

CHAP 14 Game Theory

14.1 Introduction

A. Definition

— a mathematical theory that deal with the situation that the final outcome depends primarily on the combinations of strategies selected by the adversaries in a formal abstract way

B. Objective

— to develop a rational criteria for selecting a strategy

C. Basic assumptions

- all players are rational
- all players to promote their own welfare

D. A strategy

— a predetermined value that specifies completely how one intends to respond to each possible situation at each stage of the game

E. Payoff table

— show the gain for a player that would result from each combination of strategies for all the players

strategy		player II		
		1	2	3
player I	1	5 -5		
	2			
	3			

F. A game is characterized by

- strategies of each player
- payoff table

14.2 The formulation of two-person zero-sum games

A. Only two players

B. The sum of their net winning is zero

C. Game characteristics

- strategies of player
- strategies of player
- payoff tables

D.A prototype example

—two candidates their strategies as follow

strategy 1 : spend one day in each city

strategy 2 : spend both days in city A

strategy 3 : spend both days in city B

—payoff table

strategy		player II		
		1	2	3
player I	1	1	2	4
	2	1	0	5
	3	0	1	-1

—produces of solution

step1 : dominated strategy elimination

step2 : if only one stategy left for each player,stop,otherwise go to step3

step3 : identify saddle point if it exists stop.otherwise,mixed strategy

stop4 : identify saddle point of mixed strategy combination

E.Dominated strategies

—to rule out a succession of inferior strategies only one choice remain

F.From table

—no dominated strategies for player

—strategy 3 is dominated by strategy 1

	1	2	3
1	1	2	4
2	1	0	5
3	0	1	-1

—player II strategy 3 is dominated by strategy 1

	1	2
1	1	2
2	1	0

—for player I strategy 2 is dominated by strategy 1

	1	2
1	1	2

—for player II strategy 2 is dominated by strategy 1

	1
1	1

G.Observation

- the payoff to play I when both players play optimal is referred as the value of the game
- if the value of the game is zero, it is a fair game
- most games can't identify the optimal solution by dominated strategy only

I. Variation 1 : new payoff table

	1	2	3	Min	
1	-3	-2	6	-3	
2	2	0	2	0	← maximin
3	5	-2	-4	-4	
Max	5	0	6		

↑
minimax

J.Minimax criteria

- find the saddle point
- point neither one can take the advantage of the opponent's strategy to improve his position
- procedure

- (1) identify the minimum value of each row (\bar{r})
- (2) identify the maximum value of each column (\bar{c})
- (3) identify maximum value of (\bar{r}) → maximin value
- (4) identify minimum value of (\bar{c}) → minimax value

K.Variation 2

	1	2	3	Min.	
1	0	-2	2	-2	← Maximin
2	5	4	-3	-3	
3	2	-3	-4	-4	
Max.	5	4	2		

↑
Minima
x

- no saddle point
- unstable solution
- to allow mixed strategies
- no one know in advance of what he and his opponent will do

Example:考慮下列零和二人有限賽局 $G=(X,Y,M)$ A,B二人同時出剪刀，石頭，布其輸贏如下表所示，解此賽局G

A \ B	y_1 剪刀	y_2 石頭	y_3 布
x_1 剪刀	-4	2	7
x_2 石頭	5	8	6
x_3 布	3	-2	-8

解：

A \ B	y_1 剪刀	y_2 石頭	y_3 布	
x_1 剪刀	-4	2	7	-4
x_2 石頭	5	8	6	5
x_3 布	3	-2	-8	-8

$$\max_x \min_y M(x, y) = \min_x \max_y M(x, y) = 5 = V_G$$

故此賽局之解為：第一人最佳策略為石頭 (x_2) 第二人最佳策略為剪刀 (y_1)，單純值 $V_G = 5$

14.3 Game with Mixed Strategies

A. x_i = prob. that player I uses strategy i

y_j = prob. that player II uses strategy j

B. The mixed strategies

$$(x_1, x_2, \dots, x_m) \text{ and } (y_1, y_2, \dots, y_n)$$

C. The expected payoff

$$V = \sum_{i=1}^m \sum_{j=1}^n p_{ij} x_i y_j$$

D. For player I : Maximize $V = \bar{V}$

E. For player II : Minimize $V = \underline{V}$

F. Stable condition $\underline{V} = \bar{V}$

G. Minimax theorem

if a mixed strategies is allowed, the pair of mixed strategies that is optimal provides a stable condition with $\underline{V} = \bar{V} = V$

—solution methods for mixed strategy

—2*2 strategy(x_1, x_2)(y_1, y_2)

	y_1	y_2
x_1	A	B
x_2	C	D

$$E(y_1) = A(P_1) + C(1 - P_1)$$

$$E(y_2) = B(P_1) + D(1 - P_1)$$

$$\rightarrow P_1, P_2 \left[\frac{d - c}{a + d - b - c}, \frac{a - b}{a + d - b - c} \right]$$

$$E(X_1) = A(Q_1) + B(1 - Q_1)$$

$$E(X_2) = C(Q_1) + D(1 - Q_1)$$

$$\rightarrow Q_1, Q_2 \left[\frac{d - b}{a + d - b - c}, \frac{a - c}{a + d - b - c} \right]$$

—at least one with 3 strategys

—Graphic (at least 2 strategy)

—L P model

Example: 考慮

A \ B	B_1	B_2	
A_1	-20	10	P_1
A_2	30	-40	$P_2 = 1 - P_1$

$$E(B_1) = -20P_1 + 30(1 - P_1)$$

$$E(B_2) = 10P_1 - 40(1 - P_1)$$

$$\Rightarrow E(B_1) = E(B_2)$$

$$\therefore -20P_1 + 30(1 - P_1) = 10P_1 - 40(1 - P_1)$$

$$-50P_1 + 30 = 50P_1 - 40$$

$$\Rightarrow P_1 = \frac{7}{10}, P_2 = \frac{3}{10}$$

結果顯示，A公司應採用混和策略， A_1 與 A_2 之機率為 $\frac{7}{10}$ 與 $\frac{3}{10}$

同理，B公司採用 B_1 的機率為 Q_1 ，採用 B_2 的機率為 Q_2 ：

$$E(A_1) = -20Q_1 + 10(1 - Q_1)$$

$$E(A_2) = 30Q_1 - 40(1 - Q_1)$$

$$\Rightarrow E(A_1) = E(A_2)$$

$$\therefore -20Q_1 + 10(1 - Q_1) = 30Q_1 - 40(1 - Q_1)$$

$$-30Q_1 + 10 = 70Q_1 - 40$$

$$\Rightarrow Q_1 = \frac{1}{2}, Q_2 = \frac{1}{2}$$

B公司應採用混和策略， B_1 與 B_2 之機率為 $\frac{1}{2}$ 與 $\frac{1}{2}$

A \ B	B_1	B_2	
A_1	-20	10	0.7
A_2	30	-40	0.3
	0.5	0.5	

從A公司的觀點考慮B公司採用策略 B_1 或 B_2 ，A公司的損失為：

$$\text{對局值} = -20(0.7) + 30(0.3) = -5$$

從B公司的觀點考慮A公司採用策略 A_1 或 A_2 ，B公司的收益為：

$$\text{對局值} = -20(0.5) + 10(0.5) = -5$$

雙方運用此最佳混和策略時，A公司無法將其損失再行減少，B公司也無法再將其收益增加，此賽局即告穩定，對局值為-5。

14.4 Graphic Solution

A. For two undominated strategies game for player I only

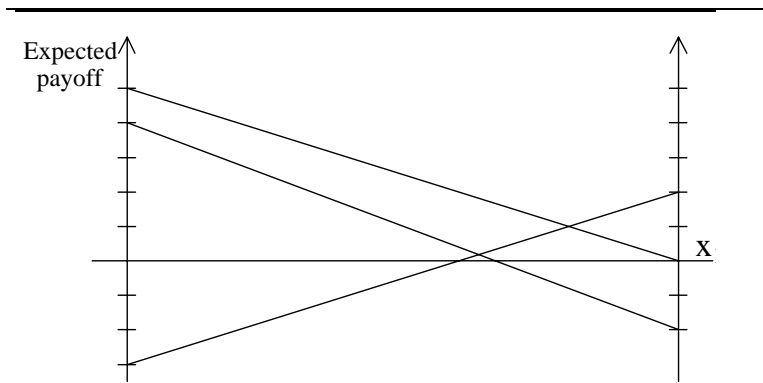
B. Plot the expected payoffs as function of X_1 for each of his opponents pure strategies

C. Identify the point that maximizes the minimum expected payoff

— example

			player II		
			y_1	y_2	y_3
			1	2	3
player I	x_1	1	0	-2	2
	x_2	2	5	4	-3

(y_1, y_2, y_3)	Expected Payoff
(1, 0, 0)	$0x_1 + 5(1 - x_1) = 5 - 5x_1$
(0, 1, 0)	$-2x_1 + 4(1 - x_1) = 4 - 6x_1$
(0, 0, 1)	$2x_1 - 3(1 - x_1) = -3 + 5x_1$



D. Expected Payoff

$$y_1(-3 - 5x_1) + y_2(4 - 6x_1) + y_3(-3 + 5x_1)$$

$$\bar{V} = V = \text{Max}\{M_m | -3 + 5x_1, 4 - 6x_1\}$$

$$\text{Maximin point is } -3 + 5x_1 = 4 - 6x_1$$

$$\Rightarrow x_1 = \frac{7}{11}, x_2 = \frac{4}{11}$$

$$\Rightarrow V = -3 + 5\left(\frac{7}{11}\right) = \frac{2}{11}$$

E. To find optimal for (y_1, y_2, y_3)

$$y_1^*(5 - 5x_1) + y_2^*(4 - 6x_1) + y_3^*(-3 + 5x_1) \leq \bar{V} = V = \frac{2}{11}$$

$$\begin{cases} \frac{20}{11}y_1^* + \frac{2}{11}y_2^* + \frac{2}{11}y_3^* = \frac{2}{11} \\ y_1^* + y_2^* + y_3^* = 1 \end{cases}$$

$$\Rightarrow y_1^* = 0$$

$$\Rightarrow y_2^*(4 - 6x_1) + y_3^*(-3 + 5x_1) \begin{cases} = \frac{2}{11}, x_1 = \frac{7}{11} \\ \leq \frac{2}{11}, \text{otherwise} \end{cases}$$

$$\Rightarrow y_2^*(4 - 6x_1) + y_3^*(-3 + 5x_1) = \frac{2}{11}$$

$$\text{use } x_1 = 0, x_2 = 1$$

$$\Rightarrow \begin{cases} 4y_2^* - 3y_3^* = \frac{2}{11} \\ -2y_2^* + 2y_3^* = \frac{2}{11} \end{cases}$$

$$\Rightarrow y_3^* = \frac{6}{11}, y_2^* = \frac{5}{11}$$

$$\Rightarrow (y_1^*, y_2^*, y_3^*) = \left(0, \frac{5}{11}, \frac{6}{11}\right)$$

F. Observations

- if there are more than two lines passing through the maximum point, so that more than two of y_j^* can be greater than zero
- solutions are obtained by assigning all but of there y_j^* equal to zero

14.5 Solving by Linear Programming

A. Original linear programming formulation

— for player I

$$\text{objective expected payoff } \sum_{i=1}^m \sum_{j=1}^n p_{ij} x_i y_j = V = \bar{V}$$

— for each strategies

$$\sum_{i=1}^m p_{ij} x_i \geq V \quad \text{for each } x_i \geq 0, i = 1, 2, \dots, m$$

$$\sum_{i=1}^m x_i = 1$$

B. Observations

— V is unknown

— no objective function in LP

C. Modifications

— replace V with x_{m+1}

$$\text{Max } x_{m+1}$$

$$\text{St } \sum_{i=1}^m p_{ij} x_i - x_{m+1} \geq 0 \quad (y_j = 1, j = 1, 2, \dots, n)$$

$$\sum_{i=1}^m x_i = 1$$

$$0 \leq x_1, \dots, x_m \leq C$$

$$x_{m+1} \text{ unrestricted}$$

— for player II

$$\text{Min. } y_{n+1}$$

$$\text{st } \sum_{j=1}^n p_{ij} y_j - y_{n+1} \leq 0$$

$$\sum_{j=1}^n y_j = 1$$

$$0 \leq y_1, \dots, y_n \leq 1$$

$$y_{n+1} \text{ unrestricted}$$

D. If is negative, we need to modify the problem by

— replace $x_{m+1}(y_{n+1})$ with $x_{m+1}^+ - x_{m+1}^-(y_{n+1}^+ - y_{n+1}^-)$

— reverse payoff table from player I to player II

– add a larger fixed constant to all entries of the payoff table

		II	
		1	2
I	1	-2	-3
	2	-1	0

		II	
		1	2
I	1	2	3
	2	1	0

		II	
		1	2
I	1	98	97
	2	99	100

– example

$$\begin{aligned}
 & \text{Max } x_3 \\
 & \text{st. } 5x_2 - x_3 \geq 0 \\
 & \quad -2x_1 + 4x_2 - x_3 \geq 0 \\
 & \quad x_1 + x_2 = 1 \\
 & \quad x_1, x_2 \geq 0 \\
 & \quad x_3 \text{ unrestricted} \\
 & \underline{(x_1, x_2, x_3) = \left(\frac{7}{11}, \frac{4}{11}, \frac{2}{11}\right)}
 \end{aligned}$$

$$\begin{aligned}
 & \text{Min } y_4 \\
 & \text{st. } -2y_2 + y_3 - y_4 \leq 0 \\
 & \quad 5y_1 + 4y_2 - 3y_3 - y_4 \leq 0 \\
 & \quad y_1 + y_2 + y_3 = 1 \\
 & \quad y_1, y_2, y_3 \geq 0 \\
 & \quad y_4 \text{ unrestricted} \\
 & \underline{(y_1, y_2, y_3, y_4) = \left(0, \frac{5}{11}, \frac{6}{11}, \frac{2}{11}\right)}
 \end{aligned}$$

Example：利用線性規劃解下列之賽局

A \ B	y_1	y_2	y_3
x_1	-1	2	1
x_2	1	-2	2
x_3	3	4	-3

為了保證 $V \geq 0$ ，將上列矩陣修正如下：

A \ B	y_1	y_2	y_3
x_1	2	5	4
x_2	4	1	5
x_3	6	7	0

第一人A之混和策略需滿足

$$2x_1 + 4x_2 + 6x_3 \geq V$$

$$5x_1 + x_2 + 7x_3 \geq V$$

$$4x_1 + 5x_2 \geq V$$

$$x_1 + x_2 + x_3 = 1$$

V unrestricted

設 $\frac{x_j}{V} = \theta_j \geq 0, j=1,2,3$ 且 $V \geq 0$ ，由 $x_1 + x_2 + x_3 = 1$ 可得 $\theta_1 + \theta_2 + \theta_3 = \frac{1}{V}$ 第一人混和策略 x ，選擇

使 V 最大，故上述問題變成下列線性規劃之問題：

$$\begin{cases} 2\theta_1 + 4\theta_2 + 6\theta_3 \geq 1 \\ 5\theta_1 + \theta_2 + 7\theta_3 \geq 1 \\ 4\theta_1 + 5\theta_2 \geq 1 \\ \theta_j \geq 0, j=1,2,3 \end{cases}$$

同理，第二人B之最佳混和策略需滿足

$$2y_1 + 5y_2 + 4y_3 \leq V$$

$$4y_1 + y_2 + 5y_3 \leq V$$

$$6y_1 + 7y_2 \leq V$$

設 $\frac{y_j}{V} = \varphi_j \geq 0, j=1,2,3$ 且 $V \geq 0$ ，由 $y_1 + y_2 + y_3 = 1$ 可得 $\varphi_1 + \varphi_2 + \varphi_3 = \frac{1}{V}$ 第二人混和策略 y ，選擇

使 V 最小，故上述問題變成下列線性規劃之問題：

$$\begin{cases} 2\varphi_1 + 5\varphi_2 + 4\varphi_3 \leq 1 \\ 4\varphi_1 + \varphi_2 + 5\varphi_3 \leq 1 \\ 6\varphi_1 + 7\varphi_2 \leq 1 \\ \varphi_j \geq 0, j=1,2,3 \end{cases}$$

上述兩問題成為對偶問題，故 $\min \theta = \max \varphi$ 利用線性規劃之單體法解之

$$\begin{cases} 2\varphi_1 + 5\varphi_2 + 4\varphi_3 + \varphi_4 = 1 \\ 4\varphi_1 + \varphi_2 + 5\varphi_3 + \varphi_5 = 1 \\ 6\varphi_1 + 7\varphi_2 + \varphi_6 = 1 \\ \varphi_j \geq 0, j=1,2,3,\dots,6 \end{cases}$$

目標函數 $Max \quad \varphi_1 + \varphi_2 + \varphi_3$

求解結果， $\varphi_1 = \frac{1}{12}, \varphi_2 = \frac{1}{14}, \varphi_3 = \frac{5}{42} \quad \frac{1}{V} = \frac{23}{84}$

$$y_1 = V\varphi_1 = \frac{84}{23} \times \frac{1}{12} = \frac{7}{23}$$

$$y_2 = \frac{6}{23}$$

$$y_3 = \frac{10}{23}$$

$$V^* = V - 3 = \frac{84}{23} - 3 = \frac{15}{23}$$

即第二人最佳混和策略為 $y = (\frac{7}{23}, \frac{6}{23}, \frac{10}{23})$ ，賽局值 $V = \frac{15}{23}$

同理，第一人最佳混和策略為 $x = (\frac{17}{46}, \frac{20}{46}, \frac{9}{46})$ ，賽局值 $V = \frac{15}{23}$

例題：

1. 主宰策略 (dominated strategy)

兩家國內電腦公司同時開發出具有相同功能的新型電腦。兩公司分別對下一季擬定了三個行銷策略，其報酬矩陣 (payoff table) 如第一個表所示 (數字之單位為百萬元)。根據主宰策略 (dominated strategy)，此問題的求解過程如下：

	$\beta 1$	$\beta 2$	$\beta 3$
$\alpha 1$	2	1	-1
$\alpha 2$	2	3	4
$\alpha 3$	1	0	5

首先，對B而言，並沒有可被主宰的策略。但對A而言，不論B的策略為何， $\alpha 2$ 必定優於 (嚴格應說不劣於) $\alpha 1$ ，故 $\alpha 1$ 可被 $\alpha 2$ 主宰。

	$\beta 1$	$\beta 2$	$\beta 3$
$\alpha 2$	2	3	4
$\alpha 3$	1	0	5

因為B知道A不會採取，所以報酬矩陣可簡化為第二個表。

	$\beta 1$	$\beta 2$
$\alpha 2$	2	3
$\alpha 3$	1	0

此時， $\beta 3$ 可被 $\beta 2$ 主宰，因此可得第三個表。

	$\beta 1$	$\beta 2$
$\alpha 2$	2	3

在第三個表中， $\alpha 3$ 可被 $\alpha 2$ 主宰，因此可得第四個表。

	$\beta 1$
$\alpha 2$	2

此時， $\beta 2$ 可被 $\beta 1$ 主宰，而可得到最後的表。

根據主宰策略，A會選擇 $\alpha 2$ ，B會選擇 $\beta 1$ ，而使得A獲得2百萬元的利益。

2. 小中取大策略 (minimax strategy)

承上，假設報酬矩陣改為下列第一個表。在此報酬矩陣中，不論對A、對B，沒有任何策略可被其他策略主宰。因此策略如右圖所示：

	$\beta 1$	$\beta 2$	$\beta 3$
$\alpha 1$	0	3	-1
$\alpha 2$	-2	-3	4
$\alpha 3$	1	2	3

	$\beta 1$	$\beta 2$	$\beta 3$	最小值
$\alpha 1$	0	3	-1	-1
$\alpha 2$	-2	-3	4	-3
$\alpha 3$	1	2	3	1
極大值	1	3	4	

∨

3.混合策略 (mixed strategy)

承上，假設報酬矩陣改為下列第一個表。根據主宰策略，我們可以依序刪除 $\beta 3$ 及 $\alpha 2$ ，而得到第三個表。

	$\beta 1$	$\beta 2$	$\beta 3$
$\alpha 1$	2	0	2
$\alpha 2$	-2	-3	2
$\alpha 3$	1	4	3

	$\beta 1$	$\beta 2$
$\alpha 1$	2	0
$\alpha 2$	-2	-3
$\alpha 3$	1	4

	$\beta 1$	$\beta 2$
$\alpha 1$	2	0
$\alpha 3$	1	4

接著，我們考慮以小中取大策略決定較低值與較高值。因為較低值與較高值不是發生在同一數字上，故此競賽沒有鞍點。

	$\beta 1$	$\beta 2$	極小值
$\alpha 1$	2	0	0
$\alpha 3$	1	4	1
極大值	2	4	

∨

此時，若A選擇 $\alpha 3$ ，則B會選擇 $\beta 1$ ，但因為彼此都知道報酬矩陣，所以A會知道B會因A選擇 $\alpha 3$ 而選擇 $\beta 1$ ，故A反而會選擇 $\alpha 1$ 。而B也會知道A會如此作，因此他反而會選擇 $\beta 2$ 。...如此彼此猜測下去，會形成循環而無法做出最後決定。因此有兩種求解方法：

(1) 圖解法

	$\beta 1$	$\beta 2$	$\beta 3$
$\alpha 1$	0	-1	2
$\alpha 2$	1	2	0
$\alpha 3$	-2	-3	2

	$\beta 1$	$\beta 2$	$\beta 3$	極小值
$\alpha 1$	0	3	-1	-1
$\alpha 2$	1	2	0	0
極大值	1	2	2	

∨

B使用不同策略時之期望收益

B之策略	期望收益
$\beta 1$	$1-X1$
$\beta 2$	$2-3X1$
$\beta 3$	$2X1$

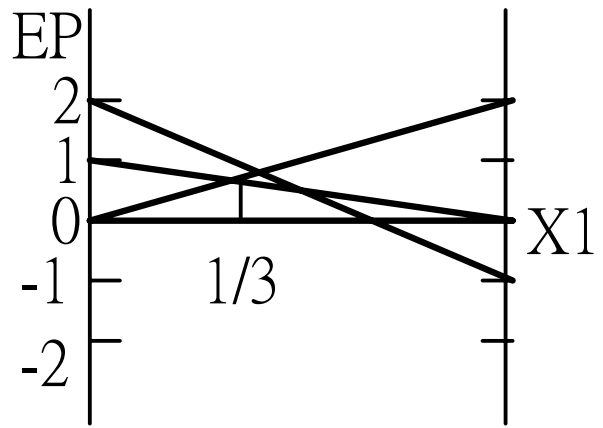
令 $X2 = 1 - X1$

這些最小值中最大值的點發生在 $1-X1$ (若B選擇 $\beta 2$) 與 $2X1$ (若B選擇 $\beta 3$) 之交點。

因此 $1-X1 = 2X1$

$X1 = 1/3 \rightarrow X2 = 2/3$ 期望收益為 $2/3$ 。

A使用不同策略時之期望收益



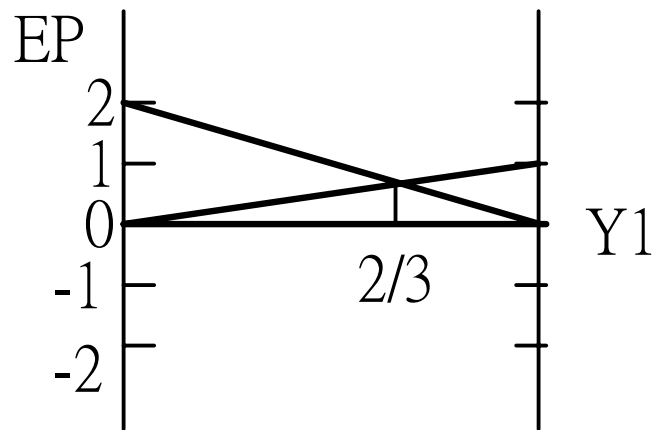
A之策略	期望收益
$\alpha 1$	$2-2Y1$
$\alpha 2$	$Y1$

$2-2Y1 = Y1$

$Y1 = 2/3$

$\rightarrow Y3 = 1/3$

期望收益則為 $2/3$ 。



(2) 線性規劃

令 $v =$ 該競賽的較低值

$\bar{v} =$ 該競賽的較高值

$V =$ 該競賽之值

	$\beta 1$	$\beta 2$	$\beta 3$	Min
$\alpha 1$	0	-1	2	-1
$\alpha 2$	1	2	0	$0 \checkmark$
$\alpha 3$	3	3	-1	-1
Max	3	3	2	

因為較低值與較高值不是發生在同一數字上。因A與B仍剩下兩個以上的策略，圖解法不再適用。為使 v 為非負，將報酬矩陣中的所有數字加上1 (非負數字-1的絕對值)。

	$\beta 1$	$\beta 2$	$\beta 3$
$\alpha 1$	1	0	3
$\alpha 2$	2	3	1
$\alpha 3$	4	4	0

求解A之最佳混合策略的線性方程式如下： 求解B之最佳混合策略的線性方程式則為：

$$\begin{aligned}
 \max \quad & \bar{v} \\
 \text{s.t.} \quad & x_1 + 2x_2 + 4x_3 \geq \bar{v} \\
 & 3x_2 + 4x_3 \geq \bar{v} \\
 & x_1 + x_2 + x_3 = 1 \\
 & 3x_1 + x_2 \geq \bar{v} \\
 & \bar{v} \text{ 無限制} \\
 & \text{在此題中 } \bar{v} \geq 0 \text{ (因為無負數)}
 \end{aligned}$$

$$\begin{aligned}
 \max \quad & \bar{v} \\
 \text{s.t.} \quad & y_1 + 3y_3 \leq \bar{v} \\
 & 2y_1 + 3y_2 + y_3 \leq \bar{v} \\
 & 4y_1 + 4y_2 \leq \bar{v} \\
 & y_1 + y_2 + y_3 = 1 \\
 & \bar{v} \text{ 無限制} \\
 & \text{在此題中 } \bar{v} \geq 0 \text{ (因為無負數)}
 \end{aligned}$$