CHAP15 Markov Chain

15.1 Introduction

A.Stochastic Process

—an indexed collection of random variable $\{X_t\}$, where the index t run through a given set T

— $\{X_t\}$represents a measurable characteristics of interest at time t

—with finite number of mutually exclusive and exhausive categories or states

—points in time maybe space equally or depends on system behavior

B.Markov Property

— $P\{X_{t+1} = j | X_0 = k_0, X_1 = k_1,..., X_t = i\} = P\{X_{t+1} = j | X_t = i\}$is the conditional prob. of any future event is independent of the past event and depends only on the present state of process

C.State Space (M)

—the set of finite states

D.Chain

—a discrete stochastic process with a limited state space

E.Conditional Probability

— $P_{ij}^{(n)} = P\{X_{t+n} = j | X_t = i\} = P\{X_n = j | X_0 = i\} \geq 0, \sum_{j=1}^{M} P_{ij}^{(n)} = 1, for \forall i$

F.Transition Matrix

$$P^{(n)} = \begin{matrix} 0 \\ 1 \\ 2 \\ . \\ . \\ m \end{matrix} \begin{bmatrix} P_{00}^{(n)} & P_{01}^{(n)} & P_{02}^{(n)} & . & . & P_{0m}^{(n)} \\ . & . & & & & . \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ P_{m0}^{(n)} & . & . & . & . & P_{mm}^{(n)} \end{bmatrix}$$

G.A Finite-State Markov Chain

—a stochastic process with $\{X_t\}$

—a finite number of states

—the Markov property

—stationary transition probabilities

—a set of initial probabilities, $P\{X_0 = i\} for \forall i$

—example:

state 0 : The stock increased today and increased yesterday

state 1 : The stock increased today and decreased yesterday state 2 : The stock decreased today and increased yesterday

state 3 : The stock decreased today and decreased yesterday

$$P = \begin{bmatrix} 0.9 & 0 & 0.1 & 0 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0.3 & 0 & 0.7 \end{bmatrix}$$

15.2 Chapman-Kolmogorov Equations

A.Definition

$$P_{ij}^{(n)} = \sum_{k=0}^{m} P_{ik}^{(\nu)} P_{kj}^{(n-\nu)} \qquad for \forall i, j, n \qquad and\, 0 \le \nu \le n$$

$$P^{(2)} = \sum_{k=0}^{m} P_{ik} P_{kj} = P^2$$

$$P^{(n)} = P \cdot P \cdot P \cdot \ldots \cdot P = P \cdot P^{n-1}$$

B.P_{ij} is conditional prob. for unconditional prob., we need information on prob. of initial state $[Qx_0(i)]$

$Qx_0(i) = P\{x_0 = i\}$ for $i = 0,1,2,\ldots,m$

$$P\{x_n = j\} = \sum_{k=0}^{m} Qx_0(i) P_{ij}^{(n)}$$

－example:

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}, Qx_0(0) = 0.5, Qx_0(1) = 0.5$$

$$P_{ij}^{(2)} = [0.5 \quad 0.5] \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}^2$$

$$= [0.5 \quad 0.5] \begin{bmatrix} 0.70 & 0.30 \\ 0.45 & 0.55 \end{bmatrix}$$

$$= [0.575 \quad 0.425]$$

15.3 Classification of states of Markov Chain

A.Accessible from i to j ($i \to j$)

If $P_{ij}^{(n)} > 0$ for some $n \geq 0$

B.Communicate state i and j ($i \leftrightarrow j$)

C.General rule of communication

$- i \leftrightarrow j$

$- i \leftrightarrow j \Rightarrow j \leftrightarrow i$

$- i \leftrightarrow j, j \leftrightarrow k \Rightarrow i \leftrightarrow k$

D.Disjoint Classes

based on communication , a Markov chain can consist of one or more disjoint classes

E.Irreducible Markov Chain

all states communicate with one another

F.Markov Chain State Type

－recurrent State

$f_{ii} = P_r$ {return to state i given it starts in state i}=1

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$$

－transient State

$$f_{ii} < 1 \Rightarrow \lim_{n \to \infty} P_{ii}^{(n)} = \infty$$

－absorbing State

$$P_{ii} = 1$$

－expect number of time-period that the process in state i

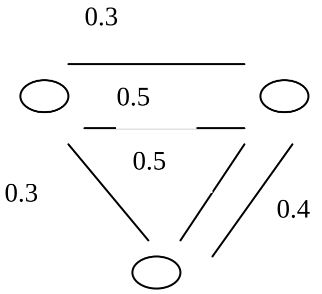
$$ie = \frac{1}{1 - f_{ii}}$$

－recurrent state is a class property

－a-finite-state Markov chain, not all states can be transit

－an-irreducible finite state Markov chain are recurrent

—example:



$$P^{(n)} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{bmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

absorbing state = 2

transient state = 3,4

recurrent state = 0,1

—the period of state i

$P_{ii}^{(n)} = 0$ for $\forall n \neq t, 2t, 3t$

—the aperiodic state

a state with period t = 1

—the positive recurrent

starting in state i, the expected time to recurrent state i is finite

—the null recurrent

if expected time to reenter is infinite

—the ergodic state

a positive recurrent state that are aperiodic

—the regular Markov Chain

an irreducible Markov Chain with all states are ergodic

—the state transition diagram

$$P = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.5 & 0 & 0.5 \\ 0 & 0.4 & 0.6 \end{bmatrix}$$

$$\left. \begin{matrix} 1 \leftrightarrow 2 \\ 2 \leftrightarrow 3 \end{matrix} \right\} 1 \leftrightarrow 3$$

—summary

$$state \begin{cases} recurrent & \begin{cases} positive & \begin{cases} t = 1 \\ t > 1 \end{cases} \\ null \end{cases} \\ transient \end{cases}$$

$$Markov\ Chain \begin{cases} Irreducible & \begin{cases} ergodic & (t = 1) \\ positive & (t > 1) \end{cases} \\ reducible & \begin{cases} two\ or\ more\ recurrent\ classes \\ at\ least\ one\ recurrent\ class \\ at\ least\ one\ transitent\ class \end{cases} \end{cases}$$

15.4 First Passage Time

A.Define

—the length of time in going from state i to state j for the first time

—the expected first passage time from state i to j

$$U_{ij} = \begin{cases} \infty & \text{if} \quad \sum_{n=1}^{\infty} f_{ij}^{(n)} < 1 \\ \sum_{n=1}^{\infty} f_{ij}^{(n)} & \text{if} \quad \sum_{n=1}^{\infty} f_{ij}^{(n)} = 1 \end{cases}$$

—where

$f_{ij}^{(n)}$ = prob. of the first passage time from state i go into state j is equal to n

$$f_{ij}^{(1)} = P_{ij}^{(1)} = P_{ij}$$

$$f_{ij}^{(2)} = P_{ij}^{(2)} - f_{ij}^{(1)} P_{ij}$$

$$|$$

$$f_{ij}^{(n)} = P_{ij}^{(n)} - f_{ij}^{(1)} P_{ij}^{(n-1)} - f_{ij}^{(2)} P_{ij}^{(n-2)} - \ldots - f_{ij}^{(n-1)} P_{ij}$$

$$= P_{ij}^{(n)} - \sum_{k=1}^{n-1} f_{ij}^{(k)} P_{ij}^{(n-k)}$$

$$\Rightarrow U_{ij} = 1 + \sum_{k \neq j} P_{ik} U$$

—example:

$$P = \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 0.080 & 0.184 & 0.368 & 0.368 \\ 0.632 & 0.368 & 0 & 0 \\ 0.264 & 0.368 & 0.368 & 0 \\ 0.080 & 0.184 & 0.368 & 0.368 \end{bmatrix}$$

$$U_{30} = 1 + P_{31}U_{10} + P_{32}U_{20} + P_{33}U_{30}$$
$$U_{20} = 1 + P_{21}U_{10} + P_{22}U_{20} + P_{23}U_{30}$$
$$U_{10} = 1 + P_{11}U_{10} + P_{12}U_{20} + P_{13}U_{30}$$

$$U_{30} = 1 + 0.184U_{10} + 0.368U_{20} + 0.368U_{30}$$
$$U_{20} = 1 + 0.368U_{10} + 0.368U_{20}$$
$$U_{10} = 1 + 0.368U_{10}$$

$$\Rightarrow U_{10} = 1.58, U_{20} = 2.51, U_{30} = 3.50$$

15.5 Long-Run Properties of Markov Chain

A. After a larger number of transitions

— there is a limited probability the system will be in state j

— the probability is independent of the initial state i.e. $\lim_{n\to\infty} P_{ij}^{(n)} = \pi$

— π_i specify the following steady-state equation

$$\pi_i > 0$$

$$\pi_i = \sum_{i=1}^{m} \pi_i P_{ij}$$

$$\sum_{j=0}^{m} \pi_i = 1$$

— π_i is the steady-state probability

$$\pi_i = \frac{1}{U_{ii}}$$

$$U_{ii} = \frac{1}{\pi_i} \quad , \left(U_{jj} = \frac{1}{\pi_j} \right)$$

B. Observations

— π_i does not imply the process settle down into one state

— in steady-state equations, there are (m+2) equations

— if i and j are recurrent states belong to different classes

$$P_{ij}^{(n)} = 0 \quad for \quad \forall n$$

— example:

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.4 \end{bmatrix} \quad \Rightarrow \begin{cases} 0.7\pi_1 + 0.4\pi_2 = \pi_1 \\ 0.3\pi_1 + 0.6\pi_2 = \pi_1 \\ \pi_1 + \pi_2 = 1 \end{cases} \quad \Rightarrow \pi_1 = \frac{4}{7}, \pi_2 = \frac{3}{7}$$

15.6 Absorption States

A.For an absorbing state k

—the first passage probability from i to k is called the probability of absorption into k (f_{ik})

— f_{ik} satisfies the following equations

$$f_{ik} = \sum_{j=0}^{m} P_{ij} f_{jk}$$

s.t. $f_{kk} = 1$, $f_{ik} = 0$ if state i is recurrent and $i \neq k$

—solution method

(1) identify i, k such that $f_{kk} = 1, f_{ik} = 0$

(2) solution the rest equation to find f_{ik}

—example: A firm with two machines. If the machine is in normal condition the probability of the machine is still in normal condition in the next day is 0.8, if the machine is breakdown, it takes two days to repair. There is only one mechanic (A) and he can only repair one machine a time. The cost of operating a machine is $500/day. If the machine is breakdown, the cost of purchase from other firm is $1,200 per day. The repair cost is $1,500 per day

(1) average daily cost

(2) if we hire mechanic B, the machine can be repaired in one day with cost of $2,500 per day

(3) if we choose to keep one machine only , the production of one machine is replaced by purchase from other firm, is it worth?

sol:

(1) Definition of states

1two machines are normal

2one normalone breakdown (but not fixed)

3one normalone under repair for one day

4two breakdown

5two breakdownone not fixed, one has been fixed for one day

$$P = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{bmatrix} 0.64 & 0.32 & 0 & 0.04 & 0 \\ 0 & 0 & 0.8 & 0 & 0.2 \\ 0.80 & 0.20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

steady-state prob. $\pi_i = [0.45 \quad 0.26 \quad 0.20 \quad 0.02 \quad 0.07]$

500*2+1500*(1-0.45)+1200*(0.26+0.20)+2400*(0.02+0.07)=2593

(2) Definition of states

1two normal

2one normal , one breakdown

3two breakdown

$$P = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 0.64 & 0.32 & 0.04 \\ 0.80 & 0.20 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

steady-state prob. $\pi_i = [0.67 \quad 0.30 \quad 0.03]$

1000+2500*(0.30+0.03)+1200*0.30+2400*0.03=2257

(3) Under mechanic A, Definition of states

1normal

2breakdown no repair

3breakdown repair one day

$$P = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0 & 1.0 \\ 1.0 & 0 & 0 \end{bmatrix}$$

steady-state prob. $\pi_i = [0.72 \quad 0.14 \quad 0.14]$

(500+1200)+1500*(1-0.72)+1200*(0.14+0.14)=2456

Under mechanic B, definition of states

1normal

2breakdown

$$P = \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{matrix} 1 \quad\; 2 \\ \begin{bmatrix} 0.8 & 0.2 \\ 1 & 0 \end{bmatrix} \end{matrix}$$

steady-state prob. $\pi_i = [0.83 \quad 0.17]$

1700+2500*(1-0.83)+1200*0.17=2329

例題1：根據市場調查顯示，目前A品牌洗髮精銷售佔市場約35% ，B品牌佔市場30% ，而其他品牌則佔35% 。A、B品牌在下次消費者購買時，各有90% 、87% 的仍會繼續購買原品牌，而使用其他品牌的消費者有73% 不會轉移至A、B品牌。詳細狀態轉移機率矩陣如下：

$$P = \begin{matrix} A \\ B \\ O \end{matrix} \begin{bmatrix} 0.90 & 0.02 & 0.08 \\ 0.04 & 0.87 & 0.09 \\ 0.15 & 0.12 & 0.73 \end{bmatrix}$$

1　求下期市場佔有率的配額？
2　某消費者此次購買B品牌，求其兩次後仍購買B品牌的機率爲何？
3　求四期後之市場佔有率配額？

SOL：

（1）$V_1 = V_0 P = \begin{bmatrix} 0.35 & 0.30 & 0.35 \end{bmatrix} \begin{bmatrix} 0.90 & 0.02 & 0.08 \\ 0.04 & 0.87 & 0.09 \\ 0.15 & 0.12 & 0.73 \end{bmatrix}$

$= \begin{bmatrix} 0.3795 & 0.3100 & 0.3105 \end{bmatrix}$

（2）$P_{22}^{(2)} = P_{21}P_{12} + P_{22}P_{22} + P_{23}P_{32} = 0.0008 + 0.7569 + 0.0108 = 0.7685$

or　$P^{(2)} = \begin{bmatrix} 0.8228 & 0.0450 & 0.1322 \\ 0.0843 & 0.7685 & 0.1472 \\ 0.2493 & 0.1950 & 0.5557 \end{bmatrix}$

（3）$V_4 = V_0 P^{(4)} = \begin{bmatrix} 0.35 & 0.30 & 035 \end{bmatrix} \begin{bmatrix} 0.90 & 0.02 & 0.08 \\ 0.04 & 0.87 & 0.09 \\ 0.15 & 0.12 & 0.73 \end{bmatrix}^4$

$= \begin{bmatrix} 0.43 & 0.32 & 0.26 \end{bmatrix}$

例題2：承接例題一，各品牌長期市場佔有率爲何？

SOL：

$$\begin{cases} \pi_1 = 0.90\pi_1 + 0.04\pi_2 + 0.15\pi_3 \\ \pi_2 = 0.02\pi_1 + 0.87\pi_2 + 0.12\pi_3 \\ \pi_3 = 0.08\pi_1 + 0.09\pi_2 + 0.73\pi_3 \\ 1 = \pi_1 + \pi_2 + \pi_3 \end{cases}$$

$\pi_1 = 0.472 \quad \pi_2 = 0.291 \quad \pi_3 = 0.237$

例題3：假設B公司不滿意長期的市場佔有率僅有29%　，故針對使用其他品牌（不包括A品牌）的消費者進行促銷，致使18% 的人會轉移至B品牌，只有9% 的人轉移至A品牌。試問長期而言，此行銷策略有何效應？

SOL：

$$P = \begin{matrix} A \\ B \\ O \end{matrix} \begin{bmatrix} 0.90 & 0.02 & 0.08 \\ 0.04 & 0.87 & 0.09 \\ 0.09 & 0.18 & 0.73 \end{bmatrix} \qquad \begin{cases} \pi_1 = 0.90\pi_1 + 0.04\pi_2 + 0.09\pi_3 \\ \pi_2 = 0.02\pi_1 + 0.87\pi_2 + 0.18\pi_3 \\ \pi_3 = 0.08\pi_1 + 0.09\pi_2 + 0.73\pi_3 \\ 1 = \pi_1 + \pi_2 + \pi_3 \end{cases}$$

$\pi_1 = 0.371 \quad \pi_2 = 0.389 \quad \pi_3 = 0.240$

例題4：某公司計劃購買一部新機器，根據供應廠商所提供的資料，公司主管欲了解其新機器的故障發生狀況如何，故將狀態分爲：

$X_1 = 1$，一週期期末，機器使用狀況最佳（購進之第一週期）

$X_2 = 2$，一週期期末，機器使用狀況普通

$X_3 = 3$，一週期期末，機器發生故障

機器若發生故障，可於下一週期開始時，到達狀態二。一部新機器大約經過多少時間會發生故障？機器經修復後，平均經過多久又會發生故障？

SOL：

$$P = \begin{bmatrix} 0.70 & 0.25 & 0.05 \\ 0 & 0.60 & 0.40 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{cases} U_{13} = 1 + 0.70U_{13} + 0.25U_{23} \\ U_{23} = 1 + 0.60U_{23} \\ U_{33} = 1 + U_{23} \end{cases} \Rightarrow U_{13} = 5.417 \quad U_{23} = 2.5 \quad U_{33} = 3.5$$

一部新機器大約經過5.4個月，發生故障現象；而機器修復後，大約2.5個月後，再度發生故障。

例題5：（1）判斷下列移轉機率矩陣是否爲吸收性馬可夫鍊，其狀態性質各爲何？

（2）被吸收前，在每一個非吸收狀態上平均各停留幾次？

（3）其中的非吸收狀態，平均要經過幾次移轉才會被吸收？

（4）被某一特定狀態吸收的機率爲多少？

SOL：$P = \begin{matrix} 2 \\ 4 \\ 1 \\ 3 \end{matrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.4 & 0.2 & 0.4 \\ 0.5 & 0.2 & 0.3 & 0 \end{bmatrix}$

$$R = \begin{bmatrix} 0 & 0.4 \\ 0.5 & 0.2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0 \end{bmatrix}$$

$$N = (I - Q)^{-1} = \begin{bmatrix} 0.8 & -0.4 \\ -0.3 & 1 \end{bmatrix}^{-1} = \begin{matrix} 1 \\ 3 \end{matrix}\begin{bmatrix} 1.47 & 0.59 \\ 0.44 & 1.18 \end{bmatrix}$$

（2）若從非吸收狀態1開始，在被吸收前，平均經過狀態1、3分別爲1.47次與0.59次。

若從非吸收狀態3開始，在被吸收前，平均經過狀態1、3分別爲0.44次與1.18次。

（3）$t = Ne = \begin{matrix} 1 \\ 3 \end{matrix}\begin{bmatrix} 1.47 & 0.59 \\ 0.44 & 1.18 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.06 \\ 1.62 \end{bmatrix}$

由非吸收狀態1開始，平均經過2.06次才會被吸收。

若由狀態3開始，則平均停留在非吸收狀態的次數爲1.62次。

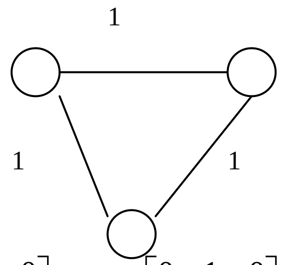
（4）$B = NR = \begin{matrix} 1 \\ 3 \end{matrix}\begin{bmatrix} \overset{1}{1.47} & \overset{3}{0.59} \\ 0.44 & 1.18 \end{bmatrix} \begin{matrix} 1 \\ 3 \end{matrix}\begin{bmatrix} \overset{2}{0} & \overset{4}{0.4} \\ 0.5 & 0.2 \end{bmatrix} = \begin{matrix} 1 \\ 3 \end{matrix}\begin{bmatrix} \overset{2}{0.295} & \overset{4}{0.706} \\ 0.590 & 0.412 \end{bmatrix}$

若從非吸收狀態1開始，被狀態2吸收的機率爲0.295，被狀態4吸收的機率爲0.706。

若從非吸收狀態3開始，被狀態2吸收的機率爲0.590，被狀態4吸收的機率爲0.412。

例題6：試繪下列移轉機率矩陣之狀態移轉圖，並判斷其狀態性質及週期

$$P=\begin{array}{c}\\1\\2\\3\end{array}\begin{bmatrix}{}^{1}&{}^{2}&{}^{3}\\0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$



$$P^2=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\quad P^3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\quad P^4=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\quad P^5=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

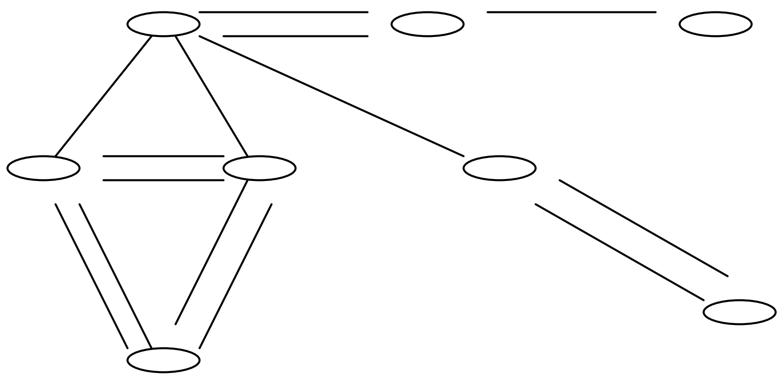
每個狀態皆無互通的情況，且均爲正返回狀態，週期爲3。

例題7：試繪下列移轉機率矩陣之狀態移轉圖，並判斷其狀態性質及族別

$$P=\begin{array}{c}1\\2\\3\\4\\5\\6\\7\\8\end{array}\begin{bmatrix}\frac{1}{5}&\frac{1}{5}&\frac{1}{5}&\frac{1}{5}&\frac{1}{5}&0&0&0\\\frac{1}{3}&\frac{1}{3}&0&0&0&\frac{1}{3}&0&0\\0&0&\frac{1}{2}&0&0&0&\frac{1}{2}&0\\0&0&0&\frac{1}{3}&\frac{1}{3}&0&0&\frac{1}{3}\\0&0&0&\frac{1}{3}&\frac{1}{3}&0&0&\frac{1}{3}\\0&0&0&0&0&1&0&0\\0&0&\frac{1}{2}&0&0&0&\frac{1}{2}&0\\0&0&0&\frac{1}{3}&\frac{1}{3}&0&0&\frac{1}{3}\end{bmatrix}$$

$\{1,2\}$一族，$\{4,5,8\}$一族，$\{3,7\}$一族，$\{6\}$一族。

$\{1,2\}$爲過渡狀態，$\{4,5,8\}$及$\{3,7\}$爲正返回狀態，$\{6\}$爲吸收狀態。



例題8：阿誠影印公司尾牙，舉行了一項摸彩活動，共有三個黑球（B），與三個白球（W），被分配到相同的罐子（1）和罐子（2）中，使得每罐只能有三個球。若每次摸彩都從罐子（1）和罐子（2）裡各取出一個來換，換完後再放回罐中（假設球都是獨立的）。今想要觀察在罐（1）中的白球數目，故令 x_n 表第n次交換後罐（1）中白球的數目，試回答下列問題：

$\{x_n=1\}$的情形

(1)請問所觀察的過程 $\{x_n : n = 0,1,...\}$ 是否為馬可夫鏈？狀態空間為何？

(2)試求過程 $\{x_n : n = 0,1,...\}$ 的一步轉移矩陣和二步轉移矩陣？

(3)試求下列機率值：

a. $\Pr\{x_1 = 2, x_2 = 2, x_3 = 2, x_4 = 1, x_5 = 0, x_6 = 1, x_7 = 2 \mid x_0 = 3\}$？

b. $\Pr\{x_4 = 2, x_6 = 3, x_7 = 2 \mid x_2 = 1\}$？

c. $\Pr\{x_2 = 1, x_3 = 2\}$？

d. $\Pr\{x_2 = 1 \mid x_3 = 2\}$？

e. $\Pr\{x_2 = 1, x_4 = 2 \mid x_3 = 2\}$？

解：

(1) 因為 $x_{n+1} = x_n + \begin{cases} 1 \\ 0 \\ -1 \end{cases}$，故可得 x_{n+1} 獨立於 $(x_0, x_1, ..., x_{n-1}) \mid x_n, \forall n$；表示 x_n 已知時，其服從馬可夫性質；x_{n+1} 便可由 x_n 決定，而與過去演進 $(x_0, x_1, ..., x_{n-1})$ 無關；此外，其狀態空間 $S = \{0,1,2,3\}$ 為有限的，轉移機率與時間無關；所以，過程 $\{x_n : n = 0,1,...\}$ 為一馬可夫鏈。有此我們可知馬可夫鏈有下列三個特徵：①只有有限個狀態；②具有馬可夫性質；③轉移機率與時間無關。

(2) 首先，由(1)可知其狀態集為 $S = \{0,1,2,3\}$，故知其邊界條件 $P_{01} = \Pr\{x_{n+1} = 0 \mid x_n = 0\} = 1$，$P_{32} = \Pr\{x_{n+1} = 2 \mid x_n = 3\} = 1$。根據轉移機率的列和為1，即可知 $P_{00} = P_{02} = P_{03} = 0$，$P_{30} = P_{31} = P_{33} = 0$。

當 $x_n = 1$時，

$$P_{01} = \Pr\{x_{n+1} = 0 \mid x_n = 1\}$$
$$= \Pr\{第n+1次交換時，罐(1)取到白球，罐(2)取到黑球\}$$
$$= \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}$$
$$P_{11} = \Pr\{x_{n+1} = 1 \mid x_n = 1\}$$
$$= \Pr\{第n+1次交換時，罐(1)與罐(2)取到相同顏色的球\}$$
$$= \frac{1}{3}\cdot\frac{2}{3} + \frac{2}{3}\cdot\frac{1}{3} = \frac{4}{9}$$

$P_{12} = \Pr\{x_{n+1} = 2 \mid x_n = 1\}$

$= \Pr\{$第$n+1$次交換時，罐（1）取到黑球，罐（2）取到白球$\}$

$= \dfrac{2}{3} \cdot \dfrac{2}{3} = \dfrac{4}{9}$

再根據轉移機率的列和便可知$P_{13} = 0$

同理，可得$x_n = 2$時，$P_{20} = 0, P_{21} = \dfrac{4}{9}, P_{22} = \dfrac{4}{9}, P_{23} = \dfrac{1}{9}$

故可得其一步轉移矩陣如下：

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/9 & 4/9 & 4/9 & 0 \\ 0 & 4/9 & 4/9 & 1/9 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

而其二步轉移矩陣爲

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/9 & 4/9 & 4/9 & 0 \\ 0 & 4/9 & 4/9 & 1/9 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/9 & 4/9 & 4/9 & 0 \\ 0 & 4/9 & 4/9 & 1/9 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/9 & 4/9 & 4/9 & 0 \\ 4/81 & 4/81 & 32/81 & 4/81 \\ 4/81 & 32/81 & 41/81 & 4/81 \\ 0 & 4/9 & 4/9 & 1/9 \end{bmatrix} \end{array}$$

(3) 下列問題皆是條件機率的基本應用，按照順序來求解

a.

$\Pr\{x_1 = 2, x_2 = 2, x_3 = 2, x_4 = 1, x_5 = 0, x_6 = 1, x_7 = 2 \mid x_0 = 3\}$

$= \Pr\{x_1 = 2 \mid x_0 = 3\} \cdot \Pr\{x_2 = 2 \mid x_1 = 2, x_0 = 3\} \cdot \Pr\{x_3 = 2 \mid x_2 = 2, x_1 = 2, x_0 = 3\}$

$\cdot \Pr\{x_4 = 1 \mid x_3 = 2, x_2 = 2, x_1 = 2, x_0 = 3\} \cdot \Pr\{x_5 = 0 \mid x_4 = 1, x_3 = 2, x_2 = 2, x_1 = 2, x_0 = 3\}$

$\cdot \Pr\{x_6 = 1 \mid x_5 = 0, x_4 = 1, x_3 = 2, x_2 = 2, x_1 = 2, x_0 = 3\}$

$\cdot \Pr\{x_7 = 2 \mid x_6 = 1, x_5 = 0, x_4 = 1, x_3 = 2, x_2 = 2, x_1 = 2, x_0 = 3\}$

$= P_{32}P_{22}P_{22}P_{21}P_{10}P_{01}P_{12} = \dfrac{256}{59049}$

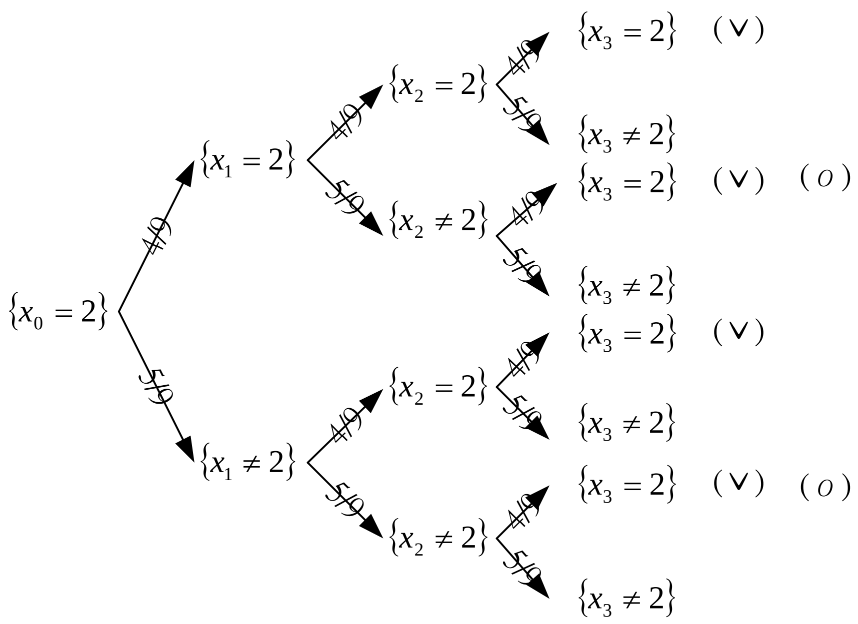
b.

$\Pr\{x_4 = 2, x_6 = 3, x_7 = 2 \mid x_2 = 1\}$

$= \Pr\{x_7 = 2 \mid x_2 = 1\} \cdot \Pr\{x_6 = 3 \mid x_4 = 2, x_2 = 1\} \cdot \Pr\{x_7 = 2 \mid x_6 = 3, x_4 = 2, x_2 = 1\}$

$= P_{12}^{(2)} P_{23}^{(2)} P_{32} = \left(\dfrac{32}{81}\right)\left(\dfrac{4}{81}\right)1 = \dfrac{128}{6561}$

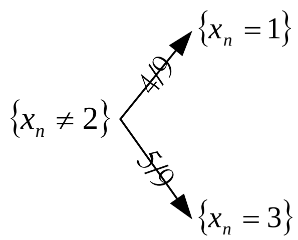
爲了求解問題c與 d。首先，我們畫出$\{x_n = 2\}$的分支樹狀圖來分析此二問題，且由於轉移機率$\Pr\{x_{n+1} = 2 \mid x_n = 2\} = \dfrac{4}{9}$，同時亦可得知$\Pr\{x_{n+1} \neq 2 \mid x_n = 2\} = \dfrac{5}{9}$



$x_n = 2$的分支樹狀圖

因此，由打（∨）部分的樹狀分支可得 $\Pr\{x_3 = 2\} = \left(\frac{4}{9}\right)^3 + 2\left(\frac{4}{9}\right)^2\left(\frac{5}{9}\right) + \left(\frac{5}{9}\right)^2\left(\frac{4}{9}\right) = \frac{324}{729}$

而在事件 $\{x_n \neq 2\}$ 中又可分解為



$\{x_n \neq 2\}$事件的分解圖

c. 問題c中之聯合機率分佈即為求樹狀圖中打（ o ）者，故

$$\Pr\{x_2 = 1, x_3 = 2\} = \left(\frac{4}{9}\right)^3 + \left(\frac{4}{9}\right)^2\left(\frac{5}{9}\right) = \frac{144}{729}$$

d. 因此由條件機率的公式可得 $\Pr\{x_2 = 1 \mid x_3 = 2\} = \frac{\Pr\{x_2 = 1, x_3 = 2\}}{\Pr\{x_3 = 2\}} = \frac{144/729}{324/729} = \frac{4}{9}$

需特別注意 $\Pr\{x_2 = 1 \mid x_3 = 2\}$ 與一般一步轉移機率之差異，切莫混淆了！

e. 由問題d可知 $\Pr\{x_2 = 1 \mid x_3 = 2\} = \frac{4}{9}$ ，又

$$\Pr\{x_2 = 1, x_4 = 2 \mid x_3 = 2\}$$
$$= \Pr\{x_2 = 1 \mid x_3 = 2\} \cdot \Pr\{x_4 = 2 \mid x_2 = 1, x_3 = 2\}$$
$$= \left(\frac{4}{9}\right) \cdot P_{22} = 0$$