, we assume that the state E_c is unstable for the ODE system of (1.1) which is equivalent to

$$\beta := 1 - a_2v_c - a_3w_c > 0. \tag{1.4}$$

Note that condition (1.4) can be achieved, e.g., when

$$a_2 + a_3 < 1. \tag{1.5}$$

Our question is to see whether there are waves connecting E_c to $(1,0,0)$. Biologically, this means that the strong alien competitor u invades the habitat of two aboriginal weak competitors v, w to wipe out these two aboriginal species. Our main existence theorem of this paper is

Theorem 1.1. *Let* $s_* := 2\sqrt{dr_1\beta}$. *Suppose, in addition to conditions* (1.3) *and* (1.4),

$$r_1\beta \geq \max\{r_2(b_1 + b_3c_2v_c), r_3[c_1 + c_2(1 - v_c)]\}. \tag{1.6}$$

Then there is a positive (for all components) solution (ϕ_1, ϕ_2, ϕ_3) *of* (1.2) *satisfying*

$$(\phi_1, \phi_2, \phi_3)(+\infty) = (0, v_c, w_c) \tag{1.7}$$

for each $s \geq s_*$. *Moreover, if we further assume that*

$$a_2b_1 \geq 1,\ a_3c_1 \geq 1,\ a_2 + a_3 < 1, \tag{1.8}$$

then $(\phi_1, \phi_2, \phi_3)(-\infty) = (1, 0, 0)$.

Biologically, condition (1.8) means that the competitions of u to v and w are strong enough ($b_1 \geq 1/a_2$ and $c_1 \geq 1/a_3$) and the competition of v and w to u is weak ($a_2 + a_3 < 1$). Intuitively, it can be expected that the strong alien competitor u will wipe out the existing two weak competitors v and w if (1.8) is enforced.

On the other hand, we have the following non-existence theorem.

Theorem 1.2. *Under conditions* (1.3) *and* (1.4), *there is no positive solution of* (1.2) *with* $(\phi_1, \phi_2, \phi_3)(+\infty) = (0, v_c, w_c)$ *if* $s < s_*$.

From the dynamical point of view, in the 6-D phase space $(\phi_1, \phi_1', \phi_2, \phi_2', \phi_3, \phi_3')$ of (1.2) for $s > 0$, with eigenfunction in the form $e^{-\lambda z}$ at $z = +\infty$ it is easy to check that there are

(1) 3-d unstable manifold and 3-d stable manifold of $(1, 0, 0, 0, 0, 0)$;
(2) 5-d unstable manifold and 1-d stable manifold of $(0, 0, 1, 0, 0, 0)$;
(3) 5-d unstable manifold and 1-d stable manifold of $(0, 0, 0, 0, 1, 0)$;
(4) 4-d unstable manifold and 2-d stable manifold of $(0, 0, v_c, 0, w_c, 0)$;

under conditions (1.3) and (1.4). Hence it is possible to have heteroclinic orbits that connect one of the states

$$\{(0, 0, 1, 0, 0, 0), (0, 0, 0, 0, 1, 0), (0, 0, v_c, 0, w_c, 0)\}$$

and $(1, 0, 0, 0, 0, 0)$. In this paper, we only consider the case with connection between $(0, 0, v_c, 0, w_c, 0)$ and $(1, 0, 0, 0, 0, 0)$. As for the range of wave speeds, it can be seen from that two roots of

$$d\lambda^2 - s\lambda + r_1\beta = 0$$

are real if and only if $s \geq s_*$.

The construction of upper-lower solutions here is motivated by the works [6, 7, 4, 5] based on choosing the appropriate tail behaviors at $z = +\infty$, the unstable tail. One of the main difficulties of this work is the verification of the wave tail limit at $z = -\infty$, the stable tail. To overcome this difficulty, we introduce in §3 a new method which is originated from the idea of contracting rectangles (cf. [8, 10, 4] and references cited therein). This is actually one of the main contributions of this work. To be more precise, the method of contracting rectangles is used to derive the tail limit of traveling wave, by constructing a sequence of rectangles which shrinks to the desired limit point. The original idea in [8, 10] is for the convergence to the positive (for all components) co-existence state. Then it was extended to the case of semi-co-existence state (with one zero component) in [4] for a 3-species predator-prey system. In [4], instead of 3-d rectangles, a sequence of 2-d contracting rectangles was adopted along with an upper bound estimate for the tail limit of wave profile corresponding to the zero component of the semi-co-existence state. In this paper, our target limit is $(1, 0, 0)$ and we introduce a sequence of intervals with one end fixed and the other end tending to 1. This idea is new in the literature.

The rest of this paper is organized as follows. In §2, we introduce the notion of upper-lower solutions and construct the upper-lower solutions for each $s \geq s_*$. These upper-lower solutions actually capture the wave tail limit at $z = +\infty$, i.e., satisfy (1.7). Although the verification is straightforward (as long as the suitable upper-lower solutions are found), we provide the details for the reader's convenience. Then we prove the wave tail limit at $z = -\infty$ in §3. This completes the proof of Theorem 1.1. Finally, we give a proof of Theorem 1.2 in §4. This gives the characterization of the minimal invading speed of the strong alien competitor. In fact, we can relax the equal diffusivities condition to, e.g., $d_1 \geq \max\{d_2, d_3\}$ for waves with super-critical speeds, where d_1, d_2, d_3 are respectively the diffusion coefficients of species u, v, w. However, the best result we are able to get for the waves with critical speed is under the equal diffusivities assumption. Therefore, we only present the case of equal diffusivities in this work.

2. Existence of wave profiles

We start with the definition of (generalized) upper-lower solutions as follows.

Definition 2.1. *Continuous functions* $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ *and* $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ *are called a pair of (generalized) upper-lower solutions of* (1.2) *if* $\overline{\phi}_i''$, $\underline{\phi}_i''$, $\overline{\phi}_i'$, $\underline{\phi}_i'$, $i = 1, 2, 3$, *are bounded in* $\mathbb{R}$ *and* *the following inequalities*

$$\mathcal{U}_1(z) := d\overline{\phi}_1''(z) + s\overline{\phi}_1'(z) + r_1\overline{\phi}_1(z)[1 - \overline{\phi}_1(z) - a_2\underline{\phi}_2(z) - a_3\underline{\phi}_3(z)] \le 0, \tag{2.1}$$

$$\mathcal{U}_2(z) := d\overline{\phi}_2''(z) + s\overline{\phi}_2'(z) + r_2\overline{\phi}_2(z)[1 - b_1\underline{\phi}_1(z) - \overline{\phi}_2(z) - b_3\underline{\phi}_3(z)] \le 0, \tag{2.2}$$

$$\mathcal{U}_3(z) := d\overline{\phi}_3''(z) + s\overline{\phi}_3'(z) + r_3\overline{\phi}_3(z)[1 - c_1\underline{\phi}_1(z) - c_2\underline{\phi}_2(z) - \overline{\phi}_3(z)] \le 0, \tag{2.3}$$

$$\mathcal{L}_1(z) := d\underline{\phi}_1''(z) + s\underline{\phi}_1'(z) + r_1\underline{\phi}_1(z)[1 - \underline{\phi}_1(z) - a_2\overline{\phi}_2(z) - a_3\overline{\phi}_3(z)] \ge 0, \tag{2.4}$$

$$\mathcal{L}_2(z) := d\underline{\phi}_2''(z) + s\underline{\phi}_2'(z) + r_2\underline{\phi}_2(z)[1 - b_1\overline{\phi}_1(z) - \underline{\phi}_2(z) - b_3\overline{\phi}_3(z)] \ge 0, \tag{2.5}$$

$$\mathcal{L}_3(z) := d\underline{\phi}_3''(z) + s\underline{\phi}_3'(z) + r_3\underline{\phi}_3(z)[1 - c_1\overline{\phi}_1(z) - c_2\overline{\phi}_2(z) - \underline{\phi}_3(z)] \ge 0, \tag{2.6}$$

hold for $z \in \mathbb{R}\backslash E$ with some finite subset E of $\mathbb{R}$.

Then, by a standard argument as that in, e.g., [11, 8]), we have the following existence theorem for system (1.2). We omit its proof here safely.

Proposition 2.2. *Given $s > 0$. Suppose that* (1.2) *has a pair of upper-lower solutions $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ such that*

$$\underline{\phi}_i \le \overline{\phi}_i,\ i = 1, 2, 3, \tag{2.7}$$

$$\lim_{z\to z^+} \overline{\phi}_i'(z) \le \lim_{z\to z^-} \overline{\phi}_i'(z),\ \lim_{z\to z^-} \underline{\phi}_i'(z) \le \lim_{z\to z^+} \underline{\phi}_i'(z),\ \forall\, z \in E,\ i = 1, 2, 3. \tag{2.8}$$

Then (1.2) *has a solution (ϕ_1, ϕ_2, ϕ_3) such that $\underline{\phi}_i \le \phi_i \le \overline{\phi}_i$, $i = 1, 2, 3$.*

2.1. Construction of upper-lower solutions: $s > s_*$.

Given $s > s_*$. Let λ_i, $i = 1, 2$, be the two positive solutions of

$$A(\lambda) := d\lambda^2 - s\lambda + r_1\beta = 0$$

such that $\lambda_1 < \lambda_2$. Note that $A(\lambda) < 0$ for $\lambda \in (\lambda_1, \lambda_2)$. We define

$$\begin{cases} \overline{\phi}_1(z) := \min\{1, e^{-\lambda_1 z}\},\ \underline{\phi}_1(z) := \max\{0, e^{-\lambda_1 z} - pe^{-\mu z}\}, \\ \overline{\phi}_2(z) := \min\{1, v_c + (1 - v_c)e^{-\lambda_1 z}\},\ \underline{\phi}_2(z) := \max\{0, v_c(1 - e^{-\lambda_1 z})\}, \\ \overline{\phi}_3(z) := \min\{1, w_c + c_2 v_c e^{-\lambda_1 z}\},\ \underline{\phi}_3(z) := \max\{0, w_c(1 - e^{-\lambda_1 z})\}, \end{cases} \tag{2.9}$$

where $\mu \in (\lambda_1, \min\{2\lambda_1, \lambda_2\})$ (so that $A(\mu) < 0$) and p satisfies

$$p > \max\{1, r_1[1 + a_2(1 - v_c) + a_3 c_2 v_c]/[-A(\mu)]\}. \tag{2.10}$$

Then we have

Lemma 2.3. *The functions defined in* (2.9) *are a pair of upper-lower solutions of* (1.2) *for a given $s > s_*$.*

Proof. To prove the lemma, it suffices to check (2.1)-(2.6) for the non-constant parts.

For $z>0$, we compute

$$\begin{aligned}\mathcal{U}_1(z) &= e^{-\lambda_1 z}(d\lambda_1^2 - s\lambda_1) + r_1 e^{-\lambda_1 z}\{1 - e^{-\lambda_1 z} - a_2 v_c(1-e^{-\lambda_1 z}) - a_3 w_c(1-e^{-\lambda_1 z})\}\\ &= -r_1\beta e^{-2\lambda_1 z} \le 0,\end{aligned}$$

using $A(\lambda_1)=0$ and $\beta = 1 - a_2 v_c - a_3 w_c$. Hence (2.1) holds for all $z \neq 0$.

For $z>0$, since $\underline{\phi}_1 \ge 0$, we have

$$\begin{aligned}\mathcal{U}_2(z) &\le (1-v_c)e^{-\lambda_1 z}(d\lambda_1^2 - s\lambda_1) + r_2\overline{\phi}_2(z)\{1 - v_c - (1-v_c)e^{-\lambda_1 z} - b_3 w_c + b_3 w_c e^{-\lambda_1 z}\}\\ &= -r_1\beta(1-v_c)e^{-\lambda_1 z} \le 0,\end{aligned}$$

using $A(\lambda_1)=0$ and $1 - v_c - b_3 w_c = 0$. Hence (2.2) holds for $z \neq 0$.

For $z>0$, we have

$$\begin{aligned}\mathcal{U}_3(z) &\le c_2 v_c e^{-\lambda_1 z}(d\lambda_1^2 - s\lambda_1) + r_3\overline{\phi}_3(z)\{1 - c_2 v_c(1-e^{-\lambda_1 z}) - w_c - c_2 v_c e^{-\lambda_1 z}\}\\ &= -r_1\beta c_2 v_c e^{-\lambda_1 z} \le 0,\end{aligned}$$

using $A(\lambda_1)=0$ and $1 - c_2 v_c - w_c = 0$. Hence (2.3) holds for $z \neq 0$.

Now, for $\underline{\phi}_1$, due to $p>1$, there is $z_0>0$ such that $\underline{\phi}_1(z)=0$ for $z \le z_0$ and $\underline{\phi}_1(z) = e^{-\lambda_1 z} - pe^{-\mu z}$ for $z > z_0$. For $z>z_0$, we compute

$$\begin{aligned}\mathcal{L}_1(z) &\ge e^{-\lambda_1 z}(d\lambda_1^2 - s\lambda_1) - pe^{-\mu z}(d\mu^2 - s\mu)\\ &\quad + r_1\underline{\phi}_1(z)\{1 - e^{-\lambda_1 z} - a_2 v_c - a_2(1-v_c)e^{-\lambda_1 z} - a_3 w_c - a_3 c_2 v_c e^{-\lambda_1 z}\}\\ &= -pe^{-\mu z}A(\mu) + r_1\underline{\phi}_1(z)\{-1 - a_2(1-v_c) - a_3 c_2 v_c\}e^{-\lambda_1 z}\\ &\ge e^{-\mu z}\{-pA(\mu) - r_1[1 + a_2(1-v_c) + a_3 c_2 v_c]e^{(\mu - 2\lambda_1)z}\}\\ &\ge -A(\mu)e^{-\mu z}\{p - r_1[1 + a_2(1-v_c) + a_3 c_2 v_c]/[-A(\mu)]\} \ge 0,\end{aligned}$$

using the choice of μ and (2.10). Hence (2.4) holds for all $z \neq z_0$.

For $z>0$, we calculate, using $1 - v_c - b_3 w_c = 0$,

$$\begin{aligned}\mathcal{L}_2(z) &= -v_c e^{-\lambda_1 z}(d\lambda_1^2 - s\lambda_1) + r_2\underline{\phi}_2(z)\{-b_1 e^{-\lambda_1 z} + v_c e^{-\lambda_1 z} - b_3 c_2 v_c e^{-\lambda_1 z}\}\\ &\ge r_1\beta v_c e^{-\lambda_1 z} - r_2 v_c(b_1 + b_3 c_2 v_c)e^{-\lambda_1 z} \ge 0,\end{aligned}$$

due to $A(\lambda_1)=0$ and (1.6). Hence (2.5) holds for all $z \neq 0$.

Finally, for $z>0$, we compute, using $A(\lambda_1)=0$ and $1 - c_2 v_c - w_c = 0$,

$$\mathcal{L}_3(z) \ge r_1\beta w_c e^{-\lambda_1 z} - r_3 w_c[c_1 + c_2(1-v_c)]e^{-\lambda_1 z} \ge 0,$$

using again (1.6). Hence (2.6) holds for all $z \neq 0$. This completes the proof of the lemma. $\square$

2.2. **Construction of upper-lower solutions:** $s = s_*$.

For $s = s_*$, $A(\lambda) = 0$ has a double root $\lambda_0 > 0$. Note that $s = s_* = 2d\lambda_0$.

Set $B := \lambda_0 e$. We define

$$
\begin{cases}
\overline{\phi}_1(z) := \begin{cases} 1, \ z \le 1/\lambda_0, \\ Bze^{-\lambda_0 z}, \ z > 1/\lambda_0, \end{cases} \quad \underline{\phi}_1(z) := \begin{cases} 0, \ z \le z_*, \\ Bze^{-\lambda_0 z} - q\sqrt{z}e^{-\lambda_0 z}, \ z > z_*, \end{cases} \\
\overline{\phi}_2(z) := \begin{cases} 1, \ z \le 1/\lambda_0, \\ v_c + (1 - v_c)Bze^{-\lambda_0 z}, \ z > 1/\lambda_0, \end{cases} \\
\underline{\phi}_2(z) := \begin{cases} 0, \ z \le 1/\lambda_0, \\ v_c(1 - Bze^{-\lambda_0 z}), \ z > 1/\lambda_0, \end{cases} \\
\overline{\phi}_3(z) := \begin{cases} 1, \ z \le 1/\lambda_0, \\ w_c + c_2 v_c Bze^{-\lambda_0 z}, \ z > 1/\lambda_0, \end{cases} \\
\underline{\phi}_3(z) := \begin{cases} 0, \ z \le 1/\lambda_0, \\ w_c(1 - Bze^{-\lambda_0 z}), \ z > 1/\lambda_0, \end{cases}
\end{cases}
\tag{2.11}
$$

where $q > B/\sqrt{\lambda_0}$ so that $z_* := (q/B)^2 > 1/\lambda_0$ and q satisfies

$$
q > \frac{4}{d} r_1 B^2 [1 + a_2(1 - v_c) + a_3 c_2 v_c] \left(\frac{7}{2\lambda_0 e} \right)^{7/2}. \tag{2.12}
$$

Then we have

Lemma 2.4. *The functions defined in* (2.11) *are a pair of upper-lower solutions of* (1.2) *for* $s = s_*$.

Proof. It suffices to check the range $z > 1/\lambda_0$ for (2.1)-(2.3) and (2.5)-(2.6); while $z > z_*$ for (2.4). The others are trivial.

We start with the computation

$$
d(Bze^{-\lambda_0 z})'' + s(Bze^{-\lambda_0 z})' = -r_1 \beta Bze^{-\lambda_0 z}, \tag{2.13}
$$

using $s = 2d\lambda_0$ and $A(\lambda_0) = 0$. The identity (2.13) shall be used in the following computations without any further mention.

For $z > 1/\lambda_0$, we compute

$$
\begin{aligned}
\mathcal{U}_1(z) &= -r_1 \beta Bze^{-\lambda_0 z} + r_1 Bze^{-\lambda_0 z}\{(1 - a_2 v_c - a_3 w_c) - Bze^{-\lambda_0 z}(1 - a_2 v_c - a_3 w_c)\} \\
&= -r_1 Bze^{-\lambda_0 z}\{\beta Bze^{-\lambda_0 z}\} \le 0,
\end{aligned}
$$

using $1 - a_2 v_c - a_3 w_c = \beta$. Hence (2.1) holds for all $z \ne 1/\lambda_0$.

For $z > 1/\lambda_0$, using $\underline{\phi}_1 \ge 0$, we have

$$
\begin{aligned}
\mathcal{U}_2(z) &\le -(1 - v_c)(r_1 \beta) Bze^{-\lambda_0 z} + r_2 \overline{\phi}_2(z)\{(1 - v_c - b_3 w_c)(1 - Bze^{-\lambda_0 z})\} \\
&= -(1 - v_c) B (r_1 \beta) ze^{-\lambda_0 z} \le 0,
\end{aligned}
$$

using $1 - v_c - b_3 w_c = 0$. Hence (2.2) holds for all $z \neq 1/\lambda_0$.

For $z > 1/\lambda_0$, using again $\underline{\phi}_1 \geq 0$, we compute

$$\mathcal{U}_3(z) \leq -c_2 v_c r_1 \beta B z e^{-\lambda_0 z} \leq 0,$$

since $1 - c_2 v_c - w_c = 0$. Hence (2.3) holds for all $z \neq 1/\lambda_0$.

Next, we note that

$$d(\sqrt{z} e^{-\lambda_0 z})'' + s(\sqrt{z} e^{-\lambda_0 z})' = -\frac{d}{4} z^{-3/2} e^{-\lambda_0 z} - r_1 \beta \sqrt{z} e^{-\lambda_0 z},\ z > 0,$$

using $s = 2d\lambda_0$. Then, for $z > z_*$, we have

$$\begin{aligned}
\mathcal{L}_1(z) \ &\geq\ -r_1 \beta B z e^{-\lambda_0 z} - q[-\frac{d}{4} z^{-3/2} e^{-\lambda_0 z} - r_1 \beta \sqrt{z} e^{-\lambda_0 z}] \\
&\quad + r_1 \underline{\phi}_1(z)\{(1 - a_2 v_c - a_3 w_c) - [1 + a_2(1 - v_c) + a_3 c_2 v_c] B z e^{-\lambda_0 z}\} \\
&=\ q \frac{d}{4} z^{-3/2} e^{-\lambda_0 z} - r_1 \underline{\phi}_1(z)[1 + a_2(1 - v_c) + a_3 c_2 v_c] B z e^{-\lambda_0 z} \\
&\geq\ \frac{d}{4} z^{-3/2} e^{-\lambda_0 z} \left\{ q - \frac{4}{d} r_1 B^2 [1 + a_2(1 - v_c) + a_3 c_2 v_c] \left(z^{7/2} e^{-\lambda_0 z} \right) \right\} \\
&\geq\ \frac{d}{4} z^{-3/2} e^{-\lambda_0 z} \left\{ q - \frac{4}{d} r_1 B^2 [1 + a_2(1 - v_c) + a_3 c_2 v_c] \left(\frac{7}{2\lambda_0 e} \right)^{7/2} \right\} \geq 0,
\end{aligned}$$

using the fact

$$\max_{z>0} \left\{ z^{7/2} e^{-\lambda_0 z} \right\} \leq \left(\frac{7}{2\lambda_0 e} \right)^{7/2}$$

and condition (2.12). Hence (2.4) holds for all $z \neq z_*$.

For $z > 1/\lambda_0$, we compute

$$\begin{aligned}
\mathcal{L}_2(z) \ &=\ r_1 \beta v_c B z e^{-\lambda_0 z} + r_2 \underline{\phi}_2(z)\{(1 - v_c - b_3 w_c) + v_c B z e^{-\lambda_0 z} - (b_1 + b_3 c_2 v_c) B z e^{-\lambda_0 z}\} \\
&\geq\ \{r_1 \beta - r_2(b_1 + b_3 c_2 v_c)\} v_c B z e^{-\lambda_0 z} \geq 0,
\end{aligned}$$

using $1 - v_c - b_3 w_c = 0$ and (1.6). Hence (2.5) holds for all $z \neq 1/\lambda_0$. Similarly, we have

$$\mathcal{L}_3(z) \geq w_c B z e^{-\lambda_0 z}\{r_1 \beta - r_3[c_1 + c_2(1 - v_c)]\} \geq 0,\ z > 1/\lambda_0.$$

Hence (2.6) also holds for all $z \neq 1/\lambda_0$. Thereby, we complete the proof of the lemma. □

By the above construction of upper-lower solutions, we see that (1.7) holds. This proves the first part of Theorem 1.1, by applying Proposition 2.2.

3. Wave tail limit at $-\infty$

In this section, we let (ϕ_1, ϕ_2, ϕ_3) be the solution obtained in the first part of Theorem 1.1 for a given $s \geq s_*$. For the left-hand tail limit at $-\infty$, we first let

$$\phi_i^- := \liminf_{z \to -\infty} \phi_i(z),\ \phi_i^+ := \limsup_{z \to -\infty} \phi_i(z),\ i = 1, 2, 3.$$

Since $\phi_i \geq 0$, $i = 1, 2, 3$, by the maximum principle we have $0 \leq \phi_i \leq 1$, $i = 1, 2, 3$. Hence

$$0 \leq \phi_i^- \leq \phi_i^+ \leq 1, \ i = 1, 2, 3.$$

Then we have

Lemma 3.1. *It holds*

$$\phi_1^- \geq \gamma_1 := 1 - a_2 - a_3 > 0, \tag{3.1}$$

provided (1.5) *is enforced.*

Proof. Since $\phi_i \leq 1$, $i = 2, 3$, $U(x,t) := \phi_1(x - st)$ satisfies

$$U_t \geq dU_{xx} + r_1 U(1 - a_2 - a_3 - U), \ x \in \mathbb{R}, \ t > 0, \ U(x, 0) = \phi_1(x).$$

It follows from [1] and the comparison principle that

$$\phi_1^- = \liminf_{z \to -\infty} \phi_1(z) = \liminf_{z \to -\infty} U(0, -z/s) \geq \gamma_1.$$

The lemma is thus proved. □

Now we define

$$m_1(\theta) := (1 - \theta)(\gamma_1 - \epsilon) + \theta, \ \theta \in [0, 1],$$

where $\epsilon \in (0, \gamma_1)$. Since $\gamma_1 < 1$, $m_1(\theta)$ is increasing in $\theta \in [0, 1]$ such that $m_1(1) = 1$. Let

$$\mathcal{A} := \{\theta \in [0, 1) \mid \phi_1^- > m_1(\theta)\}.$$

By Lemma 3.1, $0 \in \mathcal{A}$ and so the quantity $\theta_0 := \sup \mathcal{A}$ is well-defined such that $\theta_0 \in (0, 1]$. By passing to the limit, we also have

$$\phi_1^- \geq m_1(\theta_0). \tag{3.2}$$

To proceed further, we next derive better upper bounds for ϕ_i, $i = 2, 3$, as follows.

Lemma 3.2. *Under condition* (3.2)*, we have*

$$\phi_2^+ \leq M_2(\theta_0) := \max\{0, 1 - b_1 m_1(\theta_0)\}, \ \phi_3^+ \leq M_3(\theta_0) := \max\{0, 1 - c_1 m_1(\theta_0)\}.$$

Proof. We follow the proof of [4, Lemma 4.11]. For any sequence $\{z_n\}$ tending to ∞, up to a subsequence, we may assume that the function

$$\psi_2(z) := \lim_{n \to \infty} \phi_2(z + z_n), \ z \in \mathbb{R},$$

exists. Moreover, it follows from (1.2), (3.2) and $\phi_3 \geq 0$ that

$$d\psi_2'' + s\psi_2' + r_2\psi_2[1 - b_1 m_1(\theta_0) - \psi_2] \geq 0 \ \text{ in } \mathbb{R}.$$

Since $\phi_2 \leq 1$, by the parabolic comparison principle, $\psi_2(z) \leq \overline{v}(t)$ for any $z \in \mathbb{R}$ and $t > 0$, where $\overline{v}$ solves

$$\partial_t \overline{v} = r_2 \overline{v}[1 - b_1 m_1(\theta_0) - \overline{v}], \ t > 0, \ \overline{v}(t = 0) = 1.$$

Since $\overline{v}(t) \to M_2(\theta_0)$ as $t \to +\infty$, $\psi_2(z) \leq M_2(\theta_0)$ for all $z \in \mathbb{R}$. This proves that $\phi_2^+ \leq M_2(\theta_0)$. Similarly, we also have $\phi_3^+ \leq M_3(\theta_0)$ and the lemma is thus proved. $\square$

Next, we prove that $\theta_0 = 1$. We assume by contradiction that $\theta_0 \in (0,1)$. Then, by the continuity of $m_1(\theta)$, $\theta_0 \notin \mathcal{A}$ and so, by (3.2),

$$\phi_1^- = m_1(\theta_0). \tag{3.3}$$

To reach a contradiction with (3.3), we set

$$\alpha_1 := 1 - m_1(\theta_0) - a_2 M_2(\theta_0) - a_3 M_3(\theta_0).$$

We claim that $\alpha_1 > 0$. Indeed, using Lemma 3.2, it is easy to compute

$$\alpha_1 \geq \begin{cases} (1 - a_2 - a_3) + (a_2 b_1 + a_3 c_1 - 1) m_1(\theta_0), \text{ if } m_1(\theta_0) < \min\{1/b_1, 1/c_1\}, \\ (1 - a_2) + (a_2 b_1 - 1) m_1(\theta_0), \text{ if } m_1(\theta_0) \in [1/c_1, 1/b_1), \\ (1 - a_3) + (a_3 c_1 - 1) m_1(\theta_0), \quad \text{if } m_1(\theta_0) \in [1/b_1, 1/c_1). \end{cases}$$

Then $\alpha_1 > 0$ follows easily from condition (1.8).

From $\alpha_1 > 0$, we can get a contradiction following an argument as that in [6, 7]. Indeed, if $\phi_1(z)$ is monotone ultimately as $z \to -\infty$, then

$$\lim_{z \to -\infty} \phi_1(z) = m_1(\theta_0).$$

Next, an integration of ϕ_1-equation from a negative integer $-n$ to 0 gives

$$-d\phi_1'(0) + d\phi_1'(-n) - s\phi_1(0) + s\phi_1(-n) = r_1 \int_{-n}^{0} \{\phi_1(z)(1 - \phi_1 - a_2\phi_2 - a_3\phi_3)(z)\} dz. \tag{3.4}$$

Since

$$\lim_{z \to -\infty} \phi_1(z)(1 - \phi_1 - a_2\phi_2 - a_3\phi_3)(z) \geq m_1(\theta_0)\alpha_1 > 0,$$

we reach a contradiction by noting that the integral in (3.4) tends to ∞ as $n \to \infty$ and the left-hand side of (3.4) is uniformly bounded for all n.

On the other hand, if ϕ_1 is oscillatory as $z \to -\infty$, then we choose a sequence of minimal points $\{z_n\}$ tending to $-\infty$ such that $\phi_1(z_n) \to m_1(\theta_0)$ as $n \to \infty$. It follows from the ϕ_1-equation in (1.2) that

$$\begin{aligned} 0 &= d\phi_1''(z_n) + s\phi_1'(z_n) + r_1\phi_1(z_n)(1 - \phi_1 - a_2\phi_2 - a_3\phi_3)(z_n) \\ &\geq r_1\phi_1(z_n)(1 - \phi_1 - a_2\phi_2 - a_3\phi_3)(z_n), \ \forall\, n. \end{aligned}$$

This also implies a contradiction, by taking the limit inferior as $n \to \infty$ and using $\alpha_1 > 0$. Therefore, $\theta_0 = 1$. This implies that $\phi_1^- = 1$ and so $\phi_1(-\infty) = 1$.

Finally, applying Lemma 3.2 with $\theta_0 = 1$ and using $b_1, c_1 > 1$, we also conclude that $\phi_2^+ = \phi_3^+ = 0$. Hence $(\phi_2, \phi_3)(-\infty) = (0, 0)$. We conclude that $(\phi_1, \phi_2, \phi_3)(-\infty) = (1, 0, 0)$ and the proof of the second part of Theorem 1.1 is thus completed.

4. Non-existence of traveling waves

We provide in this section a proof of the non-existence of traveling waves as follows.

Proof. We follow the proof of [7, Proposition 5.1] by a contradiction argument. Suppose that there is a positive solution (ϕ_1, ϕ_2, ϕ_3) of (1.2) such that

$$(\phi_1, \phi_2, \phi_3)(+\infty) = (0, v_c, w_c) \tag{4.1}$$

for some $s \in \mathbb{R}$.

We first claim that $s > 0$. Assume for contradiction that $s \leq 0$. Then it follows from (4.1) and (1.4) that $\Phi(z) \to \beta > 0$ as $z \to \infty$, where

$$\Phi(z) := 1 - \phi_1(z) - a_2\phi_2(z) - a_3\phi_3(z).$$

Hence there is a large $K > 0$ such that

$$\Phi(z) \geq \beta/2, \ \forall\, z \geq K.$$

An integration of the ϕ_1-equation in (1.2) from $y \geq K$ to ∞ gives

$$\begin{aligned} \frac{r_1\beta}{2}\int_y^\infty \phi_1(z)dz &\leq r_1\int_y^\infty \{\phi_1(1-\phi_1-a_2\phi_2-a_3\phi_3)\}(z)dz \\ &= d\phi_1'(y) + s\phi_1(y) \leq d\phi_1'(y), \ \forall\, y \geq K. \end{aligned}$$

This implies, by an integration from K to ∞ and using $\phi_1(+\infty) = 0$, that

$$\frac{r_1\beta}{2}\int_K^\infty\int_y^\infty \phi_1(z)dzdy \leq -d\phi_1(K) < 0,$$

a contradiction to the positivity of ϕ_1 in $\mathbb{R}$. Hence we must have $s > 0$.

With this information, Theorem 1.2 can be proved by the same contradiction argument as that in [7, Proposition 5.1] with the help of the classical spreading property for scalar equations ([1]). We omit the details here. □

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