Chapter 8 Network Model

8.1 Introduction

A. Definition

— A set of nodes and a set of arcs connecting these nodes

— node：vertices , point

— arc：link , edge , line

— notation： G：(N,A)

B. Network Components

— Nodes

— Point where flow generated relayed or terminated

— Branch point of network

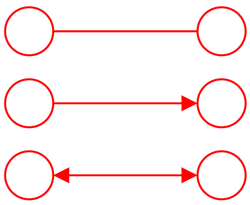
— Arc：a linkage connects two nodes

— Types

— Undirected

— Unidirectional

— Bi-directed



— Conservation of flow

total inflow + supply = total outflow

$$\sum X_{ik} + S_k = \sum X_{kj} \quad \cdots\cdots (1)$$

or

total inflow = total outflow + demand

$$\sum X_{ik} = \sum X_{kj} + D_k \quad \cdots\cdots (2)$$

— Source nodes：in Eq.（1） $\sum X_{ik} = 0$

— Sink nodes：in Eq.（2） $\sum X_{kj} = 0$

— Network representation

(1)Adjacent matrix of a directed network G

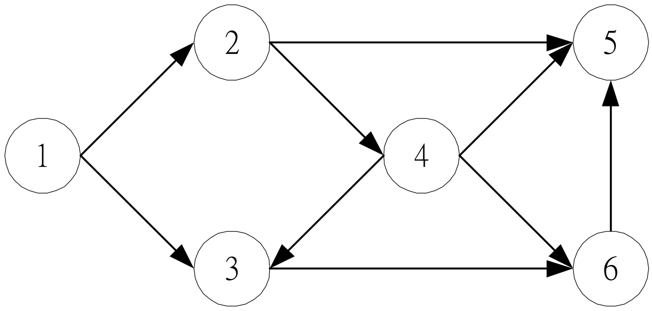
$$X_{ij} = \begin{cases} 1 & \textit{if there is arc from node } i \textit{ to node } j \\ 0 & \textit{otherwise} \end{cases}$$

(2)Node-arc incidence matrix

$$Z_{ik} = \begin{cases} 1 & \textit{node } i \textit{ is starting node of arc } k \\ -1 & \textit{node } i \textit{ is ending node of arc } k \\ 0 & \textit{otherwises} \end{cases}$$

— example：

— X_{ij}



— $X_{ij} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

1	2	3	4	5	6	7	8	9
(1,2)	(1,3)	(2,4)	(2,5)	(3,6)	(4,3)	(4,5)	(4,6)	(6,5)

— $Z_{ik} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}$

8.2 Shortest path problem

A. Introduction

— A path between two nodes is a sequence of distinct arcs connecting these two nodes.

(1)Directed path ：each arc in path toward to end point

(2)Undirected path：arcs in path can either toward or away to end point

(3)cycle：a path begins and ends at same node

— If the network contains at least one undirected path between two nodes , these two nodes are connected

— Connected network：a network that every two nodes are connected

— Shortest path

(1)origin：starting node of path

(2)destination：ending node of path

(3)objective：minimum total distance traveled

B. Solution Method

— Type

(1)one to many：Dijestita method

(2)many to many：Floyd method

— Dijestita method

(1)Definition

— SP（i）=shortest distance from origin point（No） to point I

— PN（i）=proceeding node of I

— LN（i）=label index of I

(2)Procedure

— Step 0：initialization

$SP(i)=\infty$
$PN(i)=0$
$LN(i)=0$

$LN(No)=1$, for origin node

$SP(No)=0$

set IDX=No

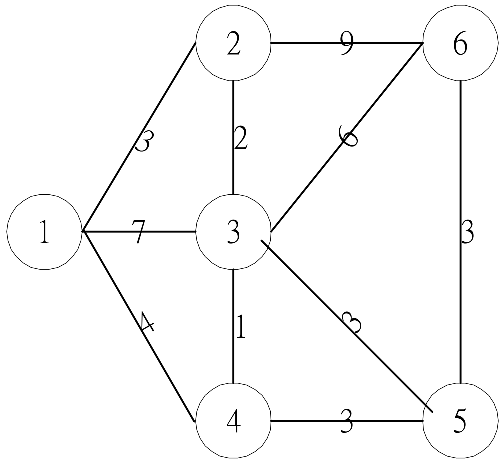
— Step 1：for all nodes with LN=0,compute the distance from associated node i to point IDX as potential $SP(i)=SP(idx)+\alpha(idx,i)$ if $SP'(i)<SP(i)$

then $SP(i) = SP'(i), PN'(i) = IDX$

— Step 2：identify the node i^* with smallest $SP(i^*)$ and $LN(i^*) = 0$,set $LN(i^*) = 1$

— Step 3：if all $LN(i) = 1$, stop. Otherwise set $IDX = i^*$,go to step 1

— Example:



Iteration 0				
	SP	PN	LN	
1	0	0	1	
2	∞	0	0	$SP(1)+(1,2)=3$
3	∞	0	0	$SP(1)+(1,3)=7$
4	∞	0	0	$SP(1)+(1,4)=4$
5	∞	0	0	
6	∞	0	0	

Iteration 1				
	SP	PN	LN	
1	0	0	1	
2	3^*	1	1	$SP(2)+(2,3)=5$
3	7	1	0	$SP(2)+(2,4)=\infty$
4	4	1	0	$SP(2)+(2,5)=\infty$
5	∞	0	0	$SP(2)+(2,6)=12$
6	∞	0	0	

Iteration 2

	SP	PN	LN	
1	0	0	1	
2	3^*	1	1	
3	5	2	0	$SP(4)+(4,3)=5$ $SP(4)+(4,5)=7$ $SP(4)+(4,6)=\infty$
4	4^*	1	1	
5	∞	0	0	
6	12	2	0	

Iteration 3

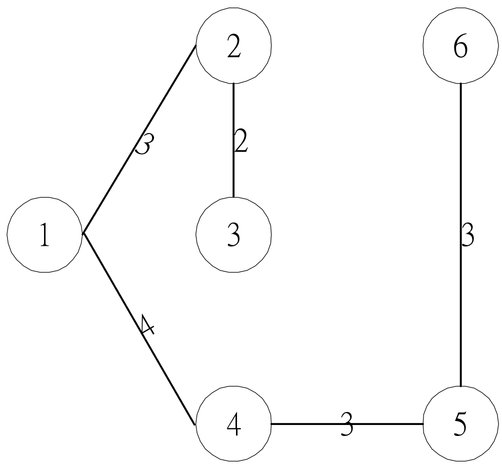
	SP	PN	LN	
1	0	0	1	
2	3^*	1	1	
3	5^*	2	1	$SP(3)+(3,5)=8$ $SP(3)+(3,6)=11$
4	4^*	1	1	
5	7	4	0	
6	12	2	0	

Iteration 4

	SP	PN	LN	
1	0	0	1	
2	3^*	1	1	
3	5^*	2	1	
4	4^*	1	1	$SP(5)+(5,6)=10$
5	7^*	4	1	
6	12	3	0	

Iteration 5				
	SP	PN	LN	
1	0	0	1	
2	3*	1	1	
3	5*	2	1	
4	4*	1	1	
5	7*	4	1	
6	10	5	1	

Shortest –path –tree



— Floyd's Algorithm

— Notation

$D_{n\times n} = \{d(i,j)\}$ ：distance matrix

$P_{n\times n} = \{p(i,j)\}$ ：predecessor matrix

— Algorithm procedure

Step 1：initialization

Set k=1

$$d_0(i,j) = \begin{cases} C(i,j) & \text{if arc } (i,j) \in A \\ 0 & \text{if } i = j \\ \infty & \text{otherwise} \end{cases}$$

$$p_0(i,j) = i \qquad \forall i, \forall j$$

Step 2： update distance matrix

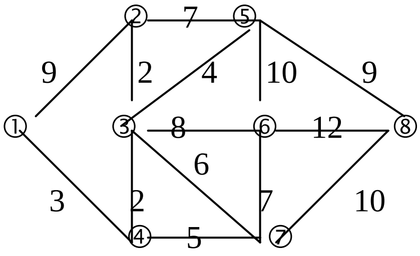
$$d_k(i,j) = \min[d_{k-1}(i,j), d_{k-1}(i,k) + d_{k-1}(k,j)]$$

Step 3：update predecessor matrix

$$p_k(i,j) = \begin{cases} k & \text{if } d_k(i,j) \neq d_{k-1}(i,j) \\ p_{k-1}(i,j) & \text{otherwise} \end{cases}$$

Step 4：if k = n stop, otherwise set k = k+1 go to step 2

example



$k=0$

$D^0(i,j)$

	1	2	3	4	5	6	7	8
1	0	9	∞	3	∞	∞	∞	∞
2	9	0	2	∞	7	∞	∞	∞
3	∞	2	0	2	4	8	6	∞
4	3	∞	2	0	∞	∞	5	∞
5	∞	7	4	∞	0	10	∞	∞
6	∞	∞	8	∞	10	0	7	∞
7	∞	∞	6	5	∞	7	0	∞
8	∞	∞	∞	∞	9	12	10	0

$P^0(i,j)$

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	1	2	3	4	5	6	7	8
3	1	2	3	4	5	6	7	8
4	1	2	3	4	5	6	7	8
5	1	2	3	4	5	6	7	8
6	1	2	3	4	5	6	7	8
7	1	2	3	4	5	6	7	8
8	1	2	3	4	5	6	7	8

$k=1$

$D^1(i,j)$

	1	2	3	4	5	6	7	8
1	0	9	∞	3	∞	∞	∞	∞
2	9	0	2	~~12~~	7	∞	∞	∞
3	∞	2	0	2	4	8	6	∞
4	3	~~12~~	2	0	∞	∞	5	∞
5	∞	7	4	∞	0	10	∞	∞
6	∞	∞	8	∞	10	0	7	∞
7	∞	∞	6	5	∞	7	0	∞
8	∞	∞	∞	∞	9	12	10	0

$P^1(i,j)$

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	1	2	3	~~1~~	5	6	7	8
3	1	2	3	4	5	6	7	8
4	1	~~1~~	3	4	5	6	7	8
5	1	2	3	4	5	6	7	8
6	1	2	3	4	5	6	7	8
7	1	2	3	4	5	6	7	8
8	1	2	3	4	5	6	7	8

$k=2$

$D^2(i,j) =$

	1	2	3	4	5	6	7	8
1	0	9	~~11~~	3	~~16~~	∞	∞	∞
2	9	0	2	12	7	∞	∞	∞
3	~~11~~	2	0	2	4	8	6	∞
4	3	12	2	0	~~19~~	∞	5	∞
5	~~16~~	7	4	~~19~~	0	10	∞	∞
6	∞	∞	8	∞	10	0	7	∞
7	∞	∞	6	5	?	7	0	∞
8	∞	∞	∞	∞	9	12	10	0

$P^2(i,j) =$

	1	2	3	4	5	6	7	8
1	1	2	~~2~~	4	~~2~~	6	7	8
2	1	2	3	1	5	6	7	8
3	~~2~~	2	3	4	5	6	7	8
4	1	1	3	4	~~2~~	6	7	8
5	~~2~~	2	3	~~2~~	5	6	7	8
6	1	2	3	4	5	6	7	8
7	1	2	3	4	5	6	7	8
8	1	2	3	4	5	6	7	8

$k=3$

$D^3(i,j) =$

	1	2	3	4	5	6	7	8
1	0	9	11	3	15	19	17	∞
2	9	0	2	4	6	10	8	∞
3	11	2	0	2	4	8	6	∞
4	3	4	2	0	6	10	5	∞
5	15	6	4	6	0	10	10	∞
6	19	10	8	10	10	0	7	∞
7	17	8	6	5	10	7	0	∞
8	∞	∞	∞	∞	9	12	10	0

$P^3(i,j) =$

	1	2	3	4	5	6	7	8
1	1	2	2	4	3	3	3	8
2	1	2	3	3	3	3	3	8
3	2	2	3	4	5	6	7	8
4	1	3	3	4	3	3	7	8
5	3	3	3	3	5	6	3	8
6	3	3	3	3	5	6	7	8
7	3	3	3	4	3	6	7	8
8	1	2	3	4	5	6	7	8

K =4

$D^4(i,j) =$

0	7	5	3	9	13	8	∞
7	0	2	4	6	10	8	∞
5	2	0	2	4	8	6	∞
3	4	2	0	6	10	5	∞
9	6	4	6	0	10	10	∞
13	10	8	10	10	0	7	∞
8	8	6	5	10	7	0	∞
∞	∞	∞	∞	9	12	10	0

$P^4(i,j) =$

	1	2	3	4	5	6	7	8
1	1	4	4	4	4	4	4	8
2	4	2	3	3	3	3	3	8
3	4	2	3	4	5	6	7	8
4	1	3	3	4	3	3	7	8
5	4	3	3	3	5	6	3	8
6	4	3	3	3	5	6	7	8
7	4	3	3	4	3	6	7	8
8	1	2	3	4	5	6	7	8

$k=5$

$D^5(i,j) =$

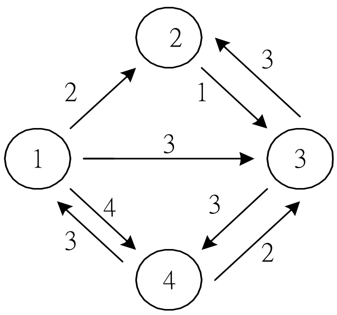
	1	2	3	4	5	6	7	8
1	0	7	5	3	9	13	8	18
2	7	0	2	4	6	10	8	15
3	5	2	0	2	4	8	6	13
4	3	4	2	0	6	10	5	15
5	9	6	4	6	0	10	10	9
6	13	10	8	10	10	0	7	12
7	8	8	6	5	10	7	0	10
8	18	15	13	15	9	12	10	0

$P^5(i,j) =$

	1	2	3	4	5	6	7	8
1	1	4	4	4	4	4	4	5
2	4	2	3	3	3	3	3	5
3	4	3	3	4	3	6	3	5
4	1	3	3	4	3	3	4	5
5	4	3	3	3	5	6	3	5
6	4	3	3	3	5	6	6	6
7	4	3	7	7	3	6	7	7
8	5	5	5	5	5	6	7	8

k = 6, k = 7, k = 8 unchanged

— example：



$$D^0 = \begin{bmatrix} 0 & 2 & 3 & 4 \\ \infty & 0 & 1 & \infty \\ \infty & 3 & 0 & 2 \\ 3 & \infty & 3 & 0 \end{bmatrix} \qquad P^0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

k=1

$$D^1 = \begin{bmatrix} 0 & 2 & 3 & 4 \\ \infty & 0 & 1 & \infty \\ \infty & 3 & 0 & 2 \\ 3 & \underline{5} & 3 & 0 \end{bmatrix} \qquad P^1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & \underline{1} & 4 & 4 \end{bmatrix}$$

k=2

$$D^2 = \begin{bmatrix} 0 & 2 & 3 & 4 \\ \infty & 0 & 1 & \infty \\ \infty & 3 & 0 & 2 \\ 3 & 5 & 3 & 0 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 1 & 4 & 4 \end{bmatrix}$$

k=3

$$D^3 = \begin{bmatrix} 0 & 2 & 3 & 4 \\ \infty & 0 & 1 & \underline{3} \\ \infty & 3 & 0 & 2 \\ 3 & 5 & 3 & 0 \end{bmatrix} \qquad P^3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & \underline{3} \\ 3 & 3 & 3 & 3 \\ 4 & 1 & 4 & 4 \end{bmatrix}$$

k=4

$$D^4 = \begin{bmatrix} 0 & 2 & 3 & 4 \\ \underline{6} & 0 & 1 & 3 \\ \underline{5} & 3 & 0 & 2 \\ 3 & 5 & 3 & 0 \end{bmatrix} \qquad P^4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \underline{4} & 2 & 2 & 3 \\ \underline{4} & 3 & 3 & 3 \\ 4 & \underline{1} & 4 & 4 \end{bmatrix}$$

8.3 Minimum Spanning Tree

A. A spanning tree

— a tree that contains every node in the networ

B. A minimum spanning tree

a spanning tree with minimum total length of arcs

C.The "greedy" algorithm

— step 1 : define two sets of nodes

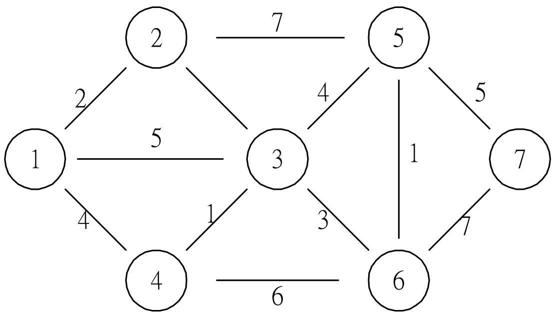
S : a set of connected nodes

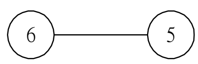
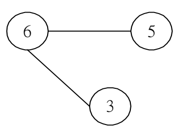
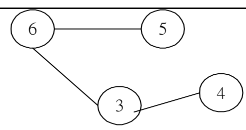
$\bar{S}$: a set of unconnected nodes

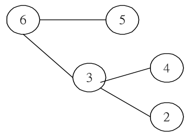
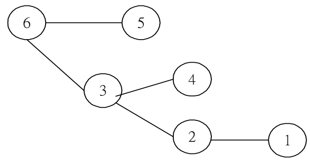
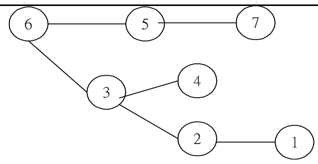
— step 2 : identify the node i^* in $\bar{S}$, closest to a node in Ss , remove i^* from $\bar{S}$ to S

—step 3 : repeat step 2 until $\bar{S}$ is empty

— example：



step 0	$\bar{s} = \Phi$ $\bar{s} = \{1,2,3,4,5,6,7\}$	
step 1	$s = \{6,5\}$ $\bar{s} = \{1,2,3,4,7\}$ T.C.=1	
step 2	$s = \{6,5,3\}$ $\bar{s} = \{1,2,4,7\}$ T.C.=1+3=4	
step 3	$s = \{6,5,3,4\}$ $\bar{s} = \{1,2,7\}$	

	T.C.=4+1=5	
step 4	$s=\{6,5,3,4,2\}$ $\bar{s}=\{1,7\}$ T.C.=5+2=7	
step 5	$s=\{6,5,3,4,2,1\}$ $\bar{s}=\{7\}$ T.C.=7+2=9	
step 6	$s=\{6,5,3,4,2,1,7\}$ $\bar{s}=\{\Phi\}$ T.C.=9+5=14	
step 7	$\bar{s}=\{\Phi\}$ STOP	

8.4 The Maximum Flow Problem

A. Definition

— under a directed and connected network, the objective is to find the feasible pattern of flow that maximize the total flow from source (S) to terminal (T) node.

B. Assumptions

— flow conservation conditions hold for the intermediate nodes of the network.

— Initinite supply in source node

C. Model

$$\text{Max} \quad \sum_{i \neq t} f_{it}$$

$$\text{St} \quad \sum_{i \neq k} f_{ik} - \sum_{j \neq k} f_{kj} = 0 \qquad \forall k \neq s,\ \forall k \neq t$$

$$0 \leq f_{ij} \leq u_{ij} \qquad (i, j) \in A$$

— Basic concept

— Flow augumenting process

— Residual capacity

$$R_{ij} = u_{ij} - f_{ij}$$
$$R_{ij} = f_{ij}$$

D. Residual network

— For all (i,j) in original network without a directed arc the opposite direction (i,j),add this arc with arc capacity of zero.

— Residual capacity represents the unused arc capacity in the original network or the amount of flow in the opposite direction is this network that can be cancelled.

E. Solution algorithm

— Step1：find any directed path from source to terminal nodes that has positive flow capacity,if it exits, go to step 2.

— Step2：identify the residual capacity c^* of this path by find the minimum residual capacity of the arc in this path increase the flow in this path by c^*.

— Decrease by c^* residual capacity of each arc in the path, increase by c^* residual capacity of each arc in the opposite direction of this path ,return to step 1.

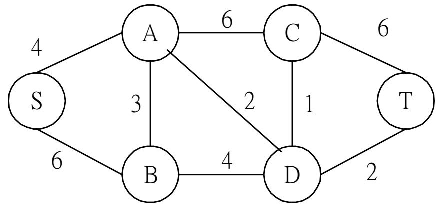
F. Procedure for finding path with positive flow capacity

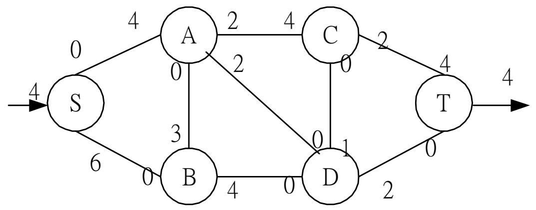
— Identify the nodes that can be reached from source node along a arc with strictly positive residual capacity.

— For each node reached identify the new nodes that can be residual capacity ≥ 0

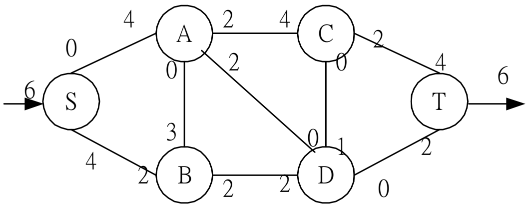
— Repeat this search until terminal is reached

example：

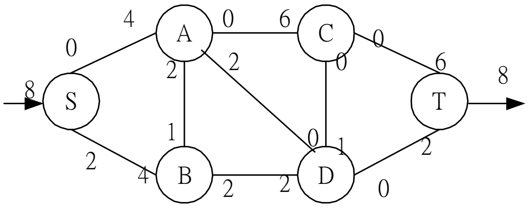




PATH S－A－C－T

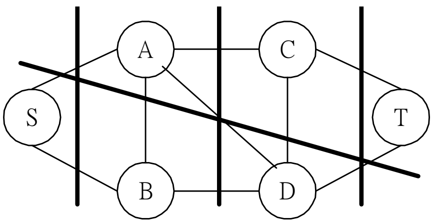


PATH S－B－D－T



PATH S－B－A－C－T

— Max-flow min-cut theorem



切面法

1. a cut is a set of directed arcs containing at least one arc from every directed path from source node to terminal node.
2. the maximum flow equals the minimum cut value for all the cut of network.
3. the optimal is reached,whenever there exist a cut in the residual network whose value is zero.

example：

an airlinecompany with six scheduled flights depart every two hours at capacities of 100,100,100,150,150,150.

The booked customers of these flights are 106,140,130,170,220,180.

If custormer is delay by 2 hours,it will be compensated with ＄200,and＄20 per additional hour delay.

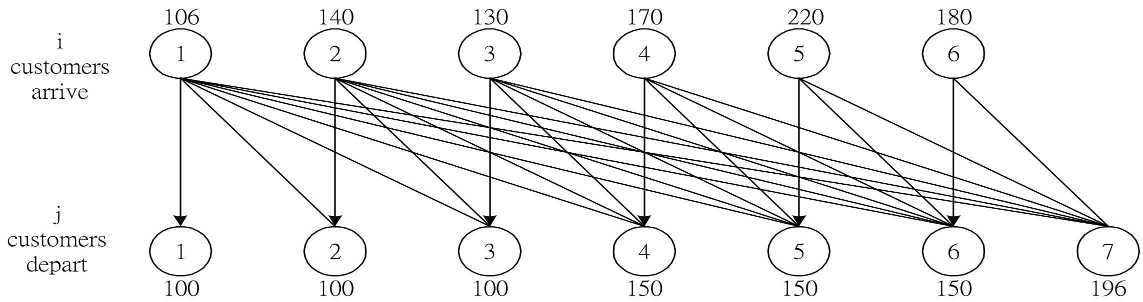
1. formulate it as network problem
2. what type of problem it is ？

solution：

node：customers arrives or departs

link：customer abord the airplane

link cost：the compensate money the airline paid



$$\sum S_i = 106 + 140 + 130 + 170 + 220 + 180 = 946$$

$$\sum D_j = 100 * 3 + 150 * 3 = 750$$

$$C_{ij} = 200 + \sum_{j=i+2}^{7} (i - j - 1) \times 20 \times 2$$

model：

$$\min \quad \sum_{i=1}^{6} \sum_{j=1}^{7} C_{ij} X_{ij}$$

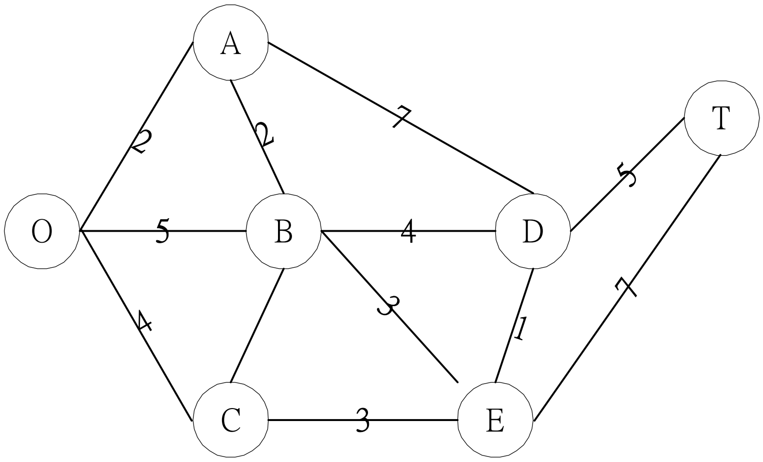
$$\text{st} \quad \sum_j X_{ij} \le S_i \qquad \forall i$$

$$\sum_i X_{ij} \le D_j \qquad \forall j$$

$$X_{ij} \ge 0 \qquad \forall i \forall j$$

Example:

西瓦達公園最近開放作有限度的觀光及露營。車輛不准進入公園。但有一些窄而彎曲的小路系統，以備管理員的吉普車通過。道路系統圖如圖所示：



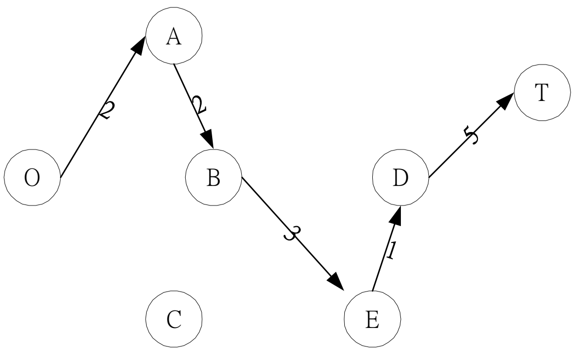
O 是公園入口；T 設有觀光景點，有少數的交通車將觀光客從入口 O 送到 T 站然後回頭；其他字母是設施或管理站位址；數字是小路的長度。

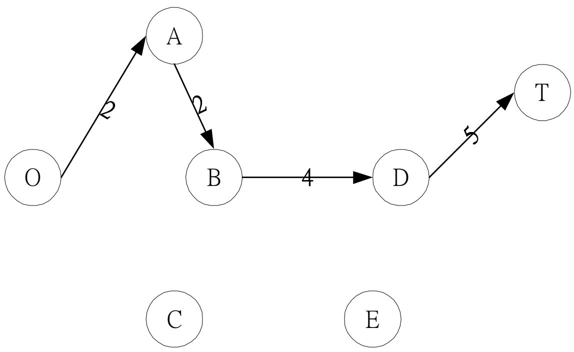
公園管理局面臨三個問題：1.決定 O 到 T 的運送交通車之最短路徑（Shortest path problem）；2.沿小路埋電話線，總電話線最小以避免破壞自然環境（Minimum Spanning Tree）；3.觀光客眾多在尖峰期無法負荷，因此每路徑通過車次各有限制以避免干擾生態，而要繞境走使每天班次運送次數最多（The Maximum Flow Problem）。

Sol:

1. 有兩組解

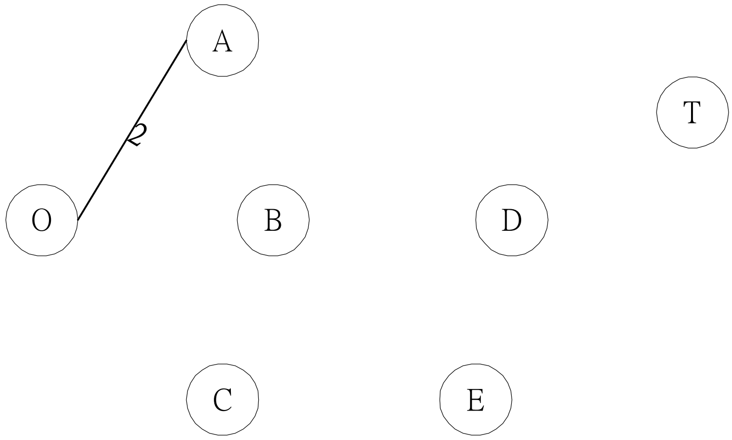
O-A-B-E-D-T　　總距離 13　　O-A-B-D-T　　總距離 13



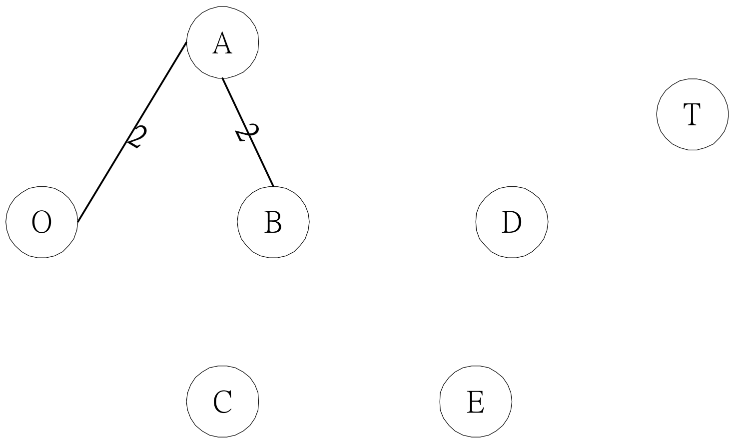


2.

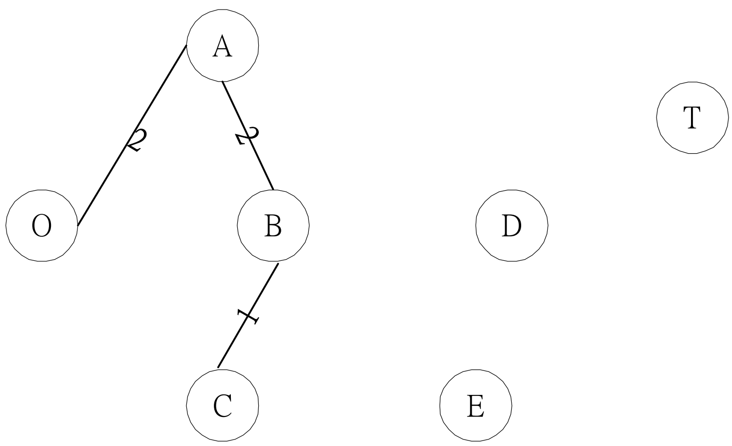
隨意選擇節點 O 為開始，將 A 連至 O



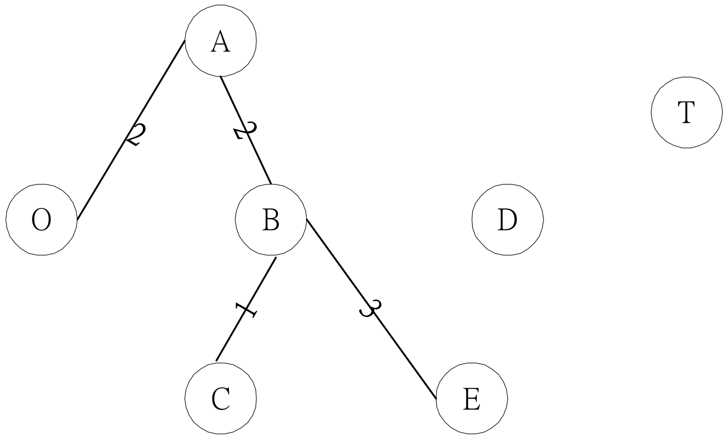
將 B 連至 A



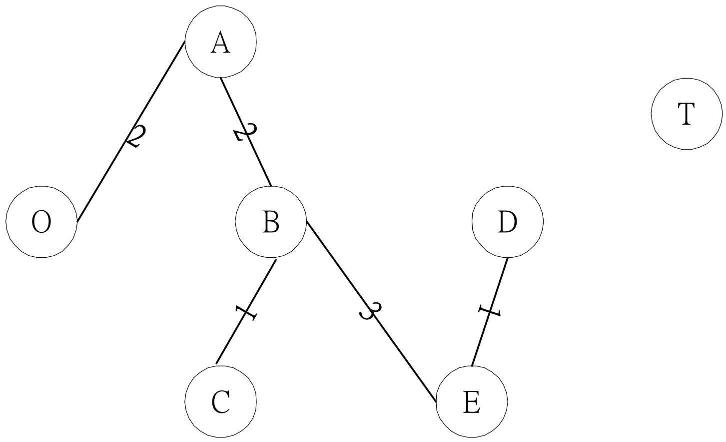
將 C 連至 B



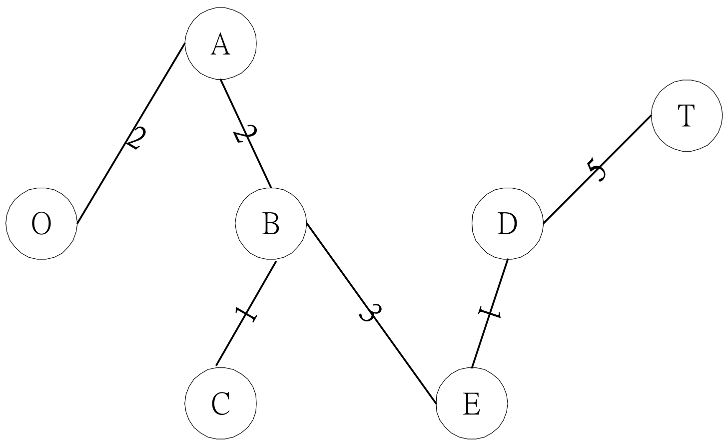
將 E 連至 B



將 D 連至 E



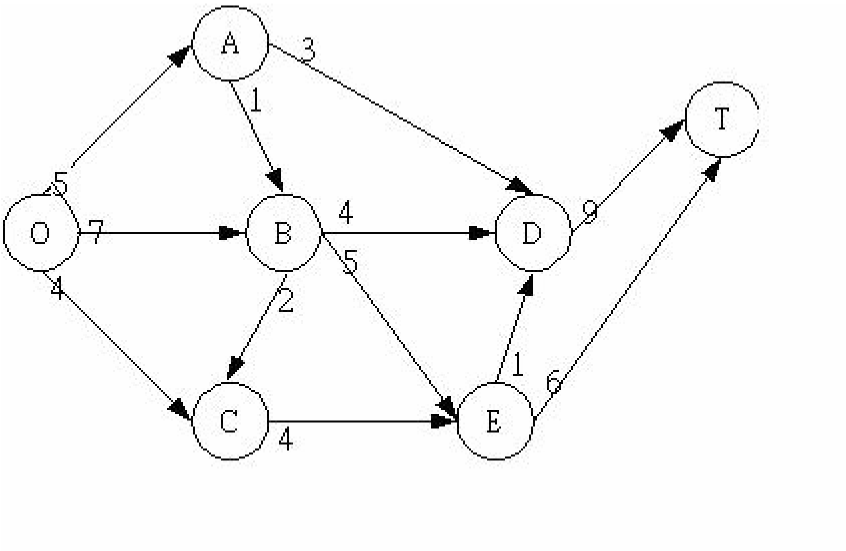
將 T 連至 D



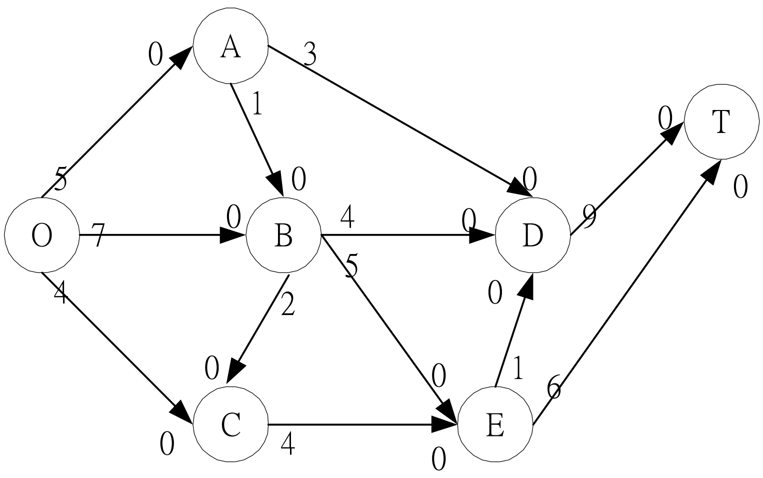
此為本題之解。總長度 14

3.

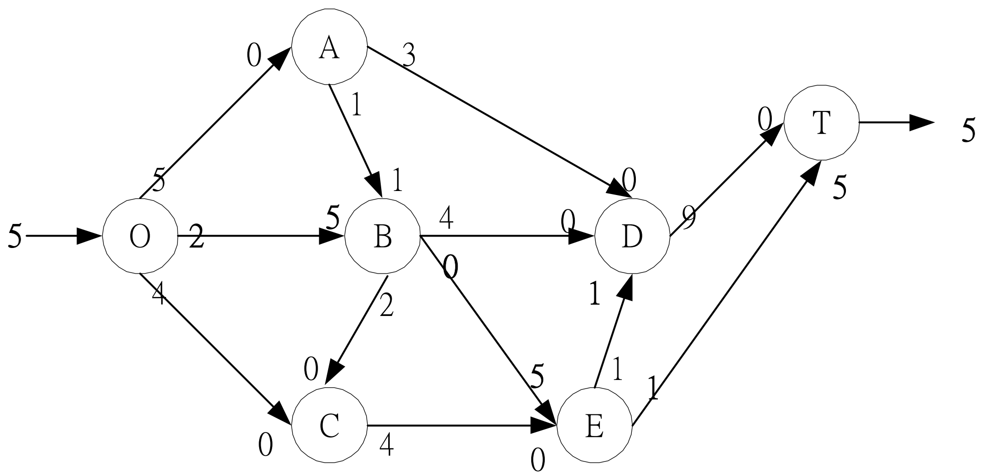
西瓦達公園最大流量問題



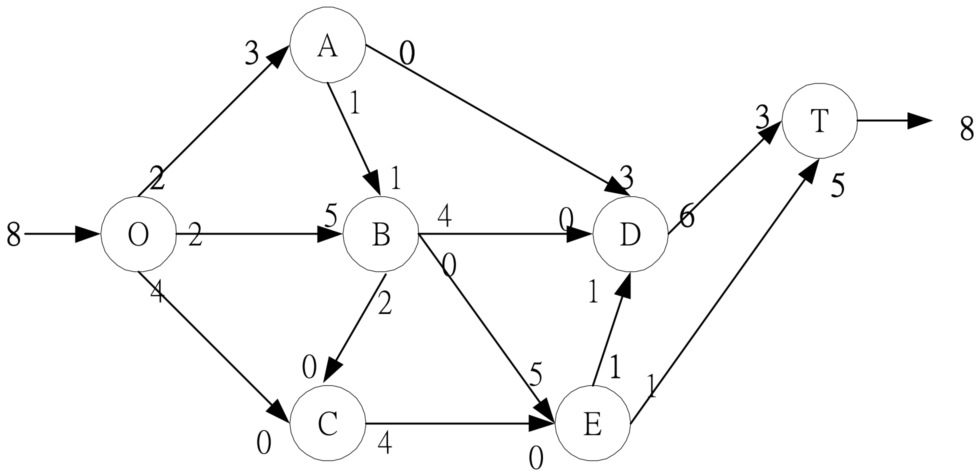
起始殘餘網路



（1）O-B-E-T　將流量 5 指派到此路徑



（2）O-A-D-T　指派量 3



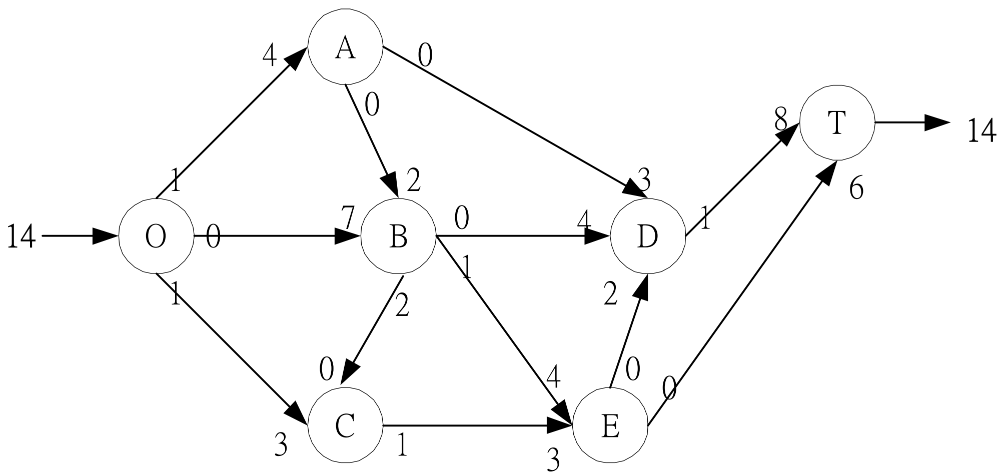
（3）O-A-B-D-T　指派量 1

（4）O-B-D-T　指派量 2

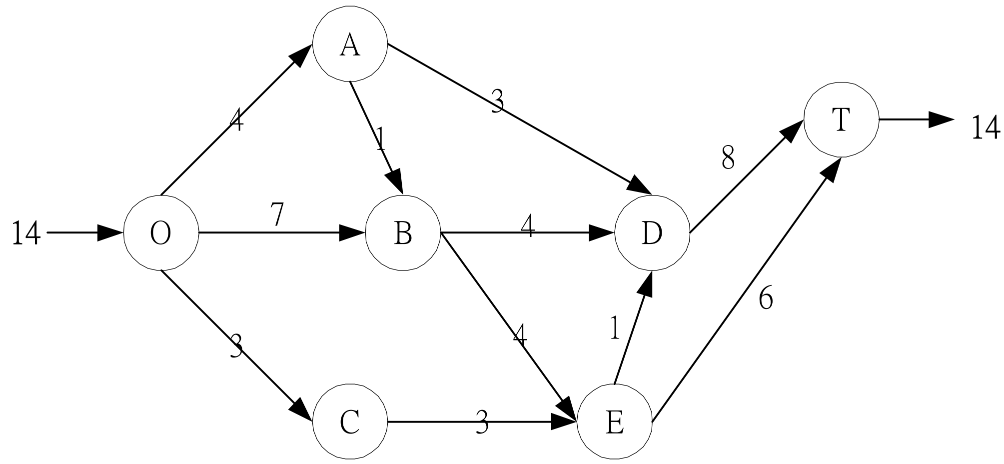
（5）O-C-E-D-T　指派量 1

（6）O-C-E-T　指派量 1

（7）O-C-E- B-D-T　指派量 1



最優解



8-24 某公園管理局所面臨的問題爲：在遠足旺季，由公園入口（圖 1 中的O站）至越野遠足起點（T站）各路線輕便車通行次數各應若干，以使每日通行次數最多。各路線各方向載人車通行次數，定有嚴格的上限。這些限制示於圖 1 中，圖中各站邊數字表示各該路線由該站出發之行車次數最高限制。求出每日通行次數最多之路線。

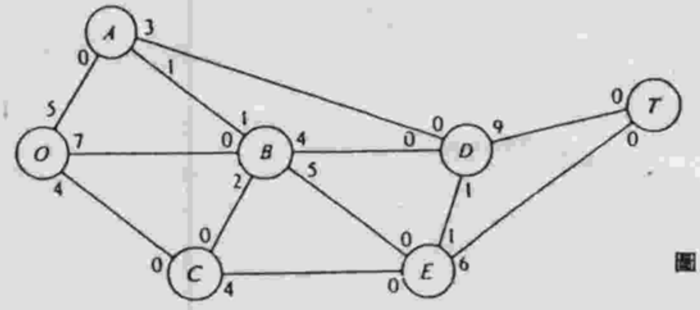
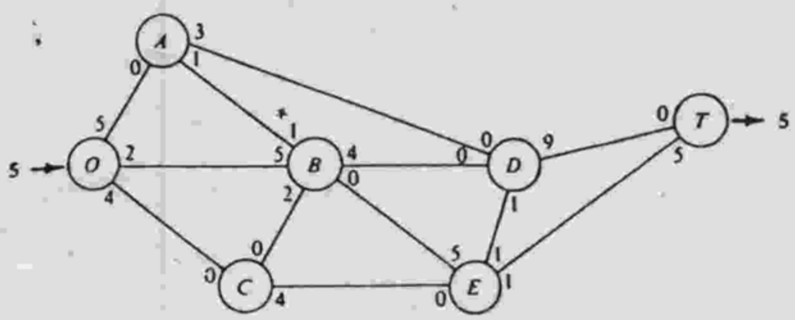


圖 1

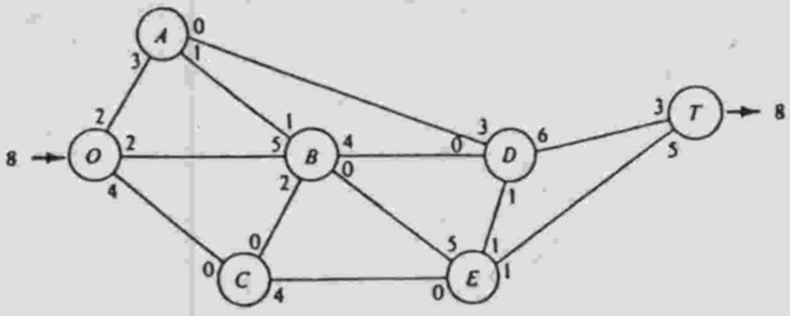
解：反覆步驟 1：

分配流量 5 予 O → B → E → T，可得下列網路



反覆步驟 2：

分配流量 3 予 O → A → D → T，可得下列網路

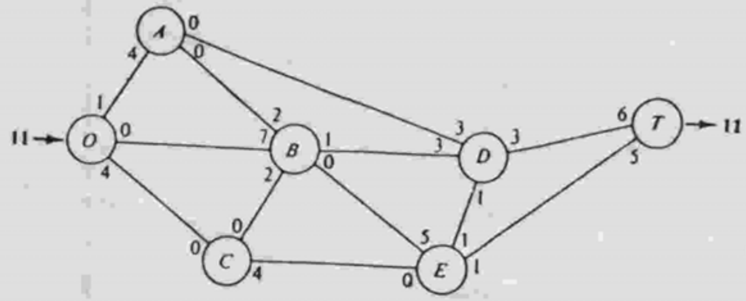


反覆步驟 3：

分配流量 1 予 O → A → B → D → T。

反覆步驟 4 :

分配流量 2 予O → B → D → T ，可得下列網路

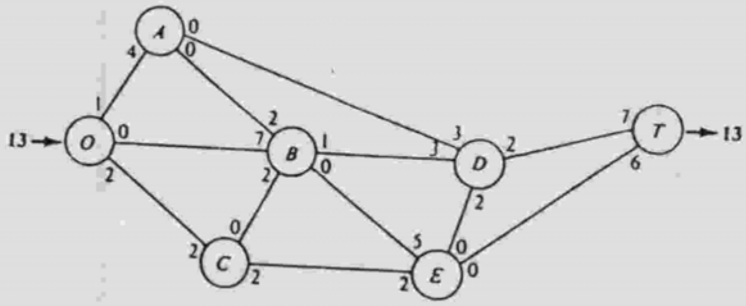


反覆步驟 5 :

分配流量 1 予O → C → E → D → T 。

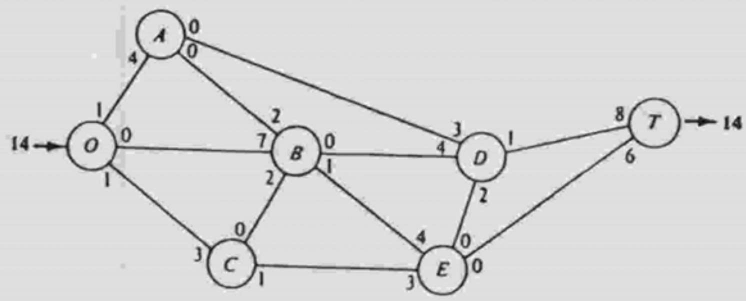
反覆步驟 6 :

分配流量 1 予O → C → E → T ，可得下列網路

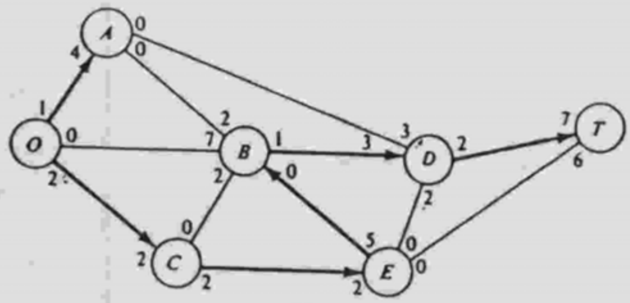


反覆步驟 7 :

分配流量 1 予O → C → E → B → D → T ，可得下列網路



至此已無具有嚴格正流通量的路線。現時的流通量型態爲最優的。



最優的路線

某公司得悉一家競爭公司正計劃生產一種有極好銷路的新產品。該公司也正在研製類似的產品，而且研究工作接近完成。要突擊趕製出這種產品，還有下表所示的四個階段的工作要做。爲了加快進度，公司撥出一千萬元資金供在每個階段上採用“優先”或“緊急”的措施，它們所需的時間（月）和費用（百萬元）如下：

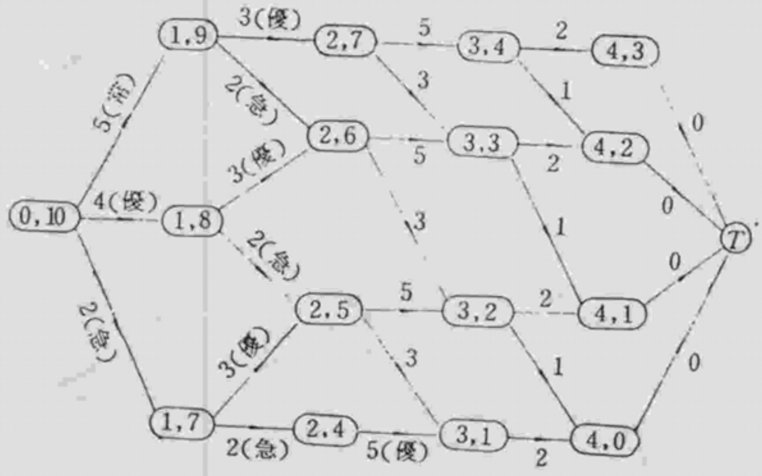
措施 \ 階段	剩餘研究		試製		工藝設計		生產與調撥	
	時間	費用	時間	費用	時間	費用	時間	費用
正常	5	1						
優先	4	2	3	2	5	3	2	1
緊急	2	3	2	3	3	4	1	2

問題是要在資金條件的限制下，每一階段應採取什麼措施才能使產品上市前的總時間最短？

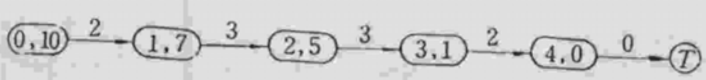
(1)把此問題化成最短路線問題；

(2)求此問題的最佳方案。

解：(1)爲了能清楚地表示工作階段和應採取的措施，網絡圖的節點編號用一對數表示，設節點 (i, j) 代表進行第 i 階段工作，有 j 百萬元剩餘資金給以後階段，節點間“距離” $d(i, j), (i+1, k)$ 爲節點 (i, j) 與節點 $(i+1, k)$ 之間由於採取花費 $(j-k)$ 百萬元資金的措施所需的時間，此外注意到化成一個“發點”和一個“收點”的網絡，我們可把這個問題表示成如下有向網絡圖：



(1)



(2)

10-3 某工廠使用一種設備，設備的購置費K_i和使用時的維修與運行費c_i如下表：

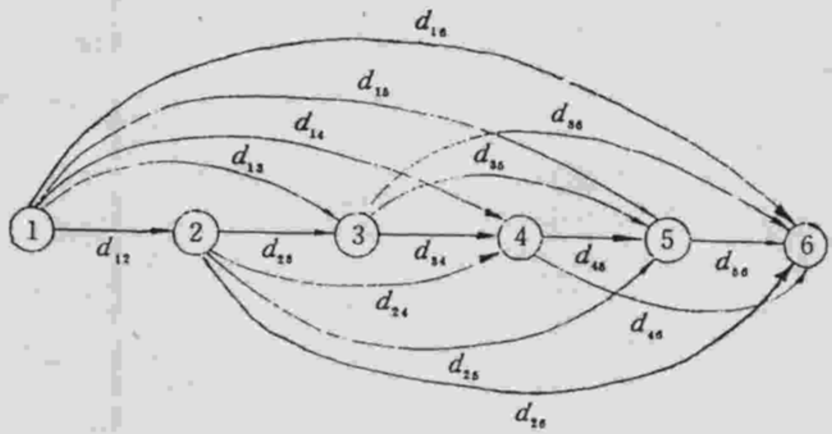
年　份 i	1	2	3	4	5
購置費 K_i	11	11	2	12	13
維修與運行費 c_i	5	6	8	11	18

問題是要決定今後五年的設備更新計劃，使包括購置費和維修與運行費在內的總費用最小。

(1)把此問題表示成最短路線問題（畫出網絡圖）

(2)求最短路線（最佳更新方案）。

解：(1)設節點 1 和 6 代表計劃期的始點和終點，每個中間節點 j（$j=2,3,4,5$）代表第 j 年開始。當 $j>i$ 時，每一節點 i 有一條有向弧與節點 j 相連。這樣可得有向網絡圖。



其中

$$d_{ij}=\begin{cases} K_i+\sum_{t=1}^{j-1} c_t \quad , & \text{當 } j>i \\ \infty & \text{當 } j\le i \end{cases}$$

把K_i和 c_i 代入上式，可得如下數據：

	1	2	3	4	5	6
1	∞	16	22	30	41	59
2	∞	∞	16	22	30	41
3	∞	∞	∞	17	23	31
4	∞	∞	∞	∞	17	23
5	∞	∞	∞	∞	∞	18
6	∞	∞	∞	∞	∞	∞

(2)用狄克斯屈拉標號算法，有如下標號結果：

$$L(0)=[\underset{*}{0},\infty,\infty,\infty,\infty,\infty]$$

$$L(1)=[\underset{*}{0},\underset{*}{16},22,30,41,57]$$

$$L(2)=[\underset{*}{0},\underset{*}{16},\underset{*}{22},30,41,53]$$

$$L(3)=[\underset{*}{0},\underset{*}{16},\underset{*}{22},\underset{*}{30},41,53]$$

$$L(4)=[\underset{*}{0},\underset{*}{16},\underset{*}{22},\underset{*}{30},\underset{*}{41},53]$$

$$L(5)=[\underset{*}{0},\underset{*}{16},\underset{*}{22},\underset{*}{30},\underset{*}{41},\underset{*}{53}]$$

最短路線爲 1 → 3 → 6 ，即在五年中第 1 年初和第 3 年初購置新設備爲最佳方案，其費用爲 53 單位 。