CHAP 11 Queuing Theory

11.1 Introduction

A.Waiting in line for service

arrival departure

○○○→ service facility ○○○→

B.Characteristics of queuing process

—arrival pattern of customers

(1) input to queuing system

(2) basic measure

· average arrivals per unit time (mean arrival rate)

· average time between successive arrival (mean inter arrival rate)

(3) type of customers

· patient customers

· impatient customer : $\begin{cases} balked \\ reneged \\ jackey \end{cases}$

(4) type of pattern

· stationary: time-independent

· non-stationary: time-dependent

—service patterns of servers

(1) service type

· single

· batch

(2) service rate

· state-dependent-depend on number in waiting

· independent of number of customers

—queuing discipline

(1) First In First Out (FIFO)

(2) First In Last Out (FILO)

(3) Service In Random Order (SIRO)

(4) Variety Priority Schemes (PRI)

－system capacity

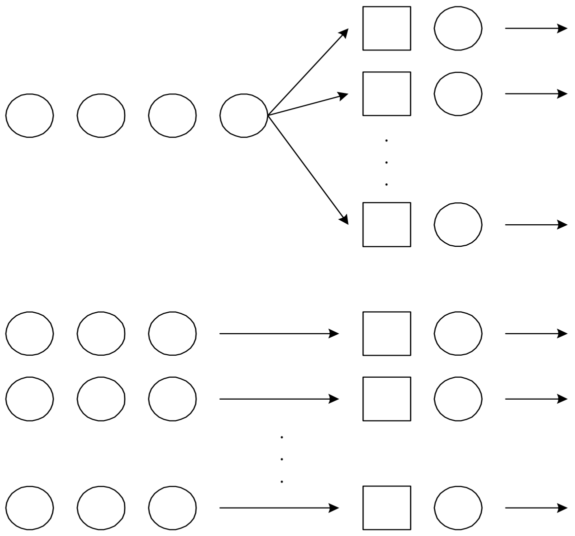
(1) finite queuing

(2) infinite

－number of service channels

(1) number of parallel service channels

(2) two types of multiple channel system



－stage of services

(1) single stage

(2) multiple stages

· recycle

· non-recycle

B.Operations Characteristics (Measure of Effectiveness)

－idle time (P_0)

－the prob. of a specific number of units in the system (P_n)

－average number of units in the system (L)

－average time each unit spends in system $(E(w))$

－average waiting time $(E(w_q))$

－average number in queue (L_q)

－prob. of a customer have to wait $(P_w = 1 - P_0)$

C.Notation (A/B/X/Y/Z)

A: interarrival time distribution

B: service time distribution

X: number of parallel service channels

Y: restriction on system capacity

Z: the queuing discipline

D.Symbol for distribution

M: Exponential distribution

D: deterministic

Ek: Erlang type k

GI: general independent for arrival

G: general for service

11.2 The Birth-and-Death Process

A.Most elementary queuing model

B.The state of the system at time t (N(t)) is the number of customers in the system at time t

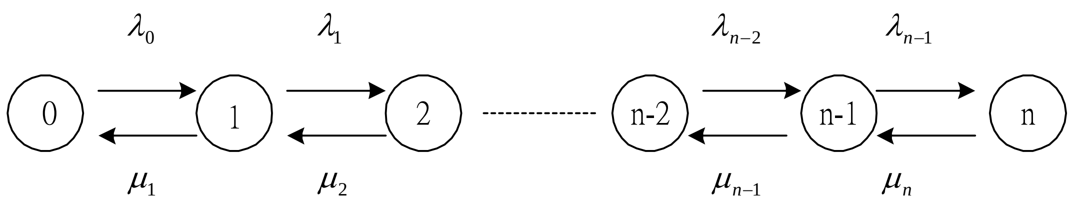
C.Three basic assumptions

－given N(t)=n, the current prob. dist. of the remaining time until the next birth (arrival) is exponential with parameter

$\lambda_n (n = 0,1..., n)$

－given N(t)=n, the current prob. dist. of the remaining time until next death (service completion) is exponential with parameter μ_n

－only one birth or death can occur at a time



－start with 0

(1) count is mode of the number of time that process enters the state and the number of times it leaves the state

· $E_n(t)$ = number of times that enters state n by time t

· $L_n(t)$ = number of times that leaves state n by time t

(2) two types of events must alternative

· $\left|E_n(t) - L_n(t)\right| \le 1$

(3) diviching both sides by t and let $t \to \infty$

· $\lim_{t\to\infty}\left|\frac{E_n(t)}{t} - \frac{L_n(t)}{t}\right| \le \frac{1}{t} = 0$

(4) rate in = rate out

· for state 0

$$In \Rightarrow \mu_1 P_1 + 0(1 - P_1) = \mu_1 P_1$$
$$Out \Rightarrow \lambda_0 P_0$$

state $0 \to \mu_1 P_1 = \lambda_0 P_0$

D.Balance equation for Birth-and-Death process

State	Rate in=Rate out
0	$\mu_1 p_1 = \lambda_0 p_0$
1	$\lambda_0 p_0 + \mu_2 p_2 = (\lambda_1 + \mu_1) p_1$
2	$\lambda_1 p_1 + \mu_3 p_3 = (\lambda_2 + \mu_2) p_2$
.	
.	
n-1	$\lambda_{n-2} p_{n-2} + \mu_n p_n = (\lambda_{n-1} + \mu_{n-1}) p_{n-1}$
n	$\lambda_{n-1} p_{n-1} + \mu_{n+1} p_{n+1} = (\lambda_n + \mu_n) P_n$

State	Prob.
0	$P_1 = \frac{\lambda_0}{\mu_1} P_0$
1	$P_2 = \frac{\lambda_1}{\mu_2} P_1 + \frac{1}{\mu_2}(\mu_1 P_1 - \lambda_0 P_0) = \frac{\lambda_1}{\mu_2} P_1 = \frac{\lambda_1 \lambda_0}{\mu_1 \mu_2} P_0$
2	$P_3 = \frac{\lambda_2}{\mu_3} P_2 + \frac{1}{\mu_3}(\mu_2 P_2 - \lambda_1 P_1) = \frac{\lambda_2}{\mu_3} P_2 = \frac{\lambda_2 \lambda_1 \lambda_0}{\mu_1 \mu_2 \mu_3} P_0$
.	
.	
n	$P_n = \frac{\lambda_{n-1}}{\mu_n} P_n + \frac{1}{\mu_n}(\mu_{n-1} P_{n-1} - \lambda_{n-2} P_{n-2}) = \frac{\lambda_{n-1}}{\mu_n} P_{n-1} = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\cdots\lambda_0}{\mu_n \mu_{n-1}\cdots\cdots\mu_1} P_0$

$\because P_n = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\cdots\lambda_0}{\mu_n \mu_{n-1}\cdots\cdots\mu_1} P_0 = C_n P_0$ (where $C_n = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\cdots\lambda_0}{\mu_n \mu_{n-1}\cdots\cdots\mu_1}$)

$$\sum_{n=0}^{\infty} P_n = 1, \quad \left(1 + \sum_{n=1}^{\infty} C_n\right) P_0 = 1$$

$$\therefore P_0 = \frac{1}{1+\sum_{n=1}^{\infty} C_n}$$

$$L = \sum_{n=0}^{\infty} nP_n$$

$$L_q = \sum_{n=s}^{\infty} (n-s)P_n$$

$$W = L/\bar{\lambda} \qquad \left(\bar{\lambda} = mean \quad arrival \quad rate = \sum_{n=0}^{\infty} \lambda_n P_n \right)$$

$$W_q = L_q/\bar{\lambda}$$

11.3 Queuing Model Based on The Birth-and-Death Process

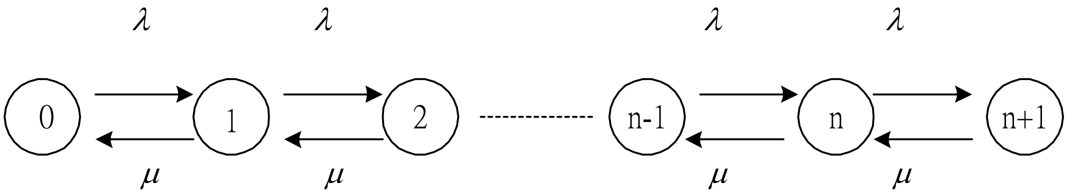
A.M/M/1/ ∞ /FIFO

－interarrival times are exponential dist. (Poisson Arrival with λ)

－service time is an exponential dist. (μ)

$$C_n = \left(\tfrac{\lambda}{\mu}\right)^n = \rho^n \qquad where \quad \rho = \tfrac{\lambda}{\mu}$$

$$\Rightarrow P_n = \rho^n P_0$$



$$P_o = \frac{1}{1+\sum_{n=1}^{\infty}\rho^n} = \left(\sum_{n=0}^{\infty}\rho^n\right)^{-1} = \left(\tfrac{1}{1-\rho}\right)^{-1} = 1-\rho$$

$$P_n = \rho^n P_o = (1-\rho)\rho^n$$

$$L = \sum_{n=0}^{\infty} n(1-\rho)\rho^n$$

$$= (1-\rho)\rho\sum_{n=0}^{\infty} n\rho^{n-1}$$

$$= (1-\rho)\rho\sum_{n=0}^{\infty} \tfrac{d}{d\rho}\rho^n$$

$$= (1-\rho)\rho\tfrac{d}{d\rho}\left(\sum_{n=0}^{\infty}\rho^n\right)$$

$$= (1-\rho)\rho\tfrac{d}{d\rho}\left(\tfrac{1}{1-\rho}\right)$$

$$= (1-\rho)\rho \frac{1}{(1-\rho)^2}$$

$$= \frac{\rho}{1-\rho}$$

$$= \frac{\lambda}{\mu-\lambda}$$

$$L_q = \sum_{n=1}^{\infty} (n-1) P_n$$

$$= \sum_{n=1}^{\infty} nP_n - \sum_{n=1}^{\infty} P_n$$

$$= L - (1 - P_o)$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)}$$

$$W = \frac{L}{\lambda}$$

$$= \frac{1}{\mu-\lambda}$$

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{\lambda}{(\mu-\lambda)\mu}$$

$$P_w = 1 - P_o = \rho = \frac{\lambda}{\mu}$$

$$P\{w > t\} = \sum_{n=0}^{\infty} P_n \times P\{S_{n+1} > t\} = e^{-\mu(1-\rho)t}$$

$$P\{w_q > t\} = \sum_{n=0}^{\infty} P_n \times P\{S_n > t\} = \rho e^{-\mu(1-\rho)t}$$

Example 1:某人開了一家一人理髮店，目前營業場所只能容納四張椅子（理髮椅不包含在內）。根據經驗，顧客到達時間間隔與服務時間大致成指數分配，顧客平均每小時4人，每人平均理髮時間爲12分鐘。店主欲知曉以下資料，以作爲是否要另聘員工擴大營業場所以維持一服務水準。

1. L, L_q, W, W_q 分別爲何？
2. 顧客不需等候即可理髮的機率
3. 顧客無座位可坐的機率
4. 顧客必須等候超過一小時才能開始理髮的機率

Sol: $\lambda = 4$ 人/小時 , $\mu = 5$人/小時 , $\rho = \dfrac{\lambda}{\mu} = \dfrac{4}{5}$

1. $L = \frac{\lambda}{\mu-\lambda} = \frac{4}{5-4} = 4$ $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{4^2}{5(5-4)} = 3.2$

   $W = \frac{1}{\mu-\lambda} = \frac{1}{5-4} = 1$（小時） $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{4}{5(5-4)} = 0.8$（小時）

2. $P_0 = 1-\rho = 1-0.8 = 0.2$

3. $P\{n \geq 5\} = \sum_{n=5}^{\infty}(1-\rho)\rho^n = \rho^5 = 0.328$

4. $P\{W_q > 1\} = \frac{4}{5}e^{-5\left(1-\frac{4}{5}\right)\times 1} = 0.294$

Example 2:汽車按照平均數爲每小時90輛的Poiisson distribution 到達車道上的一個收費站，通過收費站的平均時間是38秒，駕駛員埋怨等待時間太長。主管單位想採用新裝置，使通過該站的時間減少到平均30秒。但這只有在舊系統中等待的汽車超過平均5輛，新系統中該站的空閒時間不超過10%時才是合算的。根據這個要求，問新裝置是否合算？

Sol:

$\lambda = 90$（輛/小時） 系統服務率 $\mu_{old} = 3600/38$（輛/小時） $\mu_{new} = 3600/30$（輛/小時）

因此;

系統服務強度

$$\rho_{old} = \frac{90\times 38}{3600} = 0.95$$

$$\rho_{new} = \frac{90\times 30}{3600} = 0.75$$

舊系統等候車隊長 $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \dfrac{\rho_{old}}{1-\rho_{old}} - \rho_{old} = \dfrac{0.95}{0.05} - 0.95 = 18.05$ 輛

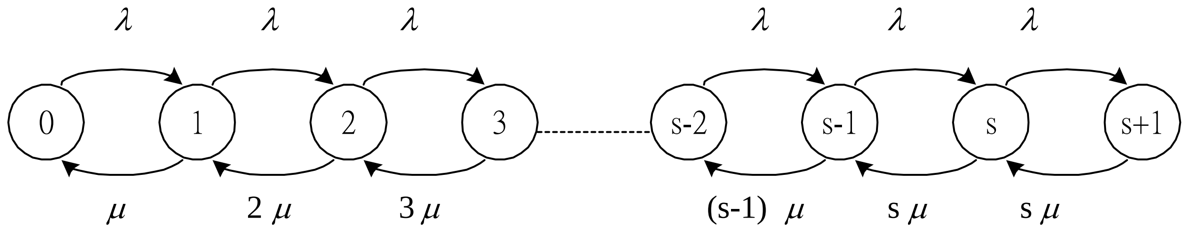
新系統空間概率 $P_0 = 1-\rho_{new} = 1-0.75 = 0.25 = 25\%$

根據舊系統中汽車等待數（=19）超過5輛的要求，新裝置是合算的；但根據新裝置的空閒時間（=25% ）不得超過10% 的要求，新裝置則是不合算的。

B.M/M/S/ ∞ /FIFO

－ S > 1

・ $C_n = \dfrac{\left(\frac{\lambda}{\mu}\right)^n}{n!}$ for n=1,2,…..,s



· $C_n = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!}\left(\frac{\lambda}{s\mu}\right)^{n-s} = \frac{\left(\frac{\lambda}{\mu}\right)^n}{s!\,s^{n-s}}$ for n=s+1,……. ,∞

－if $\lambda < s\mu$

· $P_o = \left[\sum_{n=0}^{s-1}\frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!}\sum_{n=s}^{\infty}\left(\frac{\lambda}{s\mu}\right)^{n-s}\right]^{-1}$

$= \left[\sum_{n=0}^{s-1}\frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!}\frac{1}{1-\frac{\lambda}{s\mu}}\right]^{-1}$

· $P_n = \begin{cases} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}P_o & ,0 < n \le s \\ \frac{\left(\frac{\lambda}{\mu}\right)^n}{s!\,s^{n-s}}P_o & ,n > s \end{cases}$

· $L_q = \sum_{n=s}^{\infty}(n-s)P_n = \frac{\left(\frac{\lambda}{\mu}\right)^s \lambda\mu}{(s-1)!(s\mu-\lambda)^2}P_o$

· $L = L_q + \frac{\lambda}{\mu}$

· $W_q = \frac{L_q}{\lambda}$

· $W = W_q + \frac{1}{\mu}$

· if $\lambda > s\mu$, the queue explode i.e. unstable

Example 1 :某醫院急診室星期天平均每40分鐘有一位病人到達，根據過去資料顯示，到達時間大致呈現指數分配。目前急診室僅有一位醫生，其服務時間成指數分配，平均24分鐘可醫療一位病人。為恐病人等候時間過久延誤醫治時效，所以醫院考慮增加一位或兩位醫生。試分析醫院是否需增加醫生人數？應增加幾人？

Sol:

$\lambda = 1.5$人 / 小時 $\mu = 2.5$人 / 小時

（M/M/1）: $\rho = \frac{\lambda}{s\mu} = \frac{1.5}{2.5} = 0.6$

（M/M/2）: $\rho = \frac{\lambda}{s\mu} = \frac{1.5}{2\times2.5} = 0.3$

（M/M/3）: $\rho = \frac{\lambda}{s\mu} = \frac{1.5}{3\times2.5} = 0.2$

統計量	M/M/1	M/M/2	M/M/3
P_0	0.4	0.538	0.548
P_1	0.24	0.323	0.329
P_2	0.144	0.097	0.099
P_3	0.086	0.029	0.02
L_q	0.9	0.059	0.006
L	1.5	0.659	0.606
W_q	0.6	0.039	0.004
W	1	0.439	0.404

Example 2 :某銀行有三位出納員，顧客以平均速率爲4人/分鐘的Poisson dist.到達，所有顧客排成一隊，出納員與顧客的交易時間服從平均數爲0.5分鐘的指數分配。

1. 求出銀行內顧客數的穩態機率分佈
2. 求出L, L_q, W, W_q

Sol:

這是M/M/S/∞的排隊系統，其中$\lambda = 4, \mu = 2, S = 3, \rho = \frac{\lambda}{\mu} = \frac{4}{2} = 2$

1. $P_0 = \left[1 + 2 + 2 + \frac{4}{3} \cdot \frac{1}{1-2/3}\right]^{-1} = \frac{1}{9}$

   $P_1 = P_2 = \frac{2}{9}$

   $P_n = \dfrac{2^n}{6\times 3^{n-3}} P_0 = \left(\frac{27}{6}\right)\left(\frac{2}{3}\right)^n\left(\frac{1}{9}\right) = \frac{1}{2}\left(\frac{2}{3}\right)^n$ n>=3

2. $L_q = \dfrac{\frac{1}{9}\cdot 8 \cdot \frac{2}{3}}{6\cdot\frac{1}{9}} = \frac{8}{9}$ $L = \frac{8}{9} + 2 = 2\frac{8}{9}$

   $W_q = \dfrac{\frac{8}{9}}{4} = \frac{2}{9}$ $W = \frac{2}{9} + \frac{1}{2} = \frac{13}{18}$

Example 3 假設某機車檢驗站有2部排氣檢測機，前往該站檢驗排汔的機車抵達時間間隔爲指數分配，平均抵達率爲每小時10部，每部機車檢驗時間需花費5分鐘(1)平均會有多少機車在該檢驗站？有多少機車在等候？需排多久？(2)欲至該站接受檢驗的機車，有多少機車需排隊等候？

Sol:

這是M/M/2/∞的排隊系統，其中$\lambda = 10, \mu = 60/5 = 12, S = 2$

$$\rho = \frac{\lambda}{\mu} = \frac{10}{2*12} = 0.4167$$

$$L_q = \sum_{n=s}^{\infty}(n-s)P_n = \frac{\left(\frac{\lambda}{\mu}\right)^s \lambda\mu}{(s-1)!(s\mu-\lambda)^2}P_o = \frac{\left(\frac{10}{12}\right)^2 *10*12}{(2-1)!(2*12-10)^2} * \frac{7}{17} = 0.175$$

$$P_o = \left[1 + \sum_{n=1}^{s-1}\frac{s^n\rho^n}{n!} + \frac{s^n\rho^n}{s!(1-p)}\right]^{-1} = \left[1 + \sum_{n=1}^{s-1}\frac{2^1(\frac{5}{12})^1}{1!} + \frac{2^2(\frac{5}{12})^2}{2!(1-(\frac{5}{12}))}\right]^{-1} = \frac{7}{17} = 0.4117$$

$$W = W_q + \frac{1}{\mu} = 0.0175 + \frac{1}{12} = 0.1008$$

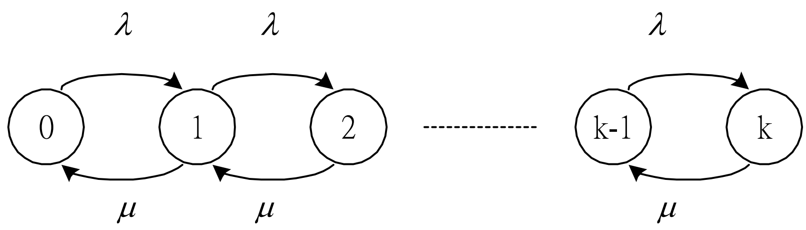
$$L = L_q + \frac{\lambda}{\mu} = 0.175 + \frac{10}{12} = 1.008 = \lambda w = 10 * 0.1008$$

C.M/M/S/K/FIFO

－finite queue

limited number of customers in the system(k)

$$\lambda_n = \begin{cases} \lambda & for \quad n = 0,1,.....,k-1 \\ 0 & for \quad n \geq k \end{cases}$$



(1) M/M/1/K/FIFO

· $C_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n = \rho^n & n = 1,2,....,k \\ 0 & n > k \end{cases}$

· $P_0 = \left[\sum_{n=0}^{k}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1} = \left[\frac{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\frac{\lambda}{\mu}}\right]^{-1} = \frac{1-\rho}{1-\rho^{k+1}}$

· $P_n = \left(\frac{1-\rho}{1-\rho^{k+1}}\right)\rho^n$

· $L = \sum_{n=0}^{k} nP_n$

$$= \frac{1-\rho}{1-\rho^{k+1}} \rho \sum_{n=0}^{k} \frac{d}{d\rho}\left(\rho^{n}\right)$$

$$= \frac{1-\rho}{1-\rho^{k+1}} \rho \frac{d}{d\rho} \sum_{n=0}^{k}\left(\rho^{n}\right)$$

$$= \frac{1-\rho}{1-\rho^{k+1}} \rho \frac{d}{d\rho}\left(\frac{1-\rho^{k+1}}{1-\rho}\right)$$

$$= \rho \frac{(-k+1)\rho^{k} + k\rho^{k+1} + 1}{\left(1-\rho^{k+1}\right)(1-\rho)}$$

$$= \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}$$

- $L_q = L - (1 - P_0)$
- $W = L/\lambda$ when $\bar{\lambda} = \sum_{n=0}^{k-1} \lambda P_n = \lambda(1 - P_K)$
- $W_q = L_q/\lambda$

Example 1 : 某人開了一家一人理髮店，目前營業場所只能容納四張椅子（理髮椅不包含在內）。K=5，根據經驗，若顧客至理髮店時，已超過系統容量則不予以等待。顧客到達時間間隔與服務時間大致成指數分配，顧客到達的平均間隔爲45分鐘，每人平均理髮時間爲30分鐘。店主欲知曉以下資料，以評估營業狀況。

1. L, L_q, W, W_q 分別爲何？
2. 理髮店空閒的機率
3. 理髮店產生等候情況的機率

Sol: $\lambda = 4/3$ 人/小時 , $\mu = 2$人/小時 , $\rho = \frac{\lambda}{\mu} = \frac{2}{3}$

1. $P_0 = \frac{1-\rho}{1-\rho^{K+1}} = 0.04481$

3. $L = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}} = 2.4812$ $L_q = L - (1 - P_0) = 1.5293$

   $\bar{\lambda} = \lambda(1 - P_K) = 0.4759$

   $W = L/\bar{\lambda} = 5.213$（小時） $W_q = L_q/\bar{\lambda} = 3.2133$（小時）

2. $1 - P_0 = 1 - 0.0481 = 0.9519$

Example 2 : 某工廠買了許多同一型式之機器，現在需要確定一名工人應看管幾台機器，機器在正常運轉時是不需要看管的。已知每台機器的正常運轉時間服從平均數爲120分鐘的指數分配，工人看管一台機器的時間服從平均數爲12分鐘的指數分配。每名工人只能看管自己的機器，工廠要求每台機器的正常運轉時間不得少於87.5% 。問在此條件之下每名工人最多能看管幾台機器？

Sol:這是M/M/1,有限來源排隊系統。每名工人看管最多台數爲 k

$\lambda = \frac{1}{2}, \mu = 5$
$\therefore \rho = \frac{1}{10}$ 因此，

$$L_s = k - \frac{\mu}{\lambda}(1 - P_0) = k - \rho(1 - P_0)$$

$$P_0 = \left[\sum_{n=0}^{k} \frac{k!}{(k-n)!}\rho^n\right]^{-1}$$

根據要求，停止運轉的機器數 $L_s \le 0.125k$ 。當m=1,2,3,4,5,時如以下之結果：

k	P_0	L_s	$L_s \le 0.125k$ ？
1	10/11	0.091	是(0.125)
2	50/61	0.197	是(0.25)
3	500/683	0.321	是(0.375)
4	1250/1933	0.467	是(0.5)
5	2500/4433	0.640	否(0.625)

可見一名工人最多只能看管四台機器

(2) M/M/S/K/FIFO

$$\cdot\ C_n = \begin{cases} \frac{(\lambda/\mu)^n}{n!} & , n = 1,2,\ldots,s \\ \frac{(\lambda/\mu)^n}{s!\,s^{n-s}} & , n = s, s+1,\ldots,k \\ 0 & , n > k \end{cases}$$

- $P_n = \begin{cases} \dfrac{(\lambda/\mu)^n}{n!} P_O & , n = 1,2,\ldots,s \\ \dfrac{(\lambda/\mu)^n}{s!\,s^{n-s}} P_O & , n = s, s+1,\ldots,k \\ 0 & , n > k \end{cases}$

- $P_O = \left[1 + \sum_{n=1}^{s} \dfrac{(\lambda/\mu)^n}{n!} + \dfrac{(\lambda/\mu)^s}{s!} \sum_{n=s+1}^{k} \left(\frac{\lambda}{s\mu}\right)^{n-s} \right]^{-1}$

- $L = \sum_{n=0}^{s-1} nP_n + L_q + s\left(1 - \sum_{n=0}^{s-1} P_n\right)$

- $L_q = \dfrac{P_0 (\lambda/\mu)^s \rho}{s!(1-\rho)^2}\left[1 - \rho^{k-s} - (k-s)\rho^{k-s}(1-\rho)\right]$

Example 1 :某加油站有四個加油位置，三個等候位置。在尖峰時段平均每兩分鐘有一輛車到達，平均每輛車的加油時間為 6 分鐘。到達時間間隔與服務時間均呈指數分配：

（１）此加油站平均有多少輛車

（２）每輛車平均在加油站內的時間是多少

（３）每小時有多少輛車因加油車位已滿而無法進入加油位

Sol:

M/M/4/7/FIFO

$\lambda = 30$ unit/hr ， $\mu = 10$ unit/hr ， $\rho = \dfrac{\lambda}{s\mu} = \dfrac{30}{4*10} = 0.75$

（1） $P_0 = 1 \Big/ \left[1 + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \frac{3^4}{4!}\left(0.75^1 + 0.75^2 + 0.75^3\right)\right] = 0.045$

$L_q = \dfrac{3^4(0.75)(0.045)}{4!(1-0.75)^2}\left[1 - (0.75)^3 - 3(0.75)^3(0.25)\right]$

=0.477

$L = 3.2845$

（2） $P_7 = \dfrac{3^7}{4^3 4!}(0.045) = 0.0641$

$\bar{\lambda} = \lambda(1 - P_7) = 28.0770$

$W = L/\bar{\lambda} = 0.1170$

（3） $\lambda P_7 = 1.9230$ =30*0.0641

Example 2 :某戲院有三個售票窗口，每個售票窗口對於每位客人之平均出售時間爲5分鐘，依指數分佈；到達該戲院每小時平均6人，依Poisson分佈。但等候列的長度爲30個人，試求下列問題？

（１）系列中有n個客人的機率

（２）系列之呼損率（系列中已滿，再到達之客人不能進入系列中而馬上離開支機率）

Sol:此題爲M/M/3/30，S=3，K=30，$\lambda = 6$（人/小時）

$$\mu = \tfrac{60}{5}(人/分) = 12(人/小時)$$

$$\rho = \tfrac{\lambda}{3\mu} = \tfrac{1}{6} < 1, 先求 P_0$$

$$\sum_{n=0}^{3}\frac{1}{n!}\left(\tfrac{1}{2}\right)^n + \frac{3^3}{3!}\sum_{n=4}^{30}\left(\tfrac{1}{6}\right)^n$$

$$= 1 + \left(\tfrac{1}{2}\right) + \tfrac{1}{2}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{6}\left(\tfrac{1}{2}\right)^3 + \tfrac{9}{2} \times \frac{\left(\frac{1}{6}\right)^4\left[1 - \left(\frac{1}{6}\right)^{27}\right]}{1 - \frac{1}{6}}$$

$$= 1.65$$

$$P_0 = \tfrac{1}{1.65} = 0.61$$

（1） $$P_n = \begin{cases} \dfrac{1}{n!}\left(\dfrac{1}{6}\right)^n \times 0.61, & 0 \le n < 3 \\ \dfrac{1}{3!3^{n-3}}\left(\dfrac{1}{6}\right)^n \times 0.61, & 3 \le n < 30 \end{cases}$$

（2） $$P_N = \frac{\rho^N}{s!s^{N-s}}P_0 = \frac{1}{3!3^{27}}\left(\frac{1}{6}\right)^{30} \times 0.61 = 0$$

11.4 Economic Analysis

A.The Real Value of Queuing Model is for Alternative Evaluation

B.Cost Component

－waiting cost

－service cost

－system idle cost

C.Steps In Solving The Problem

－identify the objective

－list the variables and establish their value

－identify specific alternatives

－compute P_0, L, L_q, W, W_q

－compute total cost

T.C.=T.C. of serving + T.C. of waiting

$$= LC_W + KC_F + \lambda C_V$$

Example:

有一超級市場在顧客的期望下，該店經理相信顧客對需排隊等候8分鐘並且共花10分鐘在等候系統上（不含真正購物時間）是難以接受的。因此經理嘗試下列兩方案中的一個來縮短顧客等待時間：（1）增聘一位員工打包貨品；（2）增加一個結帳櫃臺。

方案1：增聘一位員工

每增聘一位員工該店將增加＄150的成本，而每縮短1分鐘顧客的等候就能避免＄75的銷售損失。

原先的服務率 $\mu = 30$（平均每小時有30位顧客結完帳離去）

新雇用一名員工後的服務率 $\mu = 40$（平均每小時有40位顧客結完帳離去）

假設平均到達率保持不變，$\lambda = 24$

Sol：

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{24}{40} = 0.40$$

$$L = \frac{\lambda}{\mu - \lambda} = \frac{24}{40 - 24} = 1.5$$

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{24^2}{40(16)} = 0.9$$

$$W = \frac{1}{\mu - \lambda} = \frac{1}{40 - 24} = 0.063$$

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{24}{40(16)} = 0.038$$

$$\rho = \frac{\lambda}{\mu} = \frac{24}{40} = 0.6$$

$$I = 1 - \rho = 1 - 0.6 = 0.4$$

每位顧客的平均等候時間由8分鐘縮減爲0.038小時（2.25分鐘），所節省的金額爲：

8.00-2.25＝5.75分

5.75分×＄75／分每週＝＄431.25（每週）

而雇用一位員工每週所需成本爲＄150，因此可節省之總金額爲：

＄431.25-＄150＝＄281.25（每週）

該店經理或許認同這種節省方案而增聘一位員工。

方案2：增加一個結帳櫃臺

這項計畫之總成本為＄6000，另外加上結帳員每週＄200的薪資。假設顧客無法在兩條等候線上自由走動，每條等候線的平均到達率$\lambda = 12$，每個櫃臺的服務率仍維持不變（$\mu = 30$）。

Sol：

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{12}{30} = 0.60$$

$$L = \frac{\lambda}{\mu - \lambda} = \frac{12}{30 - 12} = 0.67$$

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{12^2}{30(18)} = 0.27$$

$$W = \frac{1}{\mu - \lambda} = \frac{1}{30 - 12} = 0.055$$

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{12}{30(18)} = 0.022$$

$$\rho = \frac{\lambda}{\mu} = \frac{12}{30} = 0.4$$

$$I = 1 - \rho = 1 - 0.4 = 0.6$$

每位顧客的平均等候時間由8分鐘縮減為0.022小時（1.33分鐘），所節省的金額為：

8.00-1.33＝6.67分

6.67分×＄75／分每週＝＄500.00（每週）

而雇用一位員工每週所需成本為＄200，因此可節省之總金額為：

＄500-＄200＝＄300（每週）

該計畫的資本支出為＄6000，需花20週的時間才能回收成本（不考慮利息）。一旦成本回收，該店選擇增加一個結帳櫃臺的策略比雇用一位新員工的策略多省下＄18.75的支出。然而，我們不能忽略需20週的時間才能回收成本的事實。