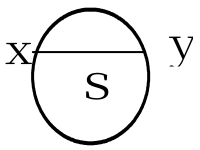
CHAP 16 Non-linear Programming

16.1 Mathematical Background

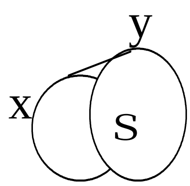
A.Convex hall

—convex set

A set S in R^n is convex if $x_0, y_0 \in s$ and $\lambda \in [0,1]$ such that $\lambda x_0 + (1-\lambda) y_0 \in s$



convex

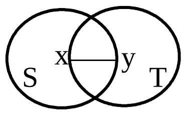


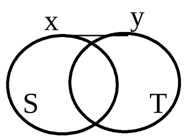
non-convex

—S and T are convex sets in R^n

$S \cap T$ is a convex set

$S \cup T$ is not necessarily a convex set





—the feasible set of LP is always a convex set

—a convex combinations of vectors ($x_1, \ldots\ldots, x_n$) is defined as

$$x = \sum_i \lambda_i x_i = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \text{ where } \lambda_i \geq 0 \ \ and \ \sum_i \lambda_i = 1$$

—convex hull

a smallest convex set containing S where S is an non-convex set in R^n

B.Interior point

—norm

Euclidean distance between two points

x and y in R^n

$x = (x_1, \ldots\ldots, x_n), y = (y_1, \ldots\ldots, y_n)$

$$norm = \|x - y\| = \left[\sum^n (x_i - y_i)^2 \right]^{\frac{1}{2}}$$

—neighborhood

(1) a set of ε neighborhood about the point x_0 is defined as $S_0 = \{ \ x \mid \|x - x_0\| < \varepsilon, \varepsilon > 0 \}$

(2) interior point : a point $x_0 \in S$ is an interior point of S if $S_0 \subset S$

C.Close set and open set

—open set : a set S is open if it is composed of interior points only

—close set : a set S is closed if it contains boundary points

D.Bounded and unbounded set

—a set is bounded if big M such that $\|x\| < M$

—a set is not bounded is an unbounded set

E.Compact set

a set is compact if it is a closed and bounded

F.If feasible set is compact then solution exists

G.Convex and concave function

—one variable

convex concave

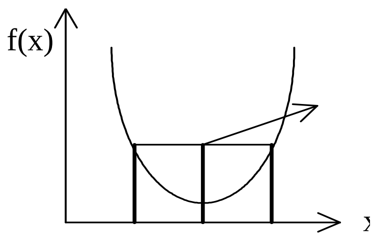
f is convex if $f''(x) \geq 0, \forall x$ (strictly without "=" sign)

f is concave if $f''(x) \leq 0, \forall x$ (strictly without "=" sign)

—definition

a function is convex if and only if two points x_1 and x_2 , and $\lambda \in [0,1]$ such that

$f[\lambda x_1 + (1-\lambda)x_2] \leq \lambda f(x_1) + (1-\lambda) f(x_2)$

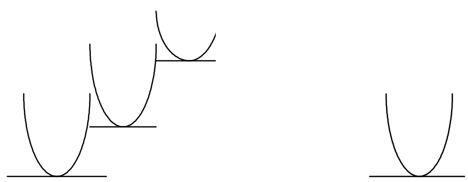


—properties of convex function

(1) f"(x)≧0

(2) the chord joing any two points on the curve lies above the curve

(3) for a convex function a local minimum is always a global minimum



non-convex concave

—the linear approximation of f(x) always under estimate the true function value

pf : from Taylor's series expansion

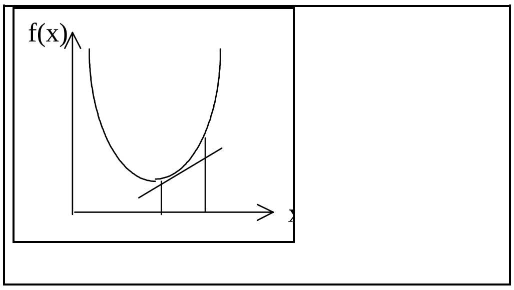
$$f(x_0+h)=\frac{1}{0!}f(x_0)+\frac{1}{1!}f^{'}(x_0)h+\frac{1}{2!}f^{''}(x_0)h^2+\frac{1}{3!}f^{'''}(x_0)h^3+\cdots$$

$let\;\; x=x_0+h \quad then$

$$f(x)=f(x_0)+f^{'}(x_0)(x-x_0)+\frac{1}{2!}f^{''}(x_0)(x-x_0)^2+\cdots$$

$$\hat{f}(x)=f(x_0)+f^{'}(x_0)(x-x_0)=f(x)$$

$$\Rightarrow f(x)\geq f(x_0)+f^{'}(x_0)(x-x_0)$$



—for many variables

gradient of a function

$$\nabla f(x)=\left\lfloor \frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\cdots,\frac{\partial f}{\partial x_n}\right\rfloor$$

Hession matrix of a function

$$H=\nabla f(x)=\left\lfloor \frac{\partial^2 f}{\partial x_1 \partial x_2}\right\rfloor$$

$$H=\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\partial x_n} \\ \frac{\partial^2 f}{\partial x_2\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial^2 f}{\partial x_n\partial x_1} & \frac{\partial^2 f}{\partial x_n\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$$H=\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$f^{''}(x)\geq 0 \rightarrow$ H is positive-semidifinite (p.s.d.)

$\rightarrow$ f(x) is a convex function

$$p.s.d \Leftrightarrow x^T Hx \geq 0, \forall x, x\neq 0$$

$f^{''}(x)>0 \rightarrow$ H is positive-definite (p.d.)

$$p.d \Leftrightarrow x^T Hx \geq 0, \forall x, x\neq 0$$

$$D_1=a_{11}$$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$D_n = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

－the leading principal minor of H matrix

H is p.d. $\Leftrightarrow D_1 > 0, D_2 > 0, \cdots, D_n > 0$

$\rightarrow$ f(x) is strictly convex function

H is p.s.d. $\Leftrightarrow D_1 \geq 0, D_2 \geq 0, \cdots, D_n \geq 0$

H is n.d. $\Leftrightarrow (-1)^n D_n > 0, (D_1 < 0, D_2 < 0, \cdots, D_n < 0)$

H is n.s.d. $\Leftrightarrow (-1)^n D_n \geq 0, (D_1 \leq 0, D_2 \leq 0, \cdots, D_n \leq 0)$

－example

$$f(x_1, x_2, x_3) = 3x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3 - 6x_1 - 4x_2 - 2x_3$$

$$\nabla f(x) = \begin{pmatrix} 6x_1 - 2x_2 - 2x_3 - 6 \\ 4x_2 - 2x_1 + 2x_3 - 4 \\ 2x_3 - 2x_1 + 2x_2 - 2 \end{pmatrix}$$

$$H = \begin{bmatrix} 6 & -2 & -2 \\ -2 & 4 & 2 \\ -2 & 2 & 2 \end{bmatrix}$$

$$D_1 = |6| > 0$$

$$D_2 = \begin{vmatrix} 6 & -2 \\ -2 & 4 \end{vmatrix} = 20 > 0$$

$$D_3 = \begin{vmatrix} 6 & -2 & -2 \\ -2 & 4 & 2 \\ -2 & 2 & 2 \end{vmatrix} = 16 > 0$$

H is p.d. $\rightarrow$ f(x) strictly convex

$\rightarrow$ minimize problem

16.2 NLP as One Dimension

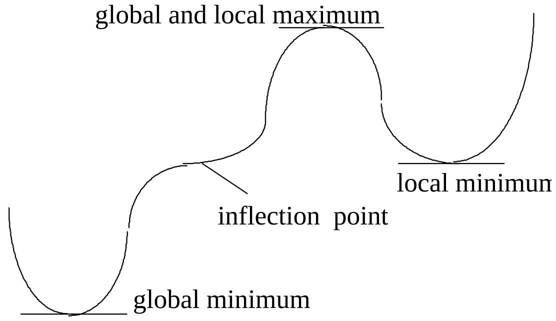
A.Unconstrained optimization

－1st order condition

(1) stationary points

(2) solution

(3) necessary condition



—2nd order condition

(1) uniqueness

(2) sufficient condition

$$\begin{cases} f''(x) > 0, \ \ \forall x \Rightarrow \min imum \\ f''(x) < 0, \ \ \forall x \Rightarrow \max imum \end{cases}$$

—theorem 1

a necessary condition for x^* to be a local minimum of f(x) on the open interval (a,b) f is twice differentiable are that $\frac{\partial f}{\partial x}\Big|_{x=x^*} = 0 \ and \ \frac{\partial^2 f}{\partial x^2}\Big|_{x=x^*} \geq 0$

—theorem 2

(1) x^* is a stationary point (f'(x)=0)

2nd order sufficient : $f''(x^*) \geq 0 \Rightarrow local \quad \min imum$

4th order sufficient : $f'(x^*) = f''(x^*) = f'''(x^*) = 0$

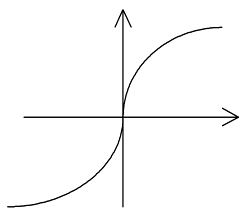
$f^4(x^*) > 0 \Rightarrow x^*$ is a local minimum

(2n)th order sufficient : $f'(x^*) = f''(x^*) = \cdots = f^{(2n-1)}(x^*) = 0$

$f^{2n}(x^*) > 0 \Rightarrow x^*$ is a local minimum

(2) example

$f(x)=x^3$
$f'(x)=3x^2$
$f''(x)=6x$
$f'''(x)=6>0$

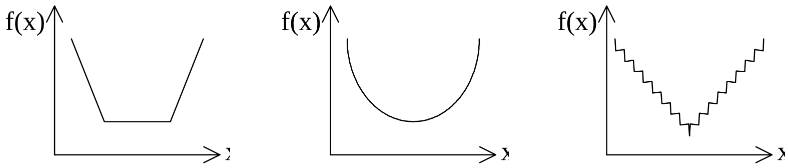


—theorem 3

x^* is a local minimum of f(x) if and only if $f(x^*) \le f(x)$ for all x with in a distance S from x^*. x^* is a global minimum of f(x) if and only if $f(x^*) \le f(x)$ for all x.

—theorem 4

a sufficient condition for x^* to be a local minimum is f(x) is strictly convex in the neighborhood of x^*



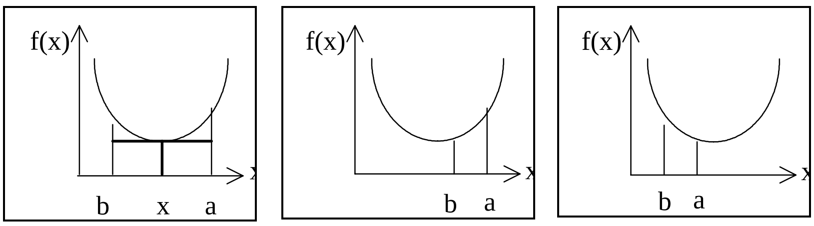
convex　　　unimodel strictly convex　　　unimodel non-convex

B.Constrained optimization

$$\min \ f(x) \quad s.t. \ b \le x \le a$$

$$\min \ f(x) \quad s.t. \ b \le x, \ x \le a$$

$$\min \ f(x) \quad s.t. \ x \le b, \ -x \le a$$



interior point $\frac{\partial f}{\partial x} = 0$　　　x≧b $\frac{\partial f(x^*)}{\partial x} \ge 0$　　　x≦a $\frac{\partial f(x^*)}{\partial x} \le 0$

—general form

$$\min \ f(x)$$
$$s.t. \ g_j(x) \ge b_j, \ \forall j$$

f'(x) and g'(x) has same sign

(1) $\frac{\partial f(x^*)}{\partial x} = 0$

(2) $\frac{\partial f(x)}{\partial x} > 0, \ \frac{\partial g_1(x)}{\partial x} = 1$

(3) $\frac{\partial f(x)}{\partial x} < 0, \ \frac{\partial g_2(x)}{\partial x} = -1$

—general Kuhn-Tucker condition (KKT)

(1) $\frac{\partial f(x)}{\partial x} = \sum_j U_j \frac{\partial g_j(x^*)}{\partial x}$

(2) $U_j \left[b_j - g_j(x) \right] = 0, \ \forall j$

(3) $U_j \geq 0, \ \forall j$

(4) $g_j(x) \geq b_j, \ \forall j$

16.3 NLP in Multiple-dimension Problem

A.Unconstrained problem

Min.f(x)

－1st order condition : $\nabla f(x) = 0$

－2nd order condition

(1) H is p.d. : unique minimum

(2) H is p.s.d. : alternative minimum

B.Constrained problem

Min. f（x）

s.t. $g_j(x) \geq b_j, \ \forall j$

－necessary condition : KKT condition

－$\nabla f(x) = U \nabla g(x)$

－$U_j \left[b_j - g_j(x) \right] = 0, \ \forall j$

－$U_j \geq 0, \ \forall j$

－$g_j(x) \geq b_j, \ \forall j$

－example

$Min. \quad f(x_1, x_2) = x_1^2 + 2x_1x_2 + 2x_2^2 - 2x_1 - 4x$

s.t. $x_1 + x_2 \geq 2$

$$\begin{cases} 2x_1^* + 2x_2^* - 2 = U \\ 2x_1^* + 4x_2^* - 4 = U \\ U(x_1^* + x_2^* - 2) = 0 \end{cases}$$

$$\left. \begin{array}{c} x_1^* + x_2^* \geq 2 \\ U \geq 0 \end{array} \right\} \Rightarrow necessary \ condition$$

$\Rightarrow (x_1^*, x_2^*) = (1,1)$

$\Rightarrow f(x) = -1, \ U = 2$

if it is unconstrained

$$\nabla f(x) = \begin{bmatrix} 2x_1 + 2x_2 - 2 \\ 4x_1 + 2x_1 - 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow (x_1^*, x_2^*) = (0,1)$

$\Rightarrow f(x) = -2$

$$H = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \ is \ \ p.d. \ \Rightarrow unique$$

－for equality constraints : use Lagrange Multipliers

Min. f（x）$= x^2 + y^2$

s.t. $g_j(x_j) = b_j, \ \forall j$

$$\Rightarrow \min. \ L = f(x) + \sum_j U_j \left[g_j(x) - b_j \right]$$

$$\Rightarrow \frac{\partial L}{\partial U_j} = 0, \ \frac{\partial L}{\partial x_j} = 0$$

$$\frac{\partial L}{\partial x} = 2x - 2\lambda_1 - 3\lambda_2 = 0$$

$$\frac{\partial L}{\partial y} = 2y - \lambda_1 + 2\lambda_2 = 0$$

$$\Rightarrow \frac{\partial L}{\partial U_1} = 2x + y - 3 = 0$$

$$\Rightarrow \frac{\partial L}{\partial U_2} = 3x + y - 6 = 0$$

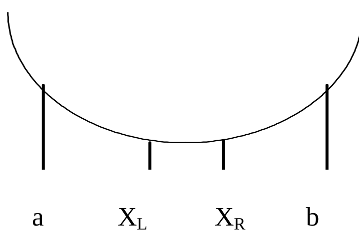
16.4 NLP Algorithms

A.Unconstrained NLP algorithms

－direct search method : one dimension

(1) Golden Section Method

Min. f(X)



$$\left. \begin{aligned} b - X_R &= X_L - a \\ X_R - X_L &= \Delta \end{aligned} \right\} \Rightarrow \begin{aligned} X_L &= a + \tfrac{b-a+\Delta}{2} \\ X_R &= a + \tfrac{b-a+\Delta}{2} \end{aligned}$$

$$\left. \begin{aligned} &if \ f(X_L) > f(X_R) \ then \ X_L = a \\ &if \ f(X_L) < f(X_R) \ then \ X_R = b \end{aligned} \right\} if \ b - a \le \Delta \ or \ \left| f(X_R - f(X_L)) \right| \le \Delta \quad \text{stop}$$

$$if \ f(X_L) = f(X_R) \ then \ X_L \le X^* \le X_R$$

. example

$$\text{Max.} \quad f(X) = \begin{cases} 3X & , \ 0 \le X \le 2 \\ -\frac{X}{3} + \frac{20}{3}, & 2 \le X \le 3 \end{cases}$$

(2) Midpoint Search Method

. step 0 : find $\overline{X}, \underline{X}$ set ε

. step 1 : let $X = \dfrac{\overline{X} + \underline{X}}{2}$

. step 2 : caculate $\dfrac{df(x')}{dx}$

. step 3 : if $\dfrac{df(x')}{dx} \geq 0 \Rightarrow \underline{X} = X'$

. step 4 :if $\dfrac{df(x')}{dx} \leq 0 \Rightarrow \overline{X} = X'$

—gradient search method : multiple dimension

(1) step 0 : select ε and initial trial solution X

(2) step 1 : set $X_j = X'_j + t\left(\dfrac{\partial f}{\partial X}\right)_{x=x'}$

(3) step 2 : substitude X_j into f(X)

(4) step 3 : use one-dimension search method find t^* that maximize $f\left(X' + t\nabla f(x)\right)$

(5) step 4 : if $\left(\dfrac{\partial f}{\partial X}\right) \leq \varepsilon$ for $\forall j$ stop

(6) step 5 : $X = X' + t^* \nabla f(x)$, go to step 1

(7) example

$$f(X) = 2X_1X_2 + 2X_2 - X_1^2 - 2X_2^2$$

$$\frac{\partial f}{\partial X_1} = 2X_2 - 2X_1$$

$$\frac{\partial f}{\partial X_2} = 2X_1 + 2 - 4X_2$$

step 0 : $\varepsilon = 0.001, \quad X' = (0,0), \quad \nabla f(0,0) = (0,2)$

step 1 : X_1=0+t(0)=0, X_2=0+t(2)=2t

step 2 : $f\left(X' + t\nabla f(x)\right) = f(0,2t) = 4t - 8t^2$

step 3 : $\frac{d}{dt}\left(4t - 8t^2\right) = 4 - 16t = 0, \;\; t^* = \frac{1}{4}$

step 4 : $\frac{\partial f}{\partial X_j}_{X=X'} = 2\left(\frac{1}{4}\right) = \frac{1}{2} > 0.001$

step 5 : $X' = (0,0) + \frac{1}{4}(0,2) = \left(0, \frac{1}{2}\right)$

step 6 : $\nabla f\left(0, \frac{1}{2}\right) = (1,0)$

step 7 :$X = (0, \frac{1}{2}) + t(1, 0) = (t, \frac{1}{2})$

	X'	$\nabla f(X')$	$X' + t\nabla f(X')$	$f\left(X' + t\nabla f(X')\right)$	t^*	$X' + t^*\nabla f(X')$
1	$(0,0)$	$(0,2)$	$(0,2t)$	$4t - 8t^2$	$\frac{1}{4}$	$\left(0, \frac{1}{2}\right)$
2	$\left(0, \frac{1}{2}\right)$	$(1,0)$	$\left(t, \frac{1}{2}\right)$	$t - t^2 + \frac{1}{2}$	$\frac{1}{2}$	$\left(\frac{1}{2}, \frac{1}{2}\right)$

$f\left(X' + t\nabla f(X')\right) = f\left(0 + t, \frac{1}{2} + 0t\right)$

$$= (2t)\tfrac{1}{2} + 2\left(\tfrac{1}{2}\right) - t^2 - 2\left(\tfrac{1}{2}\right)^2$$
$$= t - t^2 + \tfrac{1}{2}$$

step 8 : $f\left(t^*, \tfrac{1}{2}\right) = \underset{t\ge 0}{Max}\, f\left(t, \tfrac{1}{2}\right) = \underset{t\ge 0}{Max}\left\{t - t^2 + \tfrac{1}{2}\right\}$

$$\tfrac{d}{dt}\left\{t - t^2 + \tfrac{1}{2}\right\} = 1 - 2t = 0$$
$$t^* = \tfrac{1}{2}$$
$$X^{'} = \left(0, \tfrac{1}{2}\right) + \tfrac{1}{2}(1,0) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

B. Constrained NLP algorithm

— Lagrangean method

— Convert to be unconstrained problem by introducing Lagrangean multiples

$$L(x,\lambda) = f(x) - \lambda g(x)$$

— use partial derivation to identify stainary point

$$\tfrac{\partial L}{\partial \lambda} = 0$$
$$\tfrac{\partial L}{\partial x} = 0$$

— use Hession matrix to test its sufficient condition

Ex:

Min. $x_1^2 + x_2^2 + x_3^2$

s.t. $4x_1 + x_2^2 + 2x_3 - 14 = 0$

$$L(\lambda, X) = x_1^2 + x_2^2 + x_3^2 - \lambda_1\left(4x_1 + x_2^2 + 2x_3 - 14\right)$$

$$\tfrac{\partial L}{\partial x_1} = 2x_1 - 4\lambda = 0$$
$$\tfrac{\partial L}{\partial x_2} = 2x_2 - 2\lambda x_2 = 0$$
$$\tfrac{\partial L}{\partial x_3} = 2x_3 - 2\lambda = 0$$
$$\tfrac{\partial L}{\partial \lambda} = -\left(4x_1 + x_2^2 + 2x_3 - 14\right) = 0$$

$$\Rightarrow (x_0, \lambda_0)_1 = (2,2,1,1)$$
$$(x_0, \lambda_0)_2 = (2,-2,1,1)$$
$$(x_0, \lambda_0)_3 = (2.8,0,1.4,1.4)$$

$$H = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$H_1 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$D_1 = |2| = 2 > 0$

$D_2 = \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} = 0$ p.s.d $\Rightarrow$ convex $\Rightarrow$ local min.

$D_3 = 0$

$$H_2 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{vmatrix} \qquad \Rightarrow \text{p.s.d}$$

$$H_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & -0.8 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$D_1 = |2| = 2 > 0$$

$$D_2 = \begin{vmatrix} 2 & 0 \\ 0 & -0.8 \end{vmatrix} = -1.6 < 0$$

$$D_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & -0.8 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -3.2 < 0$$

16.5 Convex Programming

－min ：convex object function

－max ：concave object function

－solution approach

－Gradient algorithm： gradient search procedure

－sequential unconstrained algorithm：

use penalty function and barrier function to convert to unconstrained problem.

－sequential approach algorithm

—replace objective function by linear/guadraic approximation methods

—Frank-Wolfe Algorithm

—suitable for linear constrained convex programming

—use first-order Taylar series

$$f'(x) = f(x') + \sum_{r=1}^{n} \frac{\partial f(x')}{\partial x_j} \cdot (x - x'_j) = f(x') + \nabla f(x')(x - x')$$

—drop to give an equivalent linear objectives

$$g(x) = \nabla f'(x)x = \sum_{j=1}^{n} C_j x_j \quad \text{where } C_j = \frac{\partial f(x)}{\partial x_j} \text{ at } x = x'$$

—in linear O.F.

it increase steady as one more from x' to x_{LP}

—in convex O.F.

may be increase all the way from x' to x_{LP}

—not accept x_{LP} as next trial solution

we choose the point the maximize the nonlinear objective function along this line segment as trial solution

—procedure

step 0：find an initial trial solution $x^{(0)}$, set k=1

step 1：for r=1,2,…….,n, evaluate $\frac{\partial f(x)}{\partial x_j}$ at $x = x^{k-1}$

step 2：find an optimal $x_{LP}^{(k)}$ by LP model

Max. $g(x) = \sum_{r=1}^{n} C_j x_j$

s.t. $Ax \leq x$

step 3：for t $(0 \leq t \leq 1)$ set

$h(t) = f(x)$ for $x = x^{(k-1)} + t(x_{LP}^{(k)} - x^{(k-1)})$

—use search procedure or linear algebra to find x^k such as to maximize $h(t)$

step 4：if $x^k - x^{k-1} < \varepsilon$ stop

otherwise $k = k+1$, go to step 1

EX： Max. $f(x) = 5x_1 - x_1^2 + 8x_2 - 2x_2^2$

s.t. $3x_1 + 2x_2 \leq 6$

$x_1, x_2 \geq 0$

sol：$\frac{\partial f(x)}{\partial x_1} = 5 - 2x_1 \quad \frac{\partial f(x)}{\partial x_2} = 8 - 4x_2$

use $x^0 = (0,0)$ as $C_1 = 5,\ C_2 = 8$

solve $5x_1 + 8x_2$

s.t. $3x_1 + 2x_2 \leq 6$

$\Rightarrow x_{LP}^{(*)} = (0,3)$

$(x_1, x_2) = (0,0) + t[(0,3) - (0,0)] = (0,3t)$

$h(t) = f(0,3t) = 8(3t) - 2(3t)^2 = 24t - 18t^2$

$$\frac{\partial h(t)}{\partial t} = 24 - 36t = 0$$

$\Rightarrow t = \frac{2}{3}$

$x^{(1)} = (0,0) + \frac{2}{3}[(0,3)] - (0,0) = (0,2)$

$C_1 = 5 - 2(0) = 5$, $C_2 = 8 - 4(2) = 0$

$\Rightarrow$ solve Max. $5x_1$

s.t. $3x_1 + 2x_2 \le 6$

$\Rightarrow x_{LP}^{(2)} = (2,0)$

$X = (0,2) + t[(2,0) - (0,2)] = (2t, 2 - 2t)$

$h(t) = f(2t, 2-2t) = 5(2t) - (2t)^2 + 8(2-2t) - 2(2-2t)^2$

$= 8 + 10t - 12t^2$

$h'(t) = 10 - 24t = 0 \quad t = \frac{5}{12}$

$x^{(2)} = (0,2) + \frac{5}{12}[(2,0) - (0,2)] = \left(\frac{5}{6}, \frac{7}{6}\right)$

[例題] 求 $f(x_1,x_2)=\frac{4}{3}x_1^3+2x_1^2x_2+x_1x_2^2-16x_1-6x_2$ 之極值。

解：

$$\nabla f\left[4x_1^2+4x_1x_2+x_2^2-16 \quad 2x_1+x_1x_2-6\right]=[0,0]$$

$4x_1^2+4x_1x_2+x_2^2=16$ ……………..(4)
$2x_1^2+2x_1x_2=6$ ……………………(5)

由(4)，(5)
$2x_2^2=4$
$x_2=\pm 2$

令 $x_2=2$ 則 $x_1=1$ 或 -3
$x_2=-2$ 則 $x_1=3$ 或 -1
則 Z^* 可能為(1，2)，(-3，2)，(3，-2)，(-1，-2)

而 $H(Z)=\begin{bmatrix}8x_1+4x_2 & 4x_1+2x_2\\ 4x_1+2x_2 & 2x_2\end{bmatrix}$

$H(1,2)=\begin{bmatrix}16 & 8\\ 8 & 2\end{bmatrix}$ 為不定型，故(1，2)為鞍點。

$H(-3,2)=\begin{bmatrix}-16 & -8\\ -8 & -6\end{bmatrix}$ 為負型，故 $f(-3,2)=24$ 為局部最大值。

$H(3,-2)=\begin{bmatrix}16 & 8\\ 8 & 6\end{bmatrix}$ 為正型，故 $f(3,-2)=-24$ 為局部最小值。

$H(-1,-2)=\begin{bmatrix}-16 & -8\\ -8 & -2\end{bmatrix}$ 為不定型，故(-1，-2)為鞍點。

[例題] 在 $0\le x\le 2$ 區間內，求 $f(x)=12x-3x^4-2x^6$ 之極大值。

解：

令 $\Delta=0.01$，則

$$x_1=x_L+\frac{x_R-x_L-\Delta}{2}$$

$$x_2=x_L+\frac{x_R-x_L+\Delta}{2}$$

綜合計算如下表：

運算次數	x_L	x_R	x_1	x_2	$f(x_1)$	$f(x_2)$
0	0	2	0.995	1.005	7.0588*	6.9388
1	0	1.005	0.4975	0.5075	5.7559	5.8568*
2	0.4975	1.005	0.7463	0.7563	7.6794	7.7198*
3	0.7463	1.005	0.8707	0.8807	7.8527*	7.8303
4	0.7463	0.8807	0.8085	0.8185	7.8615	7.8742*
5	0.8085	0.8807	0.8396	0.8496	7.8838*	7.8800
6	0.8085	0.8496	0.8241	0.8341	7.8790	7.8836*
7	0.8241	0.8496	0.8319	0.8419	7.8831	7.8834*
8	0.8319	0.8496	0.8358	0.8458	7.8839*	7.8821
9	0.9319	0.8458	0.8339	0.8439	7.8836*	7.8829
10	0.8319	0.8439	0.8329	0.8429	7.8833*	7.8831
11	0.8319	0.8429	0.8324	0.8424	7.8832	7.8833*
12	0.8324	0.8429	0.8327	0.8427	7.8833*	7.8832
13	0.8324	0.8427	0.8326	0.8426	7.8832	7.8833*
14	0.8326	0.8427	0.8327	0.8427	7.8833*	7.8832

Stop

所以 $0.8326 \le x^* \le 0.8427$

取 $x^* = (0.8326 + 0.8427)/2 = 0.83765$

$f(0.83765) = 7.8839$

雖然 $x^* = 0.83765$ 不是真正最適化解，但其誤差在 $5.05 \times 10^{-3}(0.8427 - 0.83765 = 5.05 \times 10^{-3})$ 已可被我們接受了。

另解：

運算次數	$\dfrac{df(x)}{dx}$	x_L	x_R	New x'	$f(x')$
0		0	2	1	7
1	-12	0	1	0.5	5.7812
2	10.12	0.5	1	0.75	7.6948
3	4.09	0.75	1	0.875	7.8439
4	-2.19	0.75	0.875	0.8125	7.8672
5	1.31	0.8125	0.875	0.84375	7.8829
6	-0.34	0.8125	0.84375	0.828125	7.8815
7	0.51	0.828125	0.84375	0.8359375	7.8839
Stop					

$x^* \approx 0.836$

$0.828125 \le x^* \le 0.84375$

［例題］ $Max \quad f(Z) = 3x_1^2 + x_2^2 + 2x_1x_2 + 6x_1 + 2x_2$

$S.t \quad 2x_1 - x_2 = 4$

解：

$$L(Z,\lambda) = 3x_1^2 + x_2^2 + 2x_1x_2 + 6x_1 + 2x_2 - \lambda(2x_1 - x_2 - 4)$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= 6x_1 + 2x_2 + 6 - 2\lambda = 0 \\ \frac{\partial L}{\partial x_2} &= 2x_1 + 2x_2 + 2 + \lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= 2x_1 - x_2 - 4 = 0 \end{aligned}\right\} \text{解得 } x^*_1 = \frac{7}{11}，x^*_2 = -\frac{30}{11}，\lambda^* = \frac{24}{11}$$

所以 $f(\frac{7}{11}, \frac{-30}{11}) = 85.7$