CHAP 12　Dynamic Programming

12.1　Introduction

A.For making a sequence of interrelated decision

B.A systematic procedure for determing the optimal combination of decisions

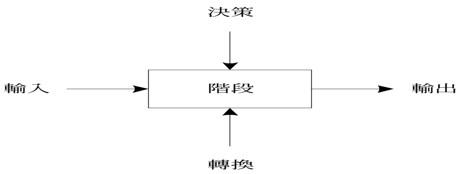
C.No standard mathematical for multistage

D.A particular equation to fit each individual problem

12.2　Characteristics of Dynamic Programming

A.The problem can be divided into stages, with a policy decision required at each stage

B.Each stage has a number of states associated with states are various possible conditions in which the system might be at that stage.



C. The effect of the policy decision at each stage is to transform the current state into a state associated with the next stage. We can use network representation

(1) nodes: a possible state

stage: a column of nodes

(2) each link as the contribution to the objective function from making that policy decision

D.The solution method is designed to find an optimal policy for overall problem

E.Given the current state, an optimal policy for the remaining stages is independent of the policy adopted in the previous stage.

F.The solution procedures begins by finding the optimal policy for the last stage

G.A recursive relationship that indetifies the optimal policy for stage n, given the optimal policy for stage n+1 is available

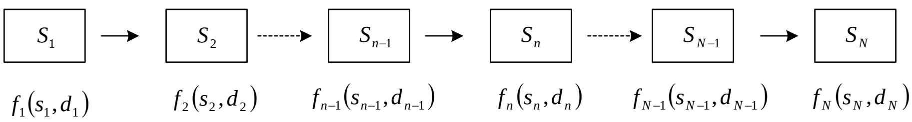
$$f_n^*(s_n) = Max(or\ \ Min)\{f_n(s_n, x_n)\}$$

where

$$f_n(s_n, x_n) = opt\{r_n(s_n, x_n) + f_{n+1}^*(s_{n+1})\}$$

H. Use recursive relationship, solution procedures move backward stage by stage until the initial stage

I. $f_N^*(s_n)$ is boundary condition



12.3 Deterministic Dynamic Problem

A. State at next stage is completely determined by the state and policy decision at the current stage

B. Classification

— by objective function

(1) minimization $\left\{ \begin{array}{l} sum \\ product \end{array} \right.$

(2) maximization $\left\{ \begin{array}{l} sum \\ product \end{array} \right.$

— by set of states

(1) discrete

(2) continuous

(3) state rector (more than one variable)

Example1:

To allocative five teams to three countries to increase life expectage.

	Additional person years life No. of team assigned					
Country	0	1	2	3	4	5
1	0	45	70	90	105	120
2	0	20	45	75	110	150
3	0	50	70	80	100	130

Sol:

(1) identify stage, state

—stage : country 1, 2, 3

—state : number of teams remain to be allocate to the remaining country

— x_n (decision variable) = the number of teams to allocate to country (stage) n

— $P_i(x_i)$ = the measureof performance from allocating x_i to country i

(2) the objective function

$$Max \sum_{i=1}^{n} P_i(x_i)$$

$$s.t. \quad \sum_{i=1}^{3} x_i = 5$$

$$x_i \geq 0$$

$$\Rightarrow f_n^*(s_n) = \max_{x_n = 0,1,\ldots,s_n} f_n(s_n, x_n)$$

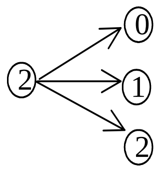
$$f_n(s_n, x_n) = P_n(x_n) + f_{n+1}^*(s_n - x_n)$$

for the last stage (n=3)

$$\Rightarrow f_3^*(s_3) = \max_{x_3 = 0,1,\ldots,s_3} P_3(x_3)$$

s_3	$f_3^*(s_3)$	x_3^*
0	0	0
1	50	1
2	70	2
3	80	3
4	100	4
5	130	5

(3) move back to stage 2, state $s_2=2$

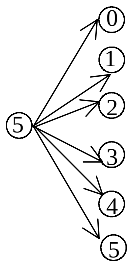


$$\left.\begin{aligned} f_2(2,0) &= P_2(2) + f_3^*(0) = 45 + 0 = 45 \\ f_2(2,1) &= P_2(1) + f_3^*(1) = 20 + 50 = 70 \\ f_2(2,2) &= P_2(0) + f_3^*(2) = 0 + 70 = 70 \end{aligned}\right\} f_2^*(2) = 70, x_2^* = 0 \ or \ 1$$

(4) proceeding in a similar way for other possible state s_2

	$f_2(s_2,x_2)=P_2(x_2)+f_3^*(s_2-x_2)$						$f_2^*(s_2)$	x_2^*
	0	1	2	3	4	5		
0	0						0	0
1	50	20					50	0
2	70	70	45				70	0 or 1
3	80	90	95	75			95	2
4	100	100	115	125	110		125	3
5	130	120	125	145	160	150	160	4

→n=1, only one state $s_1=5$



$x_1 = 5$ $\quad f_1(5,5) = P_1(5) + f_2^*(0) = 120$

$x_1 = 4$ $\quad f_1(5,4) = P_1(4) + f_2^*(1) = 155$

$x_1 = 3$ $\quad f_1(5,3) = P_1(3) + f_2^*(2) = 160$

$x_1 = 2$ $\quad f_1(5,2) = P_1(2) + f_2^*(3) = 165$

$x_1 = 1$ $\quad f_1(5,1) = P_1(1) + f_2^*(4) = 170$

$x_1 = 0$ $\quad f_1(5,0) = P_1(0) + f_2^*(5) = 160$

$x_1 = 1, f_1^*(5) = 170 \qquad , x_1^* = 1$

$x_1 = 1 \Rightarrow s_2 = 5 - 1 = 4 \Rightarrow f_2^*(4) = 125 \qquad , x_2^* = 3$

$x_2 = 1 \Rightarrow s_3 = 4 - 3 = 1 \Rightarrow f_3^*(1) = 50 \qquad , x_3^* = 1$

Example2: Data for the government space project

某科技研究由三個小組用不同方法分別研究，他們失敗的機率各為0.4，0.6，0.8；為了減少三個小組失敗的可能性，現在決定給三個小組增派二名高級科學家到各小組後，各小組科技研究失敗的機率如下：

Number of new scientist	Prob. of Failure Teams		
	1	2	3
0	0.4	0.6	0.8
1	0.2	0.4	0.5
2	0.15	0.2	0.3

問如何分派科學家才使三個小組均失敗的機率最小（即科技研究最後失敗的機率）？

Sol:

(1) goal

. to allocate the two additional scientist to minimize the prob. that all three teams will fall

用逆序算法。設：

階段	1	2	3
小組	1	2	3

(2) formulation

. stage $\rightarrow$ n research team

. state $\rightarrow S_n$: the number of scientists still available for allocation to the remaining teams

. x_i = number of additional scientists to allocate to team n

. $P_i(x_i)$ = prob. of failure for team i if it is assigned x_i

階段：每個研究小組為一個階段，且

決策變數 x_n ：分配給第 n 小組的高級科學科學家人數，對應的失敗機率為 $P_n(x_n)$

狀態變數 S_n ：再階段 n 時可分配於階段 n,n-1,..,1 的高級科學家人數

(3) model

. $Min \sum_{i=1}^{3} P_1(x_1)P_2(x_2)P_3(x_3)$

. $s.t. \sum_{i=1}^{3} x_i = 2$

遞推關係：

$$f_3^*(S_3) = \min_{x_3 \le S_3} \{P_3(x_3)\}$$

$$f_n^*(S_n) = \min_{x_n \le S_n} P_n(x_n) \cdot f_{n+1}^*(S_n - x_n) \} \quad n = 1,2$$

$$\boxed{S_n} \xrightarrow{x_n} \boxed{S_n - x_n}$$

$$f_n^*(S_n) = \min_{x_n \le s_n} f_n(s_n, x_n)$$

$$f(S_n, x_n) = P_n(x_n) \cdot f_{n+1}^*(S_n - x_n)$$

(4) solution procedure

計算：

. n = 3

s_3	$f_3^*(s_3)=\underset{x_n \le S_n}{Min}\ P_3(x_3)$	x_3^*
0	0.8	0
1	0.5	1
2	0.3	2

. n = 2

S_2 \ x_2	$f_2(s_2,x_2)=P_2(x_2)\cdot f_3^*(s_2-x_2)$			$f_2^*(s_2)$	x_2^*
	0	1	2		
0	0.48			0.48	0
1	0.30	0.32		0.30	0
2	0.18	0.20	0.16	0.16	2

$S_1 = 2$

$S_2 = 2 - X_1$

$S_3 = 2 - X_1 - X_2$

. n=1

S_1 \ x_1	$f_1(s_1,x_1)=P_1(x_1)\cdot f_2^*(s_1-x_1)$			$f_1^*(s_1)$	x_1^*
	0	1	2		
2	0.064	0.060	0.072	0.060	1

$x_1^* = 1 \Rightarrow s_2 = 2 - 1 = 1$

$s_2 = 1 \Rightarrow x_2^* = 0$

$s_3 = 2 - 1 - 0 = 1 \Rightarrow x_3^* = 1$

最佳解為 $x_1^* = 1$ ， $x_2^* = 0$ ， $x_3^* = 1$ ；科技研究失敗的機率為 0.060

Example3:

$Max\ \ 3x_1 + 5x_2$

$s.t.$

$x_1 \le 4$

$2x_2 \le 12$

$3x_1 + 2x_2 \le 18$

$x_1, x_2 \ge 0$

Sol:

(1) identify stage, state

- stages $\rightarrow$ two activity 1 & 2
- state $\rightarrow$ s_n: amount of resource still available for allocation
- $s_n = (R_{11}, R_2, R_3)$

$s_1 = (4,12,18)$

$s_2 = (4 - x_1, 12, 18 - 3x_1)$

$f_2(R_1, R_2, R_3, x_3) = 5x_2$

$f_1(4, 12, 18, x_1) = 3x_1 + \underset{x_n}{Max}\ 5x_2$

$$\Rightarrow f_n^*(R_1, R_2, R_3) = Max \quad f_n(R_1, R_2, R_3, x_n)$$

$$f_1(4,12,18, x_1) = 3x_1 + f_2^*(4 - x_1, 12, 18 - 3x_1)$$

$$f_1^*(4,12,18) = Max \quad f_1(4,12,18, x_1)$$

$$\left.\begin{array}{r} x_1 \le 4 \\ 3x_1 \le 18 \\ x_1 \ge 0 \end{array}\right\} 0 \le x_1 \le 4$$

$$x_2 \le \frac{R_2}{2}$$

$$x_2 \le \frac{R_3}{2}$$

(2) solution

- n = 2

s_1	$f_2^*(R_1, R_2, R_3)$	x_2^*
$R_i \ge 0$	$5\min\left\{\frac{R_2}{2}, \frac{R_3}{2}\right.$	$\min\left\{\frac{R_2}{2}, \frac{R_3}{2}\right.$

- n = 1

$$f_1^*(4,12,18, x_1) = 3x_1 + f_2^*(4 - x_1, 12, 18 - 3x_1)$$

$$= Max\ \{3x_1 + f_2^*(4 - x_1, 12, 18 - 3x_1)\} \qquad \begin{array}{r} x_1 \le 4 \\ 3x_1 \le 18 \\ x_1 \ge 0 \end{array}$$

$$= Max\ \left\{3x_1 + 5\min\left(\frac{12}{2}, \frac{18-3x_1}{2}\right)\right\} \qquad 0 \le x_1 \le 4$$

$$= Max\ \begin{cases} 3x_1 + 30 & 0 \le x_1 \le 2 \\ -\frac{9}{2}x_1 + 45 & 2 \le x_1 \le 4 \end{cases}$$

$\Rightarrow x_1^* = 2$

$\Rightarrow R_1 = 4 - 2 = 2 \qquad R_2 = 12 \qquad R_3 = 18 - 3(2) = 12$

$\Rightarrow n = 2, X_2^* = 6 \qquad f = 36$

Example3.1: (目標式改變，求法改變)

$$Max \quad 3x_1 \times 5x_2$$

$$s.t.$$

$$x_1 \le 4$$

$$2x_2 \le 12$$

$$2x_1 + 2x_2 \le 18$$

$$x_1, x_2 \ge 0$$

Sol:

$$f_1^*(4,12,18) = Max \;\; 3x_1 \times 5 \min\left\{6, \frac{18-3x_1}{2}\right\}$$

$$0 \le x_1 \le 4$$

(1) $0 \le x_1 \le 2$

$$3x_1 \cdot 5 \cdot 6 = 90x_1$$

$$x_1 = 2 \quad , f_1 = 180$$

(2) $2 \le x_1 \le 4$

$$3x_1 \cdot \frac{18-3x_1}{2} = 27x_1 - \frac{9}{2}x_1^2$$

$$\Rightarrow \frac{\partial f}{\partial x_1} = 0 \qquad 27x_1 - 9x_1^2 = 0 \qquad x_1 = 3$$

$$\frac{\partial f}{\partial x_1^2} = -9 \qquad x_1^* = 3$$

$$f = 81 - \frac{81}{2} = \frac{81}{2}$$

- Knapsack problem
  - Carloading problem
- To determining amount of each item such as to maximize total utility without capacity violation
- Each item with different unit weight and unit utilities

model

$$Max \;\; \sum_i^n u_i x_i$$

$$s.t. \;\; \sum_i^n W_i x_i \le W$$

Example4:

A hiker can carry up to 11 lbs of food in his knapsack .He must select amoung four items as :

Item	Unit weight	Unit utility
A	4	12
B	3	6
C	2	5
D	1	2

Sol:

Stage:item i

State s_i : weight capacity left for item to select

x_i : number of item i selected

Recorsire function

$$S_1 = 11$$
$$S_2 = 11 - 4X_1$$
$$S_3 = 11 - 4X_1 - 3X_2$$
$$S_4 = 11 - 4X_1 - 3X_2 - 2X_3$$

$$f_i(x) = \max_{0 \le Z \le S_i/W_i} u_i Z_i + f_{i+1}(S_i - W_i Z_i)$$

stage n=4 $f_4(x_4) = 2x_4$ $\quad 0 \le x_4 \le s_4 \Rightarrow x_4^* = s_4^*$

stage n=3 $f_3(x_3) = \max \ 5x_3 + f_4(s_3 - 2x_3)$

$= \max \ 5x_3 + f_4(s_3 - 2x_3)$

$= \max \ x_3 + 2s_3$ $\quad 0 \le x_3 \le \left[\frac{s_3}{2}\right]$

	0	1	2	3	4	5	6	7	8	9	10	11
f_3	0	2	5	7	10	12	15	17	20	22	25	27
x_3^*	0	0	1	1	2	2	3	3	4	4	5	5

stage n=2 $f_2(x_2) = \max \ 6x_2 + f_3(s_2 - 3x_2)$

$= \max \ 6x_2 + 2(s_2 - 3x_2) + \left[\frac{s - 3x_2}{2}\right]$

$= \max \ 2s_2 + \left[\frac{s - 3x_2}{2}\right]$ $\quad 0 \le x_2^* \le \left[\frac{s_2}{3}\right] \Rightarrow x^* = 0$

	0	1	2	3	4	5	6	7	8	9	10	11
f_2	0	2	5	7	10	12	15	17	20	22	25	27
x_2^*	0	0	0	0	0	0	0	0	0	0	0	0

stage n=1 $f_1(x_1) = \max_{0 \le Z \le [11/4]} 12x1_2 + f_2(11 - 4x_1)$

S_2	X_2 0	1	2	1	X_1^*
11	0+27	12+17	24+7	31	2

$X_1^* = 2 \Rightarrow S_2 = 3 \Rightarrow X_2^* = 0$

$S_3 = 3 \Rightarrow X_3^* = 1$

$S_4 = 3 - 1 \times 2 = 1 \Rightarrow X_4^* = 1$

Example5 :

Scheduling employment levels

Season	spring	summer	autum	winter	spring
requirement	255	220	240	200	255

— Employment should not below requirement

— employment above requirement is 200 $(x_i - x_{i-1})^2$

— fractional is possible by part=time employees

— cost can be fractional

Sol:

Stage : season 1 summer
2 autumn
3 winter
4 spring

x_n: employment level for stage n （n=1,2,3,4）

$x_4 = 255$

—why spring be last stage , optimal value of last stage must be either known or without considerly other stayer (spring is peak season)

for any other season , we must consider impact on the following seasons.

$r_n =$ min requirement for stage n

$r_1 = 220$

$r_2 = 240$

$r_3 = 200$

$r_4 = 255$

feasible range $r_n \le x_n \le 255$

n	r_n	range	$s_n = x_{n-1}$	cost
1	220	$200 < x_1 < 255$	$s_1 = 255$	$200(x_1 - 255)^2 + 2000(x_1 - 220)$
2	240	$240 < x_2 < 255$	$220 < s_2 < 255$	$200(x_1 - x_2)^2 + 2000(x_2 - 240)$
3	200	$200 < x_3 < 255$	$240 < s_3 < 255$	$200(x_2 - x_3)^2 + 2000(x_3 - 200)$
4	255	$x_4 = 255$	$200 < s_4 < 255$	$200(255 - x_3)^2$

cost for stage n $= 200(x_n - x_{n-1})^2 + 2000(x_n - r_n)$

state : $s_n = x_{n-1}$

— Model

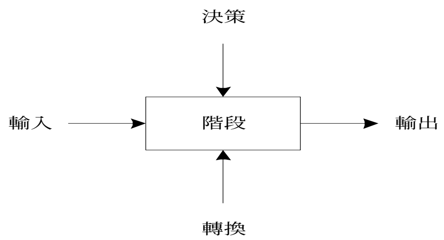
$$\text{Min} \quad \sum_{i=1}^{4}\left[\ 200(x_i - x_{i-1})^2 + 2000(x_i - r_i)\ \right] \qquad r_i \le x_i \le 255$$

$$f_n(s_n, x_n) = 200(x_n - s_n)^2 + 2000(x_n - r_n) + \min_{r_i \le x_i \le 255} \sum_{i=n+1}^{4}\left[200(x_i - x_{i-1})^2 + 2000(x_i - r_i)\right]$$

n=4 $\Rightarrow f_n^*(s_n) = \min_{r_n \le x_n \le 255} f_n(s_n, x_n)$

$$f_n(s_n, x_n) = 200(x_n - s_n)^2 + 2000(x_n - r_n) + f_{n+1}^*(x_n)$$

set $f_5 = 0$



n=4

s_4	$f_4^*(s_4)$	x_4^*
$200 \le s_4 < 255$	$200(255 - s_4)^2$	255

n=3 $f_3^*(s_3) = \min_{200 < x_3 \le 255}\left[200(x_3 - s_3)^2 + 2000(x_3 - 200) + f_4^*(x_3)\right]$

$$= \min_{200 < x_3 \le 255}\left[200(x_3 - s_3)^2 + 2000(x_3 - 200) + 200(255 - x_3)^2\ \right]$$

$$\frac{\partial f_3(s_3, x_3)}{\partial x_3} = 400(x_3 - s_3) + 2000 - 400(255 - x_3)$$

$$= 400(2x_3 - s_3 - 250) = 0$$

$$\Rightarrow x_3 = \frac{s_3 + 250}{2}$$

$$f_3^*(s_3^*) = 200\left(\frac{s_3 + 250}{2} - s_3\right)^2 + 200\left(255 - \frac{s_3 + 250}{2}\right)^2 + 2000\left(\frac{s_3 + 250}{2} - 200\right)$$

s_3	$f_3^*(s_3^*)$	x_3^*
$240 \le s_3 \le 255$	$50(250 - s_3)^2 + 50(260 - s_3)^2 + 1000(s_3 - 150)$	$\frac{s_3 + 250}{2}$

n=2 $f_2(s_2, x_2) = 200(x_2 - s_2)^2 + 2000(x_2 - r_2) + f_3^*(x_2)$

$$= 200(x_2 - s_2)^2 + 2000(x_2 - 240) + 50(250 - x_2)^2 + 50(260 - x_2)^2 + 1000(x_2 - 150)$$

$$f_2^*(s_2) = \min_{240 \le x_2 \le 255} f_2(s_2, x_2)$$

$$\frac{\partial f(s_2, x_2)}{\partial x_2} = 400(x_2 - s_2) + 2000 - 100(250 - x_2) - 100(260 - x_2) + 1000$$

$$= 200(3x_2 - 2s_2 - 240) = 0$$

$$\Rightarrow x_2 = \frac{2s_2 + 240}{3}$$

$$\frac{\partial^2 f(s_2, x_2)}{\partial x_2^2} = 600 > 0 \qquad \text{for} \quad 240 \le x_2 \le 255$$

$$\frac{\partial f(s_2, x_2)}{\partial x_2} > 0 \Rightarrow \min \quad x_2 = 240 \qquad \text{for} \quad 240 \le x_2 \le 255 \qquad \text{with} \ s_2 \le 240$$

s_2	$f_3^*(s_3^*)$	s_2^*
$220 \le s_2 \le 240$	$200(240 - s_2)^2 + 11500$	240
$240 \le s_2 \le 250$	$\frac{200}{9}\left[(240 - s_2)^2 + (250 - s_2)^2 + (270 - s_2)^2 + 2000(s_2 - 195)\right]$	

n=1

$$f_1(s_1, x_1) = 200(x_1 - s_1)^2 + 2000(x_1 - r_1) + f_2^*(x_1)$$

$$\Rightarrow r = 220$$

$$\Rightarrow 220 \le x_1 \le 255$$

$$f_1(s_1, x_1) = 200(x_1 - s_1)^2 + 2000(x_1 - 200) + 200(240 - x_1)^2 + 115000$$

$$\text{if} \quad 220 \le x_1 \le 240$$

$$= 200(x_1 - s_1)^2 + 2000(x_1 - 220) + \tfrac{200}{9}\left[(240 - x_1)^2 + (255 - x_1)^2 + (270 - x_1)^2\right] + 2000(x_1 - 195)$$

$$\text{if} \quad 240 \le x_1 \le 255$$

$$\frac{\partial f_1(s_1, x_1)}{\partial x_1} = 400(x_1 - s_1) + 2000 - 400(240 - x_1) = 400(2x_1 - s_1 - 235)$$

已知$s_1 = 255$

$$\therefore \frac{\partial f(s_1, x_1)}{\partial x_1} = 800(x_1 - 245) < 0 \qquad \text{for} \quad x_1 \le 240$$

所以 $x_1 = 240$ 是在 $220 \le x_1 \le 240$ 範圍內使 $f_1(s_1, x_1)$ 極小的值

當 $240 \le x_1 \le 255$

$$\frac{\partial f(s_1, x_1)}{\partial x_1} = 400(x_1 - s_1) + 2000 - \tfrac{400}{9}\left[(240 - x_1) + (255 - x_1) + (270 - x_1)\right] + 2000$$

$$= \tfrac{400}{3}(4x_1 - 3s_1 - 225)$$

因為 $\frac{\partial^2 f(s_1, x_1)}{\partial x_1^2} => 0 \quad \text{for all} \ x_1$

令 $\frac{\partial f(s_1, x_1)}{\partial x_1} = 0$

$$\Rightarrow x_1 = \tfrac{3s_1 + 225}{4}$$

$\because s_1 = 255$
$\therefore x_1 = 247.5$ 使 $f_1(s_1, x_1)$ 在 $240 \le x_1 \le 255$ 範圍內最小

最後以 $s_1 = 255$ 把 $x_1 = 247.5$ 代入 $f_1(255, x_1)$，以找到 $f_1^*(s_1)$

$$f_1^*(255) = 200(247.5 - 255)^2 + 2000(247.5 - 200)$$

$$+ \frac{200}{9}\left[2(250 - 247.5)^2 + (265 - 247.5)^2 + 30(742.5 - 575)\right]$$

$$= 185000$$

$x_1^* = 247.5, x_2^* = 245, x_3^* = 247.5, x_4^* = 255$ 每循環之總成本為＄185000

Example6: 某電子系統由四部分構成，需每一部份發生作用，整個系統始能發生作用，就其一部份或數部分裝置若干平行單位，可增進系統之可靠性。下表所列為各部分各有一、二、或三平行單位時，能發生作用之機率：

平行單位數	部分一	部分二	部分三	部分四
1	0.5	0.6	0.7	0.5
2	0.6	0.7	0.8	0.7
3	0.8	0.8	0.9	0.9

該系統發生作用之機率為各部分發生作用之機率之積。
各部分裝置一、二、三平行單位之成本如下：

平行單位數	部分一	部分二	部分三	部分四
1	1	2	1	2
2	2	4	3	3
3	3	5	4	4

由於預算之限制，至多僅得支用$1000。
試用動態規劃，決定各部分應裝平行單位若干，始能使該系統發生作用之機率最大。

Sol:

令 x_n 為裝設在第 n 個部分之平行單位數。

$P_n(x_n)$ 為裝設 x_n 個平行單位數之部分，其發生功能之機率。

$C_n(x_n)$ 為第 n 部分裝設 x_n 個單位之裝設成本。

S_n 表剩餘準備花費之金額。

$$f_n^*(s) = \max_{x_n=1,2,3}\left[P_n(x_n) * f_{n+1}(s - C_n(x_n))\right]$$

n=4

S	$f_4^*(S)$	x_4^*
$0 \le s \le 190$	0	0
$200 \le s \le 290$	0.5	1
$300 \le s \le 390$	0.7	2
$400 \le s \le 1000$	0.9	3

n=3

S_3 \ x_3	$f_3(s_3,x_3)=P_3(x_3)\cdot f_4^*(s_3-x_3)$				$f_2^*(s_3)$	x_3^*
	0	1	2	3		
$0 \le s \le 290$		0			0	0
$300 \le s \le 390$	0	0.35	0		0.35	1
$400 \le s \le 490$	0	0.49	0	0	0.49	1
$500 \le s \le 590$	0	0.63	0.4	0	0.63	1
$600 \le s \le 690$	0	0.63	0.56	0.45	0.63	1
$700 \le s \le 790$	0	0.63	0.72	0.63	0.72	2
$800 \le s \le 1000$	0	0.63	0.72	0.81	0.81	3

$n=2$

x_2 / S_2	$f_2(s_2, x_2)$				$f_2^*(s)$	x_2^*
	0	1	2	3		
$0 \le s \le 590$	0	0	0		0	0
$600 \le s \le 690$	0	0.294	0	0	0.294	1
$700 \le s \le 790$	0	0.378	0.245	0	0.378	1
$800 \le s \le 890$	0	0.378	0.343	0.280	0.378	1
$900 \le s \le 990$	0	0.432	0.441	0.392	0.441	2
1000	0	0.486	0.411	0.504	0.504	3

$n=1$

x_1 / S_1	$f_1(s_1, x_1)$				$f_1^*(s)$	x_1^*
	0	1	2	3		
1000	0	0.2205	0.2268	0.3024	0.3024	3

第一部份裝設 3 個平行單位
第二部份裝設 2 個平行單位
第三部份裝設 1 個平行單位
第四部份裝設 1 個平行單位

Example7:考慮下列非直線規劃問題：

Max $\quad Z = 12x_1 + 3x_1^2 - 2x_1^3 + 12x_2 - x_2^3$

St. $\quad x_1 + x_2 \le 3$

$x_1 \ge 0 \quad x_2 \ge 0$

試以動態規劃求解。

Sol:

令 R 表示在限制式 $x_1 + x_2 \le 3$ 內之差額餘數

則 $f_2^*(R) = \underset{0 \le x_2 \le R}{Max}\{12x_2 - x_2^3\}$

計算 x_2^*

$$\frac{\partial f_2(R, x_2)}{\partial x_2} = 12 - 3x_2^2 \ge 0 \quad (0 \le x_2 \le 2\text{時})$$

$$\frac{\partial^2 f_2(R, x_2)}{\partial x_2^2} = -6x_2 < 0 \quad (x_2 = 2\text{時})$$

➔ $x_2^* = \begin{cases} R(0 \le R \le 2) \\ 2(2 \le R \le 3) \end{cases}$

R	$f_2^*(R)$	x_2^*
$0 \le R \le 2$	$12R - R^3$	R
$2 \le R \le 3$	16	2

n=1 時

$$f_1^*(s_3) = \min_{0 \le x_1 \le 3}\left[12x_1 + 3x_1^2 - 2x_1^3 + f_2^*(3 - x_1)\right]$$

$$=\max \begin{bmatrix} \max_{0 \le x_1 \le 1}\left(12x_1 + 3x_1^2 - 2x_1^3 + 16\right) \\ \max_{1 \le x_1 \le 3}(12x_1 + 3x_1^2 - 2x_1^3 + 12(3 - x_1) - (3 - x_1)^3) \end{bmatrix}$$

就 $0 \le x_1 \le 1$ 而言

$$\frac{\partial f_1(3, x_1)}{\partial x_1} = -2(3x_1^2 - 3x_1 - 14) > 0$$

故 $f_1(3, x_1)$ 在 $x_1^* = 1$ 具有極大值

就 $1 \le x_1 \le 3$ 而言

$$\frac{\partial f_1(3, x_1)}{\partial x_1} = -3(x_1^2 + 4x_1 - 9)$$

令 $-3(x_1^2 + 4x_1 - 9) = 0$

$x_1 = -2 + \sqrt{13}$

$$\frac{\partial^2 f_1(3, x_1)}{\partial^2 x_1} = -3(2x_1 + 4) < 0$$

故 $f_1(3, x_1)$ 在 $x_1^* = -2 + \sqrt{13}$ 具有極大值

由上述可知最佳解為 $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 + \sqrt{13} \\ 5 - \sqrt{13} \end{pmatrix} = \begin{pmatrix} 1.606 \\ 1.394 \end{pmatrix}$

$Z = 12(1.606) + 3(1.606)^2 - 2(1.606)^3 + 12(1.394) - (1.394)^3 = 32.739$