SPREADING DYNAMICS FOR AN EPIDEMIC MODEL OF WEST-NILE VIRUS WITH SHIFTING ENVIRONMENT

INKYUNG AHN, WONHYUNG CHOI, AND JONG-SHENQ GUO

ABSTRACT. We study the disease-spreading dynamics of the West Nile virus (WNv) epidemic model under shifting climatic conditions. A WNv epidemic model is developed incorporating a shifting net growth term to depict the evolving mosquito habitat. First, we comprehensively characterize the spreading dynamics of mosquitoes for any given climate change speed compared with the intrinsic spreading speed of mosquitoes. Utilizing the results from mosquito dynamics, we determine the spreading dynamics of infected birds and mosquitoes, taking into account relationships among the shifting speed and the spreading speeds of mosquito and WNv. Ultimately, we find that infected mosquitoes and birds propagate, and their population densities converge to a stable positive endemic state. This paper provides crucial insights into the impact of climate change on the spread of vector-borne diseases such as WNv.

Keywords. Spreading dynamics, shifting speed, epidemic model, vector-borne disease

1. INTRODUCTION

The West Nile virus (WNv) stands as a quintessential example of vector-borne diseases wherein a biological organism conveys the pathogen to another species. The primary transmission method of WNv revolves around a mosquito-bird-mosquito cycle. Mosquitoes, serving as vectors, play an instrumental role in the propagation of WNv and become infected upon biting birds that harbor the virus, functioning as reservoir hosts. These avian reservoir hosts are the principal hosts for WNv, carrying the virus in their bloodstream. Consequently, the virus can be transmitted to other individuals when these infected birds are bitten by mosquitoes. The disease dynamics of WNv have been explored through various mathematical models, and we recommend references such as [4, 12, 14, 18, 19, 21] for further study on related works. Among the factors that critically influence the transmission and spread of vector-borne diseases such as WNv, the population dynamics of vectors and the environmental conditions of their habitats are particularly noteworthy.

For decades, global climate change has increasingly been recognized as a critical factor in the study of vector-borne diseases, given its influence on the expansion or contraction of vector habitats. This is primarily due to changes in climatic variables such as temperature,

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humidity, and precipitation. Mosquito-borne diseases such as WNv are particularly susceptible to these changes. Ongoing global climate change, characterized by rising temperatures, is anticipated to extend mosquito habitats northward. This, in turn, is expected to spur the emergence of diseases in these northern regions [3, 10, 15, 16, 17]. Hence, accounting for global climate change is indispensable in comprehending the disease dynamics of WNv and predicting future outbreaks.

In this paper, we consider an epidemic model under a shifting environment as follows:

$$R_t = d_1 R_{xx} - \gamma R + \alpha(1-R)V, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$V_t = d_2 V_{xx} + \delta(m-V)R - \beta V, \quad x \in \mathbb{R},\ t > 0, \tag{1.2}$$

$$m_t = d_2 m_{xx} + m[h(x-st) - m], \quad x \in \mathbb{R},\ t > 0. \tag{1.3}$$

The system represented by equations (1.1) through (1.3) is a simplified version of the model initially proposed by Wonham et al. [21]. This model excludes the larval stage of mosquitoes and adopts the following assumptions listed in [12]:

(A1) There is no avian mortality caused by WNv.
(A2) Birds that have recovered from WNv are immediately susceptible again.
(A3) Mosquitoes that have been exposed to WNv are immediately infective.

In system (1.1)-(1.3), the coefficients d_1 and d_2 are positive constants that represent the diffusion coefficients of bird and mosquito populations, respectively. The positive constant s denotes the speed at which the environment is shifting. The variables R, V, and m correspond to the population densities of infectious birds (reservoirs), infective mosquitoes (vectors), and the total mosquito population, respectively. Furthermore, α and δ represent the transmission rates via a bite to birds and mosquitoes, respectively, while γ and β denote the recovery rates for birds and mosquitoes, respectively. The term $h(x-st)$ characterizes the net growth rate of mosquitoes within a habitat that is shifting due to climate change.

It is posited that the function h satisfies the following property:

(h1) h is a monotone and continuously differentiable function in $\mathbb{R}$ such that $h(-\infty) = 1$ and $h(\infty) \in (-\infty, 0)$.

This implies that the favorable habitat for mosquitoes is expanding along the positive x-axis. It should be noted that we normalized both the total population density of birds and the maximal carrying capacity (as well as the net growth rate) of mosquitoes to 1.

It should be noted that under the condition

$$\alpha\delta - \gamma\beta > 0, \tag{1.4}$$

there exists a unique positive constant coexistence state (the endemic state)

$$E^* = (\phi_*, \psi_*),\ \phi_* := \frac{\alpha\delta - \gamma\beta}{(\alpha+\gamma)\delta} \in (0,1),\ \psi_* := \frac{\alpha\delta - \gamma\beta}{\alpha(\beta+\delta)} \in (0,1),$$

for the diffusion-free system corresponding to equations (1.1) and (1.2) with $m \equiv 1$, namely,

$$\begin{cases} R_t = d_1 R_{xx} - \gamma R + \alpha(1-R)V,\ x \in \mathbb{R}, t > 0, \\ V_t = d_2 V_{xx} + \delta(1-V)R - \beta V,\ x \in \mathbb{R}, t > 0. \end{cases} \tag{1.5}$$

For the spreading dynamics of (1.5), we recall from [12, Theorem 5.1] that the spreading speed s_* of system (1.5) is linearly determined and is defined by

$$s_* = \inf_{\mu>0} \frac{\lambda(\mu)}{\mu},$$

where $\lambda(\mu)$ represents the largest eigenvalue of the matrix

$$S(\mu) = \begin{bmatrix} d_1\mu^2 - \gamma & \alpha \\ \delta & d_2\mu^2 - \beta \end{bmatrix}.$$

More precisely, we have

Theorem 1.1 ([12]). *Let* (R, V) *be a solution of system* (1.5) *with nontrivial, nonnegative, and compactly supported initial data* $R(x, 0)$ *and* $V(x, 0)$*. Assume* (1.4) *is enforced. Then*

$$\lim_{t\to\infty} \left\{ \sup_{|x|\geq(s_*+\varepsilon)t} [R(x,t) + V(x,t)] \right\} = 0, \tag{1.6}$$

$$\lim_{t\to\infty} \left\{ \sup_{|x|\leq(s_*-\varepsilon)t} \Big[|R(x,t) - \phi_*| + |V(x,t) - \psi_*|\Big] \right\} = 0 \tag{1.7}$$

for any small $\varepsilon > 0$.

Our focus lies in the spreading dynamics of the system given by equations (1.1)-(1.3). Notably, since equation (1.3) is decoupled from equations (1.1) and (1.2), we first examine the spreading dynamics of the mosquito population. Let $s^* := 2\sqrt{d_2}$. Then, the whole picture of the spreading dynamics of equation (1.3) for any $s > 0$ is described as follows.

Theorem 1.2. *Let* m *be the solution of* (1.3) *with nontrivial, continuous and compactly supported initial data* m_0 *at* $t = 0$ *satisfying* $0 \leq m_0 \leq 1$*. Then we have*

$$\lim_{t\to+\infty} \{ \sup_{|x|\geq(s^*+\varepsilon)t} m(x,t)\} = 0 = \lim_{t\to+\infty} \{ \sup_{x\geq(s+\varepsilon)t} m(x,t)\} \tag{1.8}$$

for all $\varepsilon > 0$*. Moreover, it holds*

$$\lim_{t\to+\infty} \sup_{(-s^*+\varepsilon)t\leq x\leq(\min\{s,s^*\}-\varepsilon)t} |m(x,t) - 1| = 0 \tag{1.9}$$

for any $\varepsilon \in (0, \min\{s, s^*\})$.

Utilizing the information on the dynamics of mosquitoes, we subsequently delve into the spreading dynamics of infectious birds and infective mosquitoes. In the sequel, we let (R, V, m) be a solution of (1.1)-(1.3) with nontrivial, continuous and compactly supported initial data (R_0, V_0, m_0) at $t = 0$. Also, we let

$$X_1 := \{\phi \in C^0(\mathbb{R}) \mid 0 \leq \phi \leq 1\}.$$

First, for the disease-free case, we have

Theorem 1.3. *Assume that* $(R_0, V_0, m_0) \in X_1 \times X_1 \times X_1$*. Then we have*

$$\lim_{t\to+\infty} \left\{ \sup_{|x|\geq(s^*+\varepsilon)t} [R(x,t) + V(x,t)] \right\} = 0 = \lim_{t\to+\infty} \left\{ \sup_{x\geq(s+\varepsilon)t} [R(x,t) + V(x,t)] \right\} \tag{1.10}$$

for all $\varepsilon > 0$*. Moreover, if* $s_* < s < s^*$*, then*

$$\lim_{t\to\infty}\left\{\sup_{x\in[(-s^*+\varepsilon)t,(-s_*-\varepsilon)t]\cup[(s_*+\varepsilon)t,(s-\varepsilon)t]}\Big[R(x,t)+V(x,t)\Big]\right\}=0 \tag{1.11}$$

for any $\varepsilon \in (0,(s-s_*)/2)$*.*

For the disease spreading case, we have

Theorem 1.4. *Assume that* $(R_0, V_0, m_0) \in X_1 \times X_1 \times X_1$*. Let* (1.4) *be enforced. Then*

$$\lim_{t\to\infty}\left\{\sup_{(-\min\{s^*,s_*\}+\varepsilon)t\le x\le(\min\{s,s_*,s^*\}-\varepsilon)t}\Big[|R(x,t)-\phi_*|+|V(x,t)-\psi_*|\Big]\right\}=0 \tag{1.12}$$

for any $\varepsilon \in (0,\min\{s,s^*,s_*\})$*.*

The approach employed here is a standard one, utilizing partial comparison arguments coupled with concepts from dynamical systems theory. While applying this method, there are unique challenges to overcome that vary based on the specifics of different problems. One of the major difficulties lies in characterizing the limiting function along a particular sequence of spatial-temporal shifts in the solution. More precisely, the challenge is to characterize the entire solution set of the associated limiting system of equations.

The remainder of this paper is organized as follows. In Section 2, we provide the detailed proofs of our main results, Theorems 1.2, 1.3 and 1.4. Finally, we summarize our findings and conclude the paper in Section 3.

2. Proofs of Theorems 1.2-1.4

In this section, we provide the proofs of Theorem 1.2, 1.3 and 1.4 as follows.

2.1. **Proof of Theorem 1.2.**

Proof. First, for a constant $\tau \in (0,1]$ we consider the problem

$$\begin{cases} u_t^{(\tau)}(x,t)=d_2u_{xx}^{(\tau)}(x,t)+u^{(\tau)}(x,t)[\tau-u^{(\tau)}(x,t)],\ x\in\mathbb{R},\ t>0,\\ u^{(\tau)}(x,0)=m_0(x),\ x\in\mathbb{R}.\end{cases} \tag{2.1}$$

Then, we have (cf. [2])

$$\lim_{t\to+\infty}\{\sup_{|x|\le ct}|u^{(\tau)}(x,t)-\tau|\}=0,\quad \text{if } c<2\sqrt{d_2\tau}, \tag{2.2}$$

$$\lim_{t\to+\infty}\{\sup_{|x|\ge ct}u^{(\tau)}(x,t)\}=0,\quad \text{if } c>2\sqrt{d_2\tau}. \tag{2.3}$$

In particular, since by comparison $m(x,t) \le u^{(1)}(x,t)$, the first part of (1.8) follows.

Next, for a given $s \in (0, s^*)$, we consider a single equation

$$u_t = d_2u_{xx} + u[h(x-st)+a-u],$$

where $a \in (0,-h(+\infty))$ small enough such that $s < 2\sqrt{d_2(1+a)}$. Then, by [7, Theorem 2.1 (i)], there is a decreasing forced wave solution $u(x,t) = \theta(x-st)$ satisfying $\theta(+\infty) = 0$

and $\theta(-\infty) = 1 + a$. Since m_0 is compactly supported and $m_0(x) \leq 1$, we can choose x_0 large enough so that $m_0(x) \leq \theta(x - x_0)$ for all $x \in \mathbb{R}$. By the comparison principle, $m(x,t) \leq \theta(x - x_0 - st)$ for all $x \in \mathbb{R}$, $t > 0$. Then, as $t \to \infty$, we obtain

$$m(x,t) \leq \theta(x - x_0 - st) \to \theta(+\infty) = 0, \quad \text{if } x \geq (s + \varepsilon)t,$$

for any given $\varepsilon > 0$. Hence the second part of (1.8) follows. Note that the second part of (1.8) with $s \geq s^*$ is included in the first part of (1.8).

Finally, we give a proof of (1.9). Let $\varepsilon \in (0, \min\{s^*, s\})$. Set $a := \min\{s, s^*\} - \varepsilon$ and $b := s^* - \varepsilon$. We choose $\tau_0 \in (0,1)$ such that $s^* - \varepsilon < 2\sqrt{d_2\tau_0} < s^*$. Since $h(-\infty) = 1$, we can find a sufficiently large constant K such that

$$h(z) \geq \tau_0, \; \forall\, z \leq -K. \tag{2.4}$$

Additionally, for this τ_0, we may choose $R \gg 1$ and $\eta_0 > 0$ small enough such that

$$\frac{b^2}{4d_2} + \frac{1}{d_2R^2} - \tau_0 + \eta_0 \leq 0. \tag{2.5}$$

To proceed further, motivated by [20, 13], for the above R (fixed) we let

$$v_{c,R}(x,t) := \begin{cases} \exp\left(-\frac{c}{2d_2}(x - ct)\right)\sin\left(\frac{x-ct}{d_2R}\right), \; ct < x < ct + d_2R\pi, \\ 0, \; \text{otherwise}, \end{cases}$$

for some positive constant c. Then we introduce the function

$$v(x,t) = \begin{cases} v_{b,R}(-x,t), & -(bt + d_2R\pi) \leq x < -(bt + d_2R\pi/2), \\ \max\{v_{b,R}(-x,t), C_1\}, & -(bt + d_2R\pi/2) \leq x < -bt, \\ C_1, & -bt \leq x \leq at, \\ \max\{v_{a,R}(x,t), C_1\}, & at < x \leq at + d_2R\pi/2, \\ v_{a,R}(x,t), & at + d_2R\pi/2 \leq x \leq at + d_2R\pi, \\ 0, & \textit{otherwise}, \end{cases}$$

where C_1 is a constant satisfying

$$0 < C_1 < \min\{v_{a,R}(at + d_2R\pi/2, t), v_{b,R}(bt + d_2R\pi/2, t)\} < 1. \tag{2.6}$$

Note that v is continuous, due to (2.6).

We claim that $\eta v(x,t)$ is a (weak) sub-solution of the equation

$$u_t = d_2u_{xx} + u[h(x - st) - u], \; x \in \mathbb{R}, \; t \geq T_0, \tag{2.7}$$

for some $T_0 \gg 1$ and $0 < \eta \ll 1$. In fact, there exists a unique $x_i \in (0, d_2R\pi/2)$, $i = 1, 2$, such that

$$v_{b,R}(-(bt + x_1), t) = C_1 = v_{a,R}(at + x_2, t).$$

Hence v is a smooth function for $x \in \mathbb{R}$, $t > 0$, except on the following lines

$$x = -(bt + d_2R\pi), \; x = -(bt + x_1), \; x = at + x_2, \; x = at + d_2R\pi,$$

on which there are jump discontinuities for the first derivative of v.

When $v(x,t)=v_{a,R}(x,t)$, we compute

$$(\eta v)_t - d_2(\eta v)_{xx} - \eta v[h(x-st)-\eta v] = \eta v\Big[\frac{a^2}{4d_2}+\frac{1}{d_2R^2}-h(x-st)+\eta v\Big]. \tag{2.8}$$

Note that $v(x,t)=v_{a,R}(x,t)$ if and only if $at+x_2<x<at+d_2R\pi$, $t>0$. By choosing $T_0\gg 1$ such that $-\varepsilon T_0+d_2R\pi\le -K$, we see that

$$x-st\le (a-s)t+d_2R\pi\le -\varepsilon T_0+d_2R\pi\le -K, \text{ if } x\le at+d_2R\pi,\ t\ge T_0.$$

Recall that $a<b$. It follows from (2.4), (2.5) and (2.8) that

$$(\eta v)_t - d_2(\eta v)_{xx} - (\eta v)[h(x-st)-(\eta v)]\le 0 \tag{2.9}$$

for $at+x_2<x<at+d_2R\pi$ and $t\ge T_0$, provided $\eta\le\eta_0$. Clearly, (2.9) holds when $v=C_1$ and $\eta\le\eta_0$. Similarly, for $\eta\le\eta_0$ we can verify that (2.9) holds for

$$x\in(-\infty,0)\setminus\{-(bt+d_2R\pi),-(bt+x_1)\},\ t\ge T_0,$$

and thus, (2.9) holds for $x\in\mathbb{R}\setminus\{-(bt+d_2R\pi),-(bt+x_1),at+x_2,at+d_2R\pi\}$, $t\ge T_0$ and $\eta\le\eta_0$. This implies that ηv is a weak sub-solution of (2.7) for any positive constant $\eta\le\eta_0$.

Now, by the strong maximum principle, $m(x,t)>0$ for all $x\in\mathbb{R}$, $t>0$. We can choose a positive constant $\eta\le\eta_0$ small enough such that

$$\eta\le \min_{x\in[-(bT_0+d_2R\pi),aT_0+d_2R\pi]} m(x,T_0).$$

This implies that $m(x,T_0)\ge \eta v(x,T_0)$ for all $x\in\mathbb{R}$. It follows from the comparison principle that $m\ge\eta v$ for all $t\ge T_0$. In particular, we obtain

$$\liminf_{t\to\infty}\{\inf_{-bt\le x\le at} m(x,t)\}\ge \eta C_1>0. \tag{2.10}$$

Note that (2.10) holds for any $\varepsilon\in(0,\min\{s,s^*\})$ with different positive constants ηC_1.

To conclude the proof of (1.9), we assume for contradiction that (1.9) does not hold for some $\varepsilon_0\in(0,\min\{s,s^*\})$. Then, with $s_0:=\min\{s,s^*\}-\varepsilon_0$ and $s_1:=s^*-\varepsilon_0$, there exists a sequence $\{(x_k,t_k)\}$ such that $x_k\in[-s_1t_k,s_0t_k]$, $t_k\to\infty$ as $k\to\infty$ and $\theta:=\limsup_{k\to\infty} m(x_k,t_k)<1$. Let

$$m_k(x,t):=m(x+x_k,t+t_k).$$

Then, up to a subsequence, $m_k(x,t)$ converges locally uniformly to $m_\infty(x,t)$ as $k\to\infty$, where $m_\infty(x,t)$ satisfies

$$m_{\infty,t}(x,t)=d_2m_{\infty,xx}(x,t)+m_\infty(x,t)[1-m_\infty(x,t)],\quad (x,t)\in\mathbb{R}^2.$$

On the other hand, applying (2.10) with $\varepsilon=\varepsilon_0/2$, we can choose T large enough such that

$$m_k(x,t)\ge \eta C_1/2,\ -(s^*-\varepsilon)(t+t_k)\le x+x_k\le(\min\{s,s^*\}-\varepsilon)(t+t_k),\ t+t_k\ge T.$$

This implies that $m_\infty\ge\eta C_1/2$ in $\mathbb{R}^2$. By [8, Theorem 1.1], we obtain that $m_\infty\equiv 1$. This contradicts $m_\infty(0,0)=\theta<1$. Hence (1.9) holds. Therefore, the proof is complete. □

2.2. **Proof of Theorem 1.3.**

Proof. First, for a given $\varepsilon > 0$, we let λ_1 be the smaller positive root of

$$d_2\lambda^2 - (s^* + \varepsilon/2)\lambda + 1 = 0.$$

Since both m_0 and V_0 are compactly supported, we can choose a constant A large enough such that

$$m_0(x),\, V_0(x) \le Ae^{-\lambda_1 x} \text{ for } x \in \mathbb{R}.$$

Define $\bar{m}(x,t) = Ae^{-\lambda_1[x-(s^*+\varepsilon/2)t]}$. It is easy to check that $\bar{m}$ is a solution of

$$\bar{m}_t(x,t) = d_2\bar{m}_{xx}(x,t) + \bar{m}(x,t).$$

Thus, by the comparison principle, $m(x,t) \le \bar{m}(x,t)$ for $x \in \mathbb{R}$, $t \ge 0$. Also, we compute

$$\bar{m}_t - d_2\bar{m}_{xx} - [\delta(m-\bar{m})R - \beta\bar{m}] \ge \bar{m}[-d_2\lambda_1^2 + (s^*+\varepsilon/2)\lambda_1 + \beta] = (1+\beta)\bar{m} \ge 0.$$

By the comparison principle, $V(x,t) \le \bar{m}(x,t)$ for $x \in \mathbb{R}$ and $t \ge 0$. Hence, we obtain

$$\sup_{x \ge (s^*+\varepsilon)t} V(x,t) \le \sup_{x \ge (s^*+\varepsilon)t} Ae^{-\lambda_1[x-(s^*+\varepsilon)t]}e^{-\lambda_1\varepsilon t/2} \to 0 \text{ as } t \to +\infty.$$

Similarly, with $\hat{m}(x,t) = Ae^{-\lambda_1[-x-(s^*+\varepsilon/2)t]}$, we have also

$$\sup_{-x \ge (s^*+\varepsilon)t} V(x,t) \le \sup_{-x \ge (s^*+\varepsilon)t} Ae^{-\lambda_1[-x-(s^*+\varepsilon)t]}e^{-\lambda_1\varepsilon t/2} \to 0 \text{ as } t \to +\infty.$$

For R, we choose a small $\lambda' \in (0, \lambda_1)$ such that $d_1(\lambda')^2 - (s^*+\varepsilon/2)\lambda' := -\rho_0 < 0$. Define

$$\bar{R}(x,t) := \min\{1, Be^{-\lambda'[x-(s^*+\varepsilon/2)t]}\},$$

where B is a positive constant to be chosen later. For (x,t) satisfying

$$\bar{R}(x,t) = Be^{-\lambda'[x-(s^*+\varepsilon/2)t]} < 1,$$

we have

$$\begin{aligned}
\mathcal{N}_b(x,t) &:= \bar{R}_t - d_1\bar{R}_{xx} - [\alpha(1-\bar{R})V - \gamma\bar{R}] \\
&\ge -\bar{R}[d_1(\lambda')^2 - (s^*+\varepsilon/2)\lambda'] - \alpha Ae^{-\lambda_1[x-(s^*+\varepsilon/2)t]} \\
&\ge \bar{R}\left[\rho_0 - \frac{\alpha A}{B}(1/B)^{(\lambda_1-\lambda')/\lambda'}\right] \ge 0,
\end{aligned}$$

if we choose B to be sufficiently large. Clearly, $\mathcal{N}_b(x,t) \ge 0$ for (x,t) with $\bar{R}(x,t) = 1$. By choosing a larger B, if necessary, so that $R_0(x) \le Be^{-\lambda' x}$ for all $x \in \mathbb{R}$, we obtain that $R(x,t) \le \bar{R}(x,t)$ for $x \in \mathbb{R}$ and $t \ge 0$. Thus, we obtain $\lim_{t\to+\infty}\sup_{x\ge(s^*+\varepsilon)t} R(x,t) = 0$. A similar argument as for V, we also obtain $\lim_{t\to+\infty}\sup_{-x\ge(s^*+\varepsilon)t} R(x,t) = 0$ and so the first part of (1.10) is proved.

Next, given $\varepsilon > 0$. For $s \in (0, s^*)$, we let $\tilde{\lambda}$ be a small positive constant satisfying

$$d_1\tilde{\lambda}^2 - s\tilde{\lambda} := -\rho_1 < 0, \quad d_2\tilde{\lambda}^2 - s\tilde{\lambda} := -\rho_2 < 0.$$

Consider now the function $\check{m}(x,t) = \min\{1, e^{-\tilde{\lambda}(x-\xi_0-st)}\}$, where $\xi_0 > 0$ is large enough such that $h(\xi_0) < \rho_2$, $V_0(x) \leq \check{m}(x,0)$ for $x \in \mathbb{R}$, and $m_0(x) \leq \check{m}(x,0)$ for $x \in \mathbb{R}$. When $x > st + \xi_0$, i.e., $\check{m}(x,t) = e^{-\tilde{\lambda}(x-\xi_0-st)}$, $\check{m}(x,t)$ satisfies

$$\check{m}_t - d_2\check{m}_{xx} - \check{m}[h(x-st) - \check{m}] \geq e^{-\tilde{\lambda}(x-\xi_0-st)}[\rho_2 - h(x-st)] \geq 0,$$

using $h(x-st) \leq h(\xi_0) < \rho_2$ for $x - st > \xi_0$. Since $m \leq 1$, $m(x,t) \leq \check{m}(x,t)$ for all $x \in \mathbb{R}$, $t \geq 0$, by the comparison principle. Furthermore, we can show that $\check{m}$ is a super-solution of (1.2), using $R \geq 0$ and $m \leq \check{m}$, and so $V(x,t) \leq \check{m}(x,t)$ for all $x \in \mathbb{R}$, $t \geq 0$. This implies that $V(x,t) \leq e^{-\tilde{\lambda}(x-\xi_0-st)} \leq e^{-\tilde{\lambda}(\varepsilon t-\xi_0)}$ for all $x \geq (s+\varepsilon)t$ for $t \gg 1$ so that $\varepsilon t > \xi_0$. Hence we obtain

$$\lim_{t\to\infty} \sup_{x\geq(s+\varepsilon)t} V(x,t) = 0.$$

Now, to show the second part of (1.10) for $s \in (0, s^*)$, we define the function

$$\check{R}(x,t) = \min\{1, e^{-\tilde{\lambda}(x-\xi_1-st)}\},$$

where $\xi_1 > \xi_0$ is a sufficiently large positive constant such that $R_0(x) \leq \check{R}(x,0)$ for $x \in \mathbb{R}$. We claim that $\check{R}$ satisfies

$$\check{R}_t \geq d_1\check{R}_{xx} - \gamma\check{R} + \alpha(1-\check{R})V, \ x \in \mathbb{R}, \ t > 0, \tag{2.11}$$

For (x,t) with $x - st \leq \xi_1$, $\check{R} = 1$, and thus, (2.11) holds. For $x - st > \xi_1 > \xi_0$, we obtain

$$\begin{aligned} \check{R}_t - d_1\check{R}_{xx} - [\alpha(1-\check{R})V - \gamma\check{R}] &\geq \check{R}\rho_1 - \alpha e^{-\tilde{\lambda}(x-\xi_0-st)} \\ &= \check{R}\left[\rho_1 - \alpha e^{-\tilde{\lambda}(\xi_1-\xi_0)}\right] \geq 0, \end{aligned}$$

if we choose $\xi_1 > \xi_0$ to be larger. Therefore, (2.11) holds for $x \in \mathbb{R}$ and $t > 0$. By the comparison principle, $R(x,t) \leq \check{R}(x,t)$ for $x \in \mathbb{R}$ and $t > 0$. In particular, $R(x,t) \leq e^{-\tilde{\lambda}(x-\xi_1-st)}$ for $x \geq (s+\varepsilon)t > st + \xi_1$ with $t \gg 1$. Therefore, $\sup_{x\geq(s+\varepsilon)t} R(x,t) \to 0$ as $t \to \infty$. This proves the second part of (1.10) for $s \in (0, s^*)$ (and so for all $s > 0$).

Finally, note that $m \leq 1$. Hence, the corresponding solution (R, V) of (1.1)-(1.2) satisfies

$$\begin{cases} R_t = d_1R_{xx} - \gamma R + \alpha(1-R)V, \ x \in \mathbb{R}, \ t > 0, \\ V_t \leq d_2V_{xx} + \delta(1-V)R - \beta V, \ x \in \mathbb{R}, \ t > 0. \end{cases}$$

Since (1.5) is a cooperative system, (1.11) follows from the comparison principle and (1.6). This completes the proof of Theorem 1.3. □

2.3. **Proof of Theorem 1.4.**

The proof of Theorem 1.4 is quite similar to the method used in [1, 5], but with some *nontrivial* modifications. Therefore, we provide the details of the proof here.

Following [1, 5], for $0 \leq c_1 < c_2$ we define $\omega_{[c_1,c_2]}$ as the set of the functions $(\tilde{R}, \tilde{V}, \tilde{m}) : \mathbb{R}^2 \to \mathbb{R}^3$ such that there exist sequence $\{t_n\} \subset [0,\infty)$ and $\{x_n\} \subset \mathbb{R}$ with $t_n \to \infty$ as $n \to \infty$ and $x_n \in [c_1t_n, c_2t_n]$ for all $n \geq 0$ such that

$$(\tilde{R}, \tilde{V}, \tilde{m})(x,t) = \lim_{n\to\infty}(R, V, m)(x + x_n, t + t_n) \text{ locally uniformly for } (x,t) \in \mathbb{R}^2.$$

For $0 < \varepsilon_0 \ll 1$, $c \in \mathbb{R}$ and $L > 0$, let $\lambda = \lambda_{1,L} := \lambda_{1,L}(c, \varepsilon_0)$ be the principal eigenvalue of

$$\begin{cases} -d_1\phi_{xx} - c\phi_x + \gamma\phi - \alpha(1-\varepsilon_0)\psi = \lambda\phi, \ x \in (-L, L), \\ -d_2\psi_{xx} - c\psi_x + \beta\psi - \delta(1-2\varepsilon_0)\phi = \lambda\psi, \ x \in (-L, L), \\ \phi(\pm L) = \psi(\pm L) = 0. \end{cases} \tag{2.12}$$

Recall from [5, Proposition 3.5] (see also [9, Theorem 4.2]), for a given constant c with $|c| < s_*$, $\lambda_{1,L}(c, \varepsilon_0) < 0$ for large L and small ε_0.

First, we prepare the following lemma.

Lemma 2.1. *Let $0 \le c_1 < \min\{s, s_*, s^*\} - \varepsilon := c_2$. For any $c \in [c_1, c_2)$, there exists $\delta_1(c) > 0$ such that for each nontrivial initial data $(R_0, V_0, m_0) \in X_1 \times X_1 \times X_1$, the solution (R, V, m) of* (1.1)-(1.3) *satisfies*

$$\limsup_{t\to\infty}(R + V)(ct, t) \ge \delta_1(c), \tag{2.13}$$

and for any $(\tilde{R}, \tilde{V}, \tilde{m}) \in \omega_{[c_1,c_2]}$ with $\tilde{R} \ne 0, \tilde{V} \ne 0$ and $\tilde{m} \ne 0$, one has

$$\limsup_{t\to\infty}(\tilde{R} + \tilde{V})(ct, t) \ge \delta_1(c). \tag{2.14}$$

Proof. We only give proof of (2.14). The proof of (2.13) can be done by a similar argument.

Suppose that (2.14) does not hold. Then, there exist $c \in [c_1, c_2]$, $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ and $\{(\tilde{R}_n, \tilde{V}_n, \tilde{m}_n)\} \subset \omega_{[c_1,c_2]}$ with $\tilde{R}_n \ne 0, \tilde{V}_n \ne 0$ and $\tilde{m}_n \ne 0$ for all $n \ge 0$ such that

$$\lim_{n\to\infty} \sup_{t\ge t_n}(\tilde{R}_n + \tilde{V}_n)(ct, t) = 0.$$

Then we have

$$\lim_{n\to\infty} \sup_{t\ge t_n} \tilde{R}_n(ct, t) = 0, \quad \lim_{n\to\infty} \sup_{t\ge t_n} \tilde{V}_n(ct, t) = 0. \tag{2.15}$$

We claim that

$$\lim_{n\to\infty} \{ \sup_{|x-ct|\le L, t\ge t_n} \tilde{R}_n \} = 0, \quad \lim_{n\to\infty} \{ \sup_{|x-ct|\le L, t\ge t_n} \tilde{V}_n \} = 0 \tag{2.16}$$

for any $L > 0$. Assume for contradiction that for some $L > 0$ there exists $\{x_n\} \subset [-L, L]$ and $\{\tau_n\}$ with $\tau_n \ge t_n$ such that

$$\liminf_{n\to\infty} \tilde{R}_n(x_n + c\tau_n, \tau_n) > 0.$$

Without loss of generality, up to subsequence, we may assume that $x_n \to x_0$ as $n \to \infty$ for some $x_0 \in [-L, L]$. Then, up to subsequence, by the standard parabolic estimate we have

$$(\tilde{R}_n, \tilde{V}_n, \tilde{m}_n)(x + c\tau_n, t + \tau_n) \to (R_\infty, V_\infty, 1)(x, t) \text{ locally uniformly in } \mathbb{R}^2 \text{ as } n \to \infty,$$

where (R_∞, V_∞) is an entire solution of

$$\begin{cases} R_t = d_1 R_{xx} - \gamma R + \alpha(1 - R)V, \\ V_t = d_2 V_{xx} + \delta(1 - V)R - \beta V, \end{cases} \tag{2.17}$$

such that $0 \le R_\infty, V_\infty \le 1$ in $\mathbb{R}^2$. Since $R_\infty(0, 0) = 0$, by (2.15), and R_∞ satisfies

$$R_t \ge d_1 R_{xx} - \gamma R \ \text{ in } \mathbb{R}^2,$$

we obtain that $R_\infty(\cdot,t)\equiv 0$ for all $t\le 0$ by the strong maximum principle. Similarly, we also have $V_\infty(\cdot,t)\equiv 0$ for all $t\le 0$. Since (2.17) is a cooperative system, the uniqueness of solutions to (2.17) gives $R_\infty=V_\infty=0$ in $\mathbb{R}^2$. This contradicts $R_\infty(x_0,0)>0$. Hence, we conclude that $\lim_{n\to\infty}\{\sup_{|x-ct|\le L,t\ge t_n}\tilde{R}_n\}=0$ for any $L>0$. Similarly, we also have $\lim_{n\to\infty}\{\sup_{|x-ct|\le L,t\ge t_n}\tilde{V}_n\}=0$ for any $L>0$. Hence, (2.16) follows.

Recall from (1.9) that $\lim_{n\to\infty}\{\sup_{|x-ct|\le L,t\ge t_n}|\tilde{m}_n-1|\}=0$. For a given small $\varepsilon_0>0$ and large L with $\lambda_{1,L}(c,\varepsilon_0)<0$, there is a fixed sufficiently large n such that $(\tilde{R}_n,\tilde{V}_n)$ satisfies

$$\begin{cases}(\tilde{R}_n)_t\ge d_1(\tilde{R}_n)_{xx}-\gamma\tilde{R}_n+\alpha(1-\varepsilon_0)\tilde{V}_n,\\ (\tilde{V}_n)_t\ge d_2(\tilde{V}_n)_{xx}+\delta(1-2\varepsilon_0)\tilde{R}_n-\beta\tilde{V}_n,\end{cases}$$

for $t\ge t_n$ and $|x-ct_n|\le L$. Let (ϕ,ψ) be a positive eigenfunction of (2.12) corresponding to the principal eigenvalue $\lambda_{1,L}=\lambda_{1,L}(c,\varepsilon_0)$. It follows from the comparison principle that

$$\tilde{R}_n(x+ct,t)\ge\kappa e^{-\lambda_{1,L}t}\phi(x),\ \ \tilde{V}_n(x+ct,t)\ge\kappa e^{-\lambda_{1,L}t}\psi(x),\ \ -L\le x\le L,\ t\ge t_n,$$

if $\kappa>0$ is chosen small enough such that

$$\tilde{R}_n(x+ct_n,t_n)\ge\kappa e^{-\lambda_{1,L}t_n}\phi(x),\ \ \tilde{V}_n(x+ct_n,t_n)\ge\kappa e^{-\lambda_{1,L}t_n}\psi(x),\ \ -L\le x\le L.$$

This implies $\tilde{R}_n(ct,t)\to\infty$ and $\tilde{V}_n(ct,t)\to\infty$ as $t\to\infty$, which is a contradiction. Hence we conclude the proof of the lemma. □

We remark that, from the same argument as in Lemma 2.1, for any $c\in[0,\min\{s_*,s^*\})$ there exists $\hat{\delta}(c)>0$ such that

$$\limsup_{t\to\infty}(R+V)(ct,t)\ge\hat{\delta}(c) \tag{2.18}$$

for any nontrivial solution $(R,V)\in X_1\times X_1$ of (2.17).

Next, we derive a weak spreading property for the infectious birds and infective mosquitoes. Hereafter, we let $\hat{s}:=\min\{s_*,s^*,s\}$ for the given $s>0$.

Lemma 2.2. *For any $c\in[0,\hat{s})$, there exists $\delta_2(c)>0$ such that for each nontrivial initial data $(R_0,V_0,m_0)\in X_1\times X_1\times X_1$, the solution (R,V,m) of (1.1)-(1.3) satisfies*

$$\liminf_{t\to\infty}(R+V)(ct,t)\ge\delta_2(c), \tag{2.19}$$

Proof. We assume by contradiction that there exists a sequence $\{t_n\}$ with $t_n\to\infty$ as $n\to\infty$ such that

$$\lim_{n\to\infty}(R+V)(ct_n,t_n)=0.$$

By Lemma 2.1, we can also choose a sequence $\{t_n'\}$ with $t_n'<t_n$ and $t_n'\to\infty$ such that

$$(R+V)(ct_n',t_n')\ge\frac{\delta_1(c)}{2}\ \text{ for all } n\ge 0.$$

Define

$$\tau_n:=\sup\{t\ge t_n':(R+V)(ct,t)\ge\rho\},\quad \rho:=\min\{\delta_1(c),\hat{\delta}(c)\}/2>0.$$

Then we have

$$(R+V)(ct,t)\le\rho\ \text{ for } t\in(\tau_n,t_n),\ \ (R+V)(c\tau_n,\tau_n)=\rho. \tag{2.20}$$

Also, taking the limit (up to subsequence), we have

$$(R,V,m)(x+c\tau_n,t+\tau_n)\to(R_\infty,V_\infty,m_\infty)(x,t) \text{ locally uniformly in } \mathbb{R}^2,$$

where (R_∞,V_∞) satisfies (2.17) such that $0\le R_\infty,V_\infty\le 1$.

If $t_n-\tau_n$ converges to t_0 as $n\to\infty$ for some $t_0\in\mathbb{R}$, then

$$\begin{aligned}(R_\infty+V_\infty)(ct_0,t_0) &= \lim_{n\to\infty}(R+V)(c(t_n-\tau_n)+c\tau_n,(t_n-\tau_n)+\tau_n)\\ &= \lim_{n\to\infty}(R+V)(ct_n,t_n)=0.\end{aligned}$$

Hence $R_\infty(ct_0,t_0)=V_\infty(ct_0,t_0)=0$. It follows from the same argument as in Lemma 2.1 that both $R_\infty=0$ and $V_\infty=0$. This contradicts $(R_\infty+V_\infty)(0,0)=\rho>0$. Therefore, $t_n-\tau_n\to\infty$ as $n\to\infty$. Then, from (2.20), we have $(R_\infty+V_\infty)(ct,t)\le\rho$ for all $t\ge 0$. However, (R_∞,V_∞) is a solution of (2.17), and thus, by (2.18),

$$\hat{\delta}(c)\le\limsup_{t\to\infty}(R_\infty+V_\infty)(ct,t)\le\rho\le\hat{\delta}(c)/2,$$

which is a contradiction. Hence the proof is done. □

Now we show a uniform persistence result as follows.

Theorem 2.3. *Let (R,V,m) be a solution of (1.1)-(1.3) with nontrivial initial data $(R_0,V_0,m_0)\in X_1\times X_1\times X_1$. Then, for any $\varepsilon\in(0,\hat{s})$, there is a positive constant θ such that*

$$\liminf_{t\to\infty}\left\{\inf_{0\le x\le(\hat{s}-\varepsilon)t}R(x,t)\right\}\ge\theta, \tag{2.21}$$

$$\liminf_{t\to\infty}\left\{\inf_{0\le x\le(\hat{s}-\varepsilon)t}V(x,t)\right\}\ge\theta. \tag{2.22}$$

Proof. We first show that

$$\liminf_{t\to\infty}\left\{\inf_{0\le x\le(\hat{s}-\varepsilon)t}(R+V)(x,t)\right\}\ge\tilde{\theta}, \tag{2.23}$$

for some positive constant $\tilde{\theta}$. Let $\hat{c}=\hat{s}-\varepsilon$. We assume by contradiction that there exist sequences $\{c_k\}\subset[0,\hat{c})$ and $\{t_k\}$ such that $c_k\to\tilde{c}\in[0,\hat{c})$, $t_k\to\infty$, and $(R+V)(c_kt_k,t_k)\to 0$ as $k\to\infty$.

Define a sequence $\{t'_k\}$ by $t'_k:=\frac{c_kt_k}{\hat{c}}<t_k$. We first show that $t'_k\to\infty$. Suppose c_kt_k converges to $x_\infty\in[0,\infty)$ as $k\to\infty$ (up to subsequence if necessary). Consider the sequence of functions

$$(R_k,V_k,m_k)(x,t):=(R,V,m)(x,t+t_k),$$

which converges to $(R_\infty,V_\infty,m_\infty)\in\omega_{[0,\hat{c}]}$. Note that $m_\infty=1$, due to (1.9). Then we have

$$(R_\infty+V_\infty)(x_\infty,0)=\lim_{k\to\infty}(R+V)(c_kt_k,t_k)=0.$$

It follows from the strong maximum principle that $R_\infty=0$ and $V_\infty=0$, as in Lemma 2.1. However, from Lemma 2.2 with $c=0$, we have

$$(R+V)(0,t_k)\ge\frac{3}{4}\delta_2(0),\quad\forall k\gg 1.$$

This implies that $(R_\infty + V_\infty)(0,0) > 0$, a contradiction. Therefore, $c_k t_k \to \infty$, and thus $t'_k \to \infty$ as $k \to \infty$. Then, from Lemma 2.2 with $c = \hat{c}$, we have

$$(R+V)(\hat{c}t'_k, t'_k) \geq \frac{1}{2}\delta_2(\hat{c}), \quad \forall k \gg 1.$$

Now, let us consider

$$\tau_k := \sup\{t \geq t'_k : (R+V)(c_k t_k, t) \geq \eta\}, \quad \eta := \min\{\delta_2(\hat{c}), \delta_1(0)\}/2 > 0.$$

Then $\tau_k < t_k$ for large k, and as in the proof of Lemma 2.2, $t_k - \tau_k \to \infty$ as $k \to \infty$. For large k, we obtain from the definition of τ_k that

$$(R+V)(c_k t_k, t' + \tau_k) \leq (R+V)(c_k t_k, \tau_k) = \eta, \quad \forall t' \in [0, t_k - \tau_k]. \tag{2.24}$$

Moreover, up to subsequence, we may assume that

$$(R_k, V_k, m_k)(x,t) \to (R_\infty, V_\infty, 1)(x,t) \text{ locally uniformly for } (x,t) \in \mathbb{R}^2 \text{ as } k \to \infty,$$

where

$$(R_k, V_k, m_k)(x,t) := (R, V, m)(x + c_k t_k, t + \tau_k).$$

From (2.24) and the fact that $t_k - \tau_k \to \infty$ as $k \to \infty$, we have

$$(R_\infty + V_\infty)(0,0) = \eta, \quad (R_\infty + V_\infty)(0,t') \leq \eta, \quad \forall t' \geq 0.$$

However, since $0 \leq c_k t_k = \hat{c}t'_k \leq \hat{c}\tau_k$, $(R_\infty, V_\infty, m_\infty) \in \omega_{[0,\hat{c}]}$. Thus it follows from Lemma 2.1 that

$$\eta \geq \limsup_{t\to\infty} (R_\infty + V_\infty)(0,t) \geq \delta_1(0) \geq 2\eta,$$

which is a contradiction. Hence, (2.23) follows.

Next, we show (2.21). Suppose not, we can find sequences $\{c_k\} \subset [0, \hat{c})$ and $\{t_k\}$ such that $c_k \to \tilde{c} \in [0, \hat{c})$, $t_k \to \infty$, and $R(c_k t_k, t_k) \to 0$ as $k \to \infty$. From (2.23),

$$\liminf_{k\to\infty} V(c_k t_k, t_k) \geq \tilde{\theta}.$$

Let us consider the sequence of functions

$$(R_k, V_k, m_k)(x,t) := (R, V, m)(x + c_k t_k, t + t_k).$$

Taking the limit, we have (up to extraction of a subsequence)

$$(R_k, V_k, m_k)(x,t) \to (R_\infty, V_\infty, 1)(x,t) \text{ locally uniformly for } (x,t) \in \mathbb{R}^2 \text{ as } k \to \infty,$$

where (R_∞, V_∞) is an entire solution of (2.17) such that $0 \leq R_\infty, V_\infty \leq 1$. Note that $R_\infty(0,0) = 0$. The strong maximum principle implies that $R_\infty(\cdot, t) \equiv 0$ for all $t \leq 0$. Hence, V_∞ satisfies

$$V_t = d_2 V_{xx} - \beta V, \; x \in \mathbb{R}, \; t \leq 0,$$

or

$$(e^{\beta t} V)_t = (e^{\beta t} V)_{xx}, \; x \in \mathbb{R}, \; t \leq 0.$$

Then, we have

$$\begin{aligned} V_\infty(0,0) &= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi(-\tau)}} \exp\left\{-\frac{(-y)^2}{4(-\tau)}\right\} e^{\beta\tau} V_\infty(y,\tau)dy \\ &\leq e^{\beta\tau} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi(-\tau)}} \exp\left\{-\frac{(-y)^2}{4(-\tau)}\right\} dy = e^{\beta\tau} \end{aligned}$$

for any $\tau < 0$, using $V_\infty \leq 1$. This is impossible, since $V_\infty(0,0) \geq \tilde{\theta} > 0$. Hence, the proof of (2.21) is done.

Similarly, we can also show that (2.22) holds. Therefore, we complete the proof. □

Note that the same argument as that in Theorem 2.3, using (1.9), also leads to the following result for $c \in [-\min\{s_*, s^*\} + \varepsilon, 0]$.

Theorem 2.4. *Let (R, V, m) be a solution of* (1.1)-(1.3) *with a nontrivial initial data $(R_0, V_0, m_0) \in X_1 \times X_1 \times X_1$. Then, for any $\varepsilon \in (0, \min\{s_*, s^*\})$, there is a positive constant θ such that*

$$\liminf_{t\to\infty} \left\{ \inf_{-(\min\{s_*,s^*\}-\varepsilon)t \leq x \leq 0} R(x,t) \right\} \geq \theta, \tag{2.25}$$

$$\liminf_{t\to\infty} \left\{ \inf_{-(\min\{s_*,s^*\}-\varepsilon)t \leq x \leq 0} V(x,t) \right\} \geq \theta. \tag{2.26}$$

Finally, we are ready to prove the disease spreading property as described in Theorem 1.4.

Proof of Theorem 1.4. We apply a contradiction argument used in [8]. Suppose that there exists a positive constant δ and a sequence $\{(x_n, t_n)\}$ with $t_n \to \infty$ as $n \to \infty$ and

$$x_n \in [-(\min\{s_*, s^*\} - \varepsilon)t_n, (\hat{s} - \varepsilon)t_n],\ \forall\, n,$$

such that

$$|R(x_n, t_n) - \phi_*| + |V(x_n, t_n) - \psi_*| \geq \delta,\ \forall\, n. \tag{2.27}$$

Then, up to extracting a subsequence, we obtain that

$$(R, V, m)(x + x_n, t + t_n) \to (R_\infty, V_\infty, m_\infty)(x, t) \text{ as } n \to \infty,$$

locally uniformly for $(x, t) \in \mathbb{R}^2$, where $m_\infty \equiv 1$, by (1.9), and (R_∞, V_∞) is an entire solution of (2.17). Moreover, by Theorems 2.3 and 2.4, it holds

$$1 \geq R_\infty(x,t) \geq \theta,\ \ 1 \geq V_\infty(x,t) \geq \theta \text{ for all } (x,t) \in \mathbb{R}^2.$$

Now, we consider a Lyapunov functional for (2.17)

$$F(R, V) := \frac{\psi_*}{2\gamma\phi_*}(R - \phi_*)^2 + \frac{\phi_*}{2\beta\psi_*}(V - \psi_*)^2.$$

To apply Theorem 1.1 in [8], we calculate the directional derivative of F along the vector

$$(-\gamma R + \alpha(1 - R)V, -\beta V + \delta(1 - V)R)$$

as

$$\mathcal{D}F := \frac{\psi_*}{\gamma\phi_*}(R - \phi_*)\Big[-\gamma R + \alpha(1 - R)V\Big] + \frac{\phi_*}{\beta\psi_*}(V - \psi_*)\Big[-\beta V + \delta(1 - V)R\Big].$$

Since $-\gamma\phi_* + \alpha(1-\phi_*)\psi_* = 0$, we have

$$\begin{aligned}-\gamma R + \alpha(1-R)V &= -\gamma(R-\phi_*) - \alpha(1-\phi_*)\psi_* + \alpha(1-R)V\\ &= -\gamma(R-\phi_*) + \alpha(V-\psi_*) + \alpha[-\phi_*(V-\psi_*) - V(R-\phi_*)]\\ &= -(\gamma+\alpha V)(R-\phi_*) + \alpha(1-\phi_*)(V-\psi_*)\\ &= -(\gamma+\alpha V)(R-\phi_*) + \frac{\gamma\phi_*}{\psi_*}(V-\psi_*).\end{aligned}$$

Similarly, we have

$$-\beta V + \delta(1-V)R = -(\beta+\delta R)(V-\psi_*) + \frac{\beta\psi_*}{\phi_*}(R-\phi_*).$$

Let $A := R - \phi_*$ and $B := V - \psi_*$. Then, for $(R,V) \in [\theta,1]\times[\theta,1]$, we compute

$$\begin{aligned}\mathcal{D}F &= -\frac{\psi_*}{\gamma\phi_*}(\gamma+\alpha V)A^2 + 2AB - \frac{\phi_*}{\beta\psi_*}(\beta+\delta R)B^2\\ &= -\Big[\frac{\alpha\psi_* V}{\gamma\phi_*}A^2 + \frac{\delta\phi_* R}{\beta\psi_*}B^2\Big] - \Big[\frac{\psi_*}{\phi_*}A^2 - 2AB + \frac{\phi_*}{\psi_*}B^2\Big]\\ &\le -\Big[\frac{\alpha\psi_*\theta}{\gamma\phi_*}A^2 + \frac{\delta\phi_*\theta}{\beta\psi_*}B^2\Big] - \Big(\sqrt{\frac{\psi_*}{\phi_*}}A - \sqrt{\frac{\phi_*}{\psi_*}}B\Big)^2\\ &\le -\Big[\frac{\alpha\psi_*\theta}{\gamma\phi_*}A^2 + \frac{\delta\phi_*\theta}{\beta\psi_*}B^2\Big] \le -\nu F(R,V),\end{aligned}$$

for all $\nu \in (0, 2\theta\min\{\alpha,\delta\})$. Then it follows from [8, Theorem 1.1] that

$$(R_\infty, V_\infty) = (\phi_*, \psi_*),$$

a contradiction to (2.27). Hence the proof is complete. □

3. Summary

In this paper, we explored the dynamics of disease spread in a West Nile virus (WNv) epidemic model under the influence of shifting climates. Given the significant impact of climate change on the habitats of disease vectors, considering these effects has become imperative in the study of vector-borne diseases. To this end, we developed a WNv epidemic model incorporating a shifting net growth term to illustrate the changing mosquito habitat. This model was derived from a previously introduced model by Wonham et al. [21] and simplified using the hypothesis adopted in [12]. Our mathematical approach hinged on partial comparison arguments and concepts from dynamical systems theory. However, we encountered certain challenges when applying these methods, specifically in characterizing the entire solution of the limiting system of equations during the analysis of the disease spread scenario. These challenges were addressed in the paper, and their resolution necessitated our use of the methodologies employed in [1, 5].

Throughout the course of this paper, we achieved a comprehensive understanding of the spreading dynamics of mosquitoes for any given climate change speed, denoted as $s > 0$, compared with the intrinsic spreading speed of mosquitoes, represented by s^*. This is detailed in Theorem 1.2. Leveraging the obtained results concerning mosquitoes, we were able to determine the spreading dynamics of infectious birds and mosquitoes under conditions contingent upon the relationships among shifting speeds s, s^*, and s_*, where s^* and s_* represent the spreading speeds of mosquitoes and WNv, respectively. The quantity s_*, derived from Theorem 1.1 and the results of [12], assume that the mosquito population density is given by 1. Naturally, a disease-free condition occurs when the mosquitoes are unable to survive, as indicated in Theorem 1.3. We identified moving regions where mosquito spread occurs, yet the disease remains absent, as highlighted in (1.11). Finally, Theorem 1.4 demonstrates that infected mosquitoes and birds spread, and their population densities converge to a positive constant endemic state.

In conclusion, this study delivers predictions concerning the spread of infection under the influence of global climate change and the expansion of mosquito habitats. While the model established here is a simplified version, the results and methodologies presented should prove beneficial for future research into other vector-borne diseases in the context of climate change. Moreover, we identify future work in the form of more comprehensive models, such as those incorporating the larval stage of mosquitoes and infection delays attributable to exposure stages.

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