

Chapter 7 Integer Programming

7.1 Introduction

A. Definition

- A L.P with some of variables must be integer.

B. Characteristics

- Cost indivisibility
- Complex solution method
- Finite number of solutions
- Optimal solution are limited to one ,several or none
- LP relaxation

C. Type of L.P

- All integer problem

$$\begin{aligned} \text{Min} \quad & 6X_1 + 4X_2 + 3X_3 \\ \text{St} \quad & 2X_1 + 3X_2 + 4X_3 \geq 600 \\ & 5X_1 + 7X_2 - X_3 \geq 70 \\ & X_1 \quad X_2 \quad X_3 \geq 0 \quad \text{and integer} \end{aligned}$$

- Mixed integer program (MIP) same as above except

$$X_1 \quad X_2 \quad X_3 \geq 0 \quad \text{and } X_2 \text{ is integer}$$

- Binary integer problems (BIP)

$$\begin{aligned} \text{Min} \quad & \sum_i \sum_j X_{ij} \\ \text{St} \quad & \sum_j X_{ij} = 1, \quad \forall i \\ & \sum_i X_{ij} = 1, \quad \forall j \\ & X_{ij} = \begin{cases} 0 \\ 1 \end{cases} \end{aligned}$$

D. Example of integer programming

- capital budget

$$X_{ij} = \begin{cases} 0, & \text{if plant } i \text{ is built} \\ 1, & \text{otherwise} \end{cases}$$

$$\text{Max } 90X_1 + 40X_2 + 10X_3 + 37X_4$$

$$\begin{aligned} \text{St } 15X_1 + 10X_2 + \quad & 15X_4 \leq 40 \\ & 20X_1 + 15X_2 + \quad & 10X_4 \leq 50 \\ & 20X_1 + 20X_2 + \quad & 10X_4 \leq 40 \\ & 15X_1 + 5X_2 + 4X_3 + 10X_4 \leq 35 \end{aligned}$$

$$X_1, X_2, X_3, X_4 = \begin{cases} 1 \\ 0 \end{cases}$$

(1) multiple-choice constraint

$$X_1 + X_2 + X_3 = 1$$

(2) mutually-exclusive constraint

$$X_1 + X_2 + X_3 \leq 1$$

(3) k out of n constraint

$$X_1 + X_2 + X_3 = 2$$

(4) contingent constraint

$$X_1 \leq X_2$$

(5) corequisite constraint

$$X_1 = X_2$$

(6) or constraint

$$3X_1 + 2X_2 \leq 18$$

$$\text{or } X_1 + 4X_2 \leq 4$$

— Distribution system design

$$Y_i = \begin{cases} 1, \text{ plant } i \text{ is built} \\ 0, \text{ o, w} \end{cases}$$

$$\text{Min } \sum_i^m \sum_j^n C_{ij} X_{ij} + \sum_{i=1}^m f_{ij} y_{ij}$$

$$\text{St } \sum_j^n X_{ij} \leq s_{ij} y_{ij}, \forall i$$

$$\sum_j^m X_{ij} \leq D_j, \forall j$$

$$3X_1 + 2X_2 \leq 18 + My$$

$$X_1 + 4X_2 \leq 4 + M(1 - y)$$

$$X_1, X_2 \geq 0 \quad Y = \{0,1\}$$

例題

某一鐵路管理局提供下列等級的車輛列車：

- (a)
- 4高級與4普通級車廂
 - 6高級與3普通級車廂
 - 2高級與5普通級車廂

而各類型列車的收收入比例為5：6：4。該管理當局共有22節高級車廂與25節普通級車廂可供其調配。試問各類型的列車應如何分派？

(b)改變為下列型態的列車，求解問題

- 6高級與4普通級車廂
- 6高級與6普通級車廂
- 5高級與7普通級車廂

且各類型列車的收收入比例為26：30：19，而管理當局共有33節高級車廂與37節普通級車廂可供調配。試問各類型的列車應如何分派？

$$\begin{aligned} & \text{Max} \quad 5X_1 + 6X_2 + 4X_3 \\ \text{(a)} \quad & 4X_1 + 6X_2 + 2X_3 \leq 22 \\ & 4X_1 + 3X_2 + 5X_3 \leq 25 \\ & X_j \geq 0, \text{整數} \end{aligned}$$

Ans: $X_1 = 1, X_2 = 2, X_3 = 3$ 或

$$X_1 = 5, X_2 = 0, X_3 = 1$$

目標函數：29

其中，並未所有車廂皆用盡。

$$\begin{aligned} & \text{Max} \quad 26X_1 + 30X_2 + 19X_3 \\ \text{(b)} \quad & 6X_1 + 6X_2 + 5X_3 \leq 33 \\ & 4X_1 + 6X_2 + 7X_3 \leq 37 \\ & X_j \geq 0, \text{整數} \end{aligned}$$

Ans: $X_1 = 0, X_2 = 5, X_3 = 0$

目標函數：150

其中，並未所有車廂皆用盡。

管理當局可能開出第一種類型的列車1班次、第二種類型的列車2班次、第三種類型的列車3班次，如此方能用盡所有車廂，但此時的收入為143，而非最佳解150。

7.2 Characteristic in I.P

A. The value of O.V of the associated L.P relaxation is an upper bound on the O.V of I.P.

B. Factors associated with computational difficulty in I.P.

- The number of integer variables.
- The structure of problem.

C. Solution methods

- Enumeration
- Rounding non-integer solution
- Branch and bound algorithm
- Zero-one Algorithm
- Cutting plane algorithm
- Heuristic methods

D. Problem in solving I.P by L.P then rounded the solution.

- Not necessarily feasible after it is round
- No guarantee it is optimal

E. Enumeration

- Exponential growth of problem
- May remove corner-point

F. Heuristic method

- Efficient
- No guarantee optimal

7.3 Branch and Bound Method

A. Concept

- Divide and conque
- Branch to divide the set into two subsets
- Bounding is to determine how good , the subset can be
- Discard it when it can't contain optimal solution

B. Procedure

- Set $z^* = -\infty$ (Max) or ∞ (min)
- Solving the LP-relaxation
- If all variables satisfy constraint stop
- Branching
 - Among the remain sub-problems select the one that recent create , branching from this node into two new sub-problem.
 - (1) Adding constraints $X_j \leq [X^*]$ and $X_j \geq [X^*]+1$
 - (2) It is binary integer problem (BIP) , $X_j = 0$ and $X_j = 1$
- Bounding
 - Solve the LP-relaxation , and use its O.V as upper bounding (UB)
- Fathoming
 - (1) Its bound $\leq Z^*$
 - (2) No feasible solution
 - (3) LP-relaxation is integer and its $o.v \geq Z^*$, set $Z^* = o.v$

Example

$$\text{Max } 2X_1 + 3X_2$$

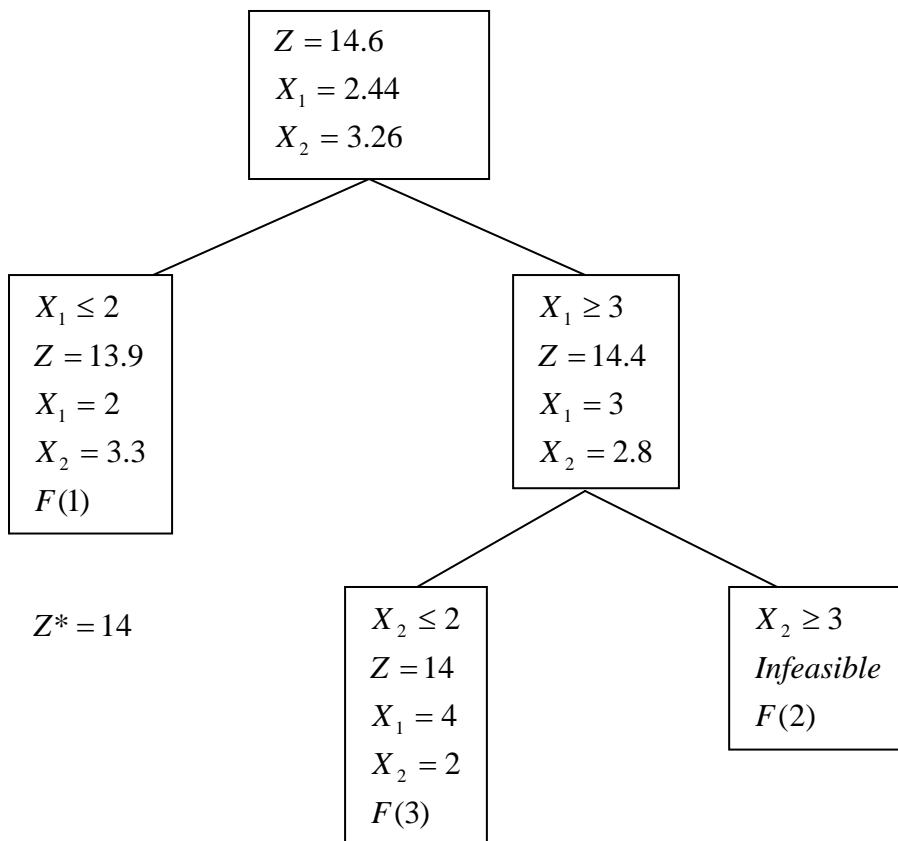
$$\text{St } 195X_1 + 273X_2 \leq 1365$$

$$4X_1 + 40X_2 \leq 140$$

$$X_1 \leq 4$$

$$X_1, X_2 \geq 0, \text{ and integer}$$

$$z^* = -\infty$$



- Max $4X_1 - 2X_2 + 7X_3 - X_4$

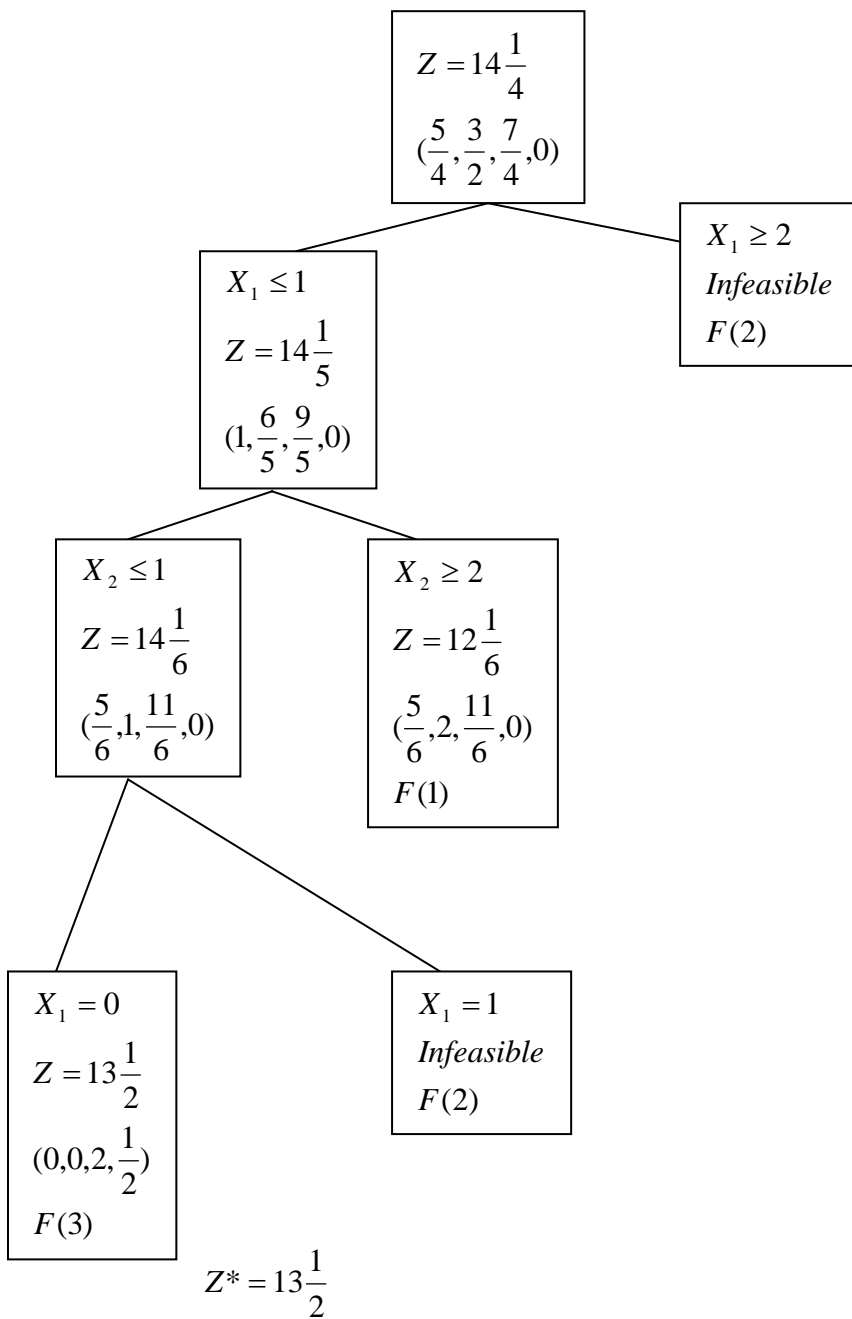
St $X_1 + 5X_3 \leq 10$

$X_1 + X_2 - X_3 \leq 1$

$-X_1 + 2X_2 - 2X_4 \leq 3$

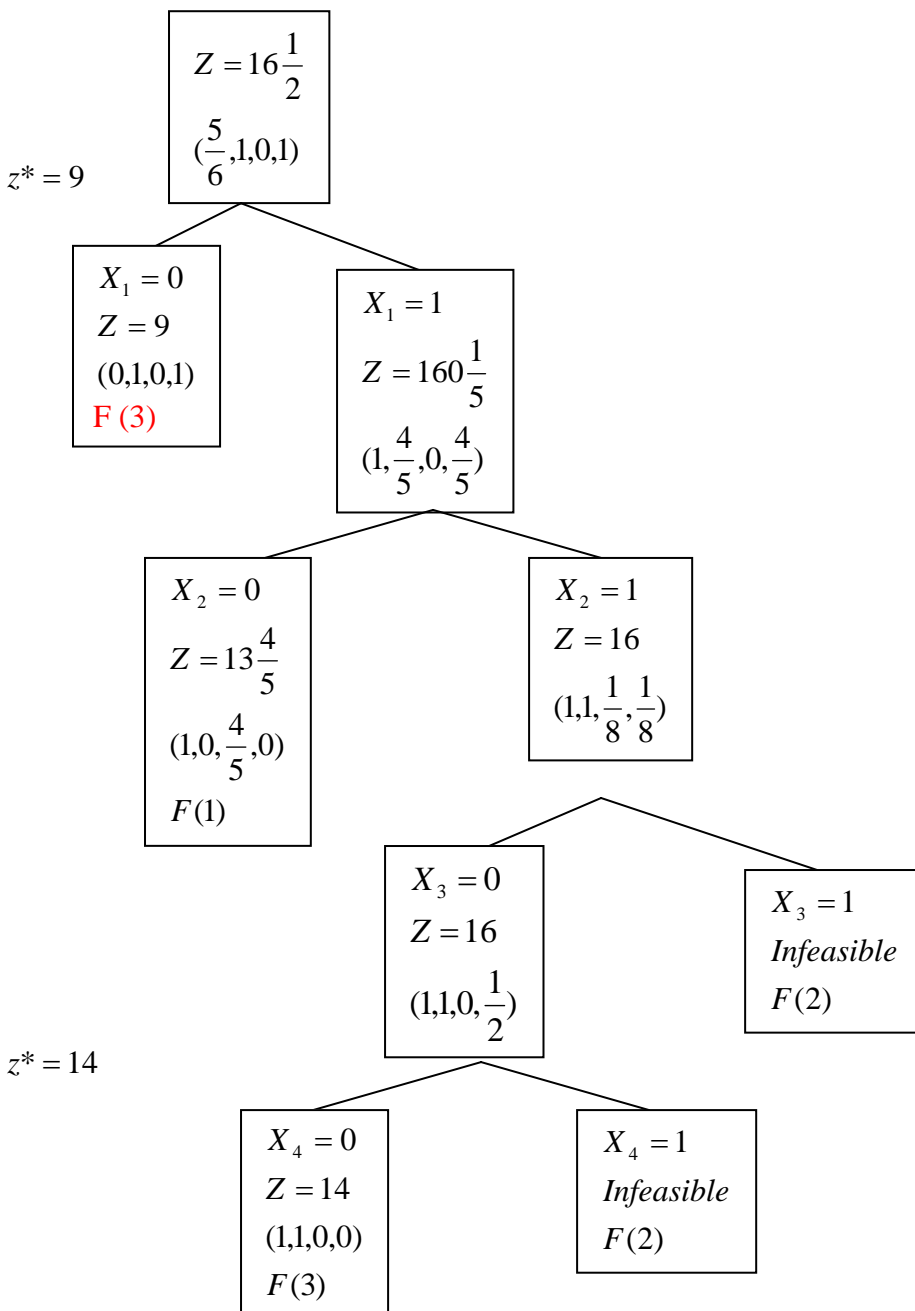
$X_1, X_2, X_3, X_4 \geq 0$ and X_1, X_2, X_3 are Integers

$z^* = -\infty$



$$\begin{aligned}
 & - \text{Max } 9X_1 + 5X_2 + 6X_3 + 4X_4 \\
 & \text{St } 6X_1 + 3X_2 + 5X_3 + 2X_4 \leq 10 \\
 & \quad X_3 + X_4 \leq 1 \\
 & \quad -X_1 + X_3 \leq 0 \\
 & \quad -X_2 + X_4 \leq 0 \\
 & \quad X_i = \begin{cases} 0 \\ 1 \end{cases}, \forall_i
 \end{aligned}$$

$z^* = -\infty$



7.4 Converting the MIP into BIP

Require integer variable has upper $X_k \leq U$

Bound let

$$X_k = \sum_{i=1}^p 2^{i-1} y_i$$

p : smallest integer such that $2^p - 1 \geq u$

Ex. $X_i \leq 7$

$$X_i = \sum_{i=1}^{p=3} 2^{i-1} y_i$$

$$\longrightarrow X_i = y_1 + 2y_2 + 4y_3$$

X_1	y_1	y_2	y_3
0	0	0	0
1	1	0	0
2	0	1	0
3	1	1	0
4	0	0	1
5	1	0	1
6	0	1	1
7	1	1	1

7.5 Additive Algorithm (pure BIP)

A. Introduction

- Primal approach: maintain feasibility but improve optimality
- Dual approach: maintain optimality but improve feasibility
- Star with dual-feasible: optimal but infeasible additive algorithm
- 0-1 algorithm
- Implicit Enumeration
- Ballas Algorithm

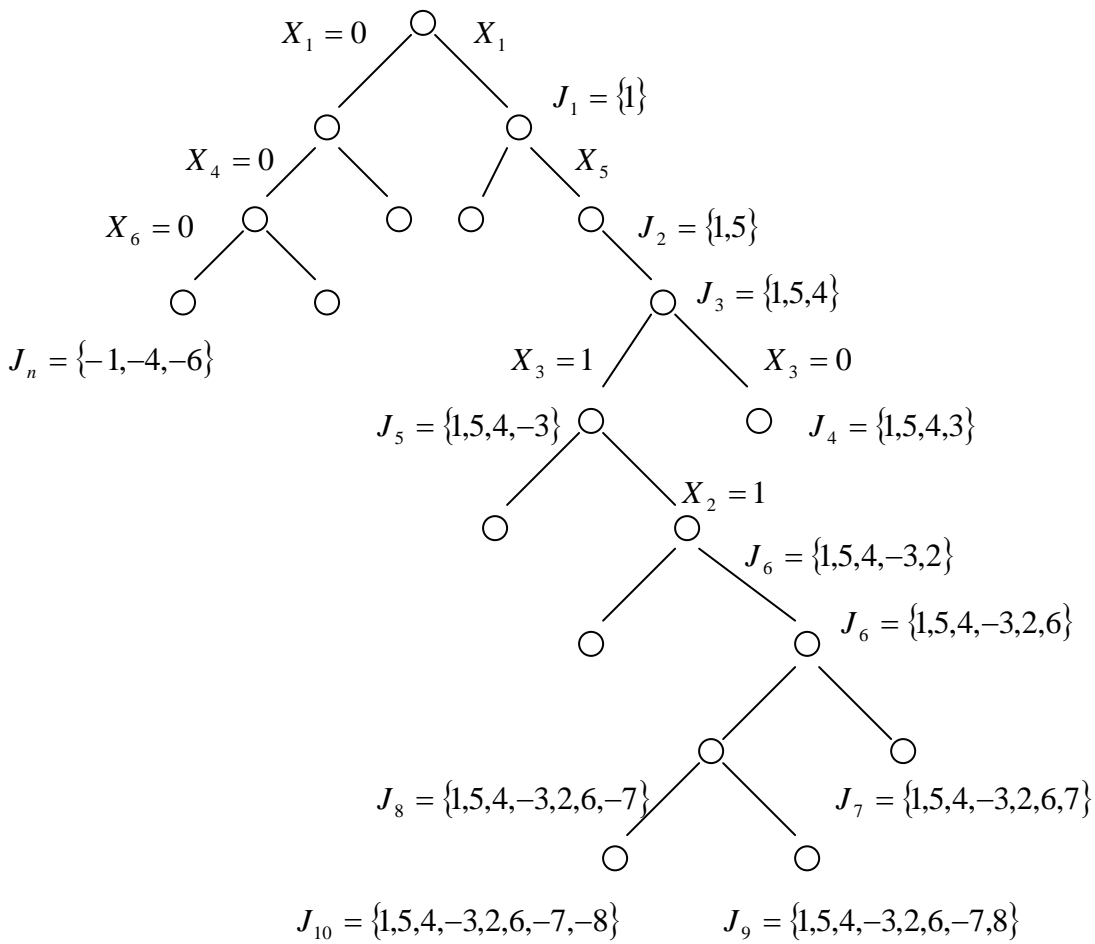
$$\text{Min } \sum_{j=1} C_j X_j$$

- 1. set all constraint be the type of (\leq)
- 2. get all $C_{ij} \geq 0$ if $C_{ij} \leq 0 \Rightarrow X_j' = 1 - X_j, C_j \geq 0$
- Start with setting all $X_j = 0$
- Some $S_i \leq 0 \Rightarrow$ infeasible
- Evaluate one X_j at a time toward feasibility

B. Special terms

- Free variables: binary variable not fixed to any branch leading to this node
- Partial solution: a specific binary assignment for some variables
- 1. J_t : the partial solution of the nodes
 - +J $\rightarrow X_j = 1$
 - J $\rightarrow X_j = 0$
- 2. J_{t+1} : a new element is augmented to the right of the J_t
- 3. J_{t+1} is fathomed if
 - (a) it can not lead to a better objective value
 - (b) it can not lead to a feasible solution

$$J_0 = \phi(\dots\dots)$$



— 4. if all the element of a fathomed J_t is negative , the enumeration is finished . Otherwise , select the rightmost positive element , complement it and delete all the negative elements to its right

— example :

$$J_t = \{1,5,4,-3,2,6,-7,-8\}$$

$$J_{t+1} = \{1,5,4,-3,2,6\}$$

C. Basic tests

- A. For any free variables X_r , if $a_{ir} \geq 0$ for all i corresponding to $S_i^t \leq 0 \Rightarrow X_r$ can not improve the feasibility, X_r is discard
- B. For any free variables X_r , $C_r + Z^t \geq Z_u \Rightarrow X_r$ is discard
- C. For the remaining free variables X , for at least one $S_i^t < 0$:

J_t is fathomed

$$\sum_{j \in N_t}^m \min\{0, a_{ij}\} \geq S_i^t$$

- D. Measure of the total infeasibility resulting from set free X_r to 1

$N_t \neq \emptyset$, branch variables X is select as the one

$$V_k^t = \text{Max}_{j \in N_t} \{V_j^t\}, \text{ where } V_j^t = \min_{i \in I} \{0, S_i^t - a_{ij}\}$$

if $V_k^t = 0, X_k^t = 1$, J_t is improve feasible solution

J_{t+1} is defined by J_t^* with k on the right is fathomed

D. Solution procedure

- Step0 set $J_0 = \emptyset$ $Z_u = \infty$

$$\text{Set all } X_i = 0 \quad \forall_i \in n$$

$$S_j = b_j \quad \forall_j \in m$$

$$t = 0 \quad Z^0 = 0$$

- Step1 identify the branching variable X_k by the basic tests A,B,C,D if no free variables, J_t is fathomed, go to step3
- Step2 X_k is identified, $J_{t+1} = J_t \cup \{k\}$, if $V_k^t = 0, Z < Z_u$
Set $Z_u = Z$ from J_{t+1} , and J_{t+1} is fathomed. Otherwise, go to step1
- Step3 if all $j \leq 0$ in the remaining unfathomed J_t , \forall_t stop, otherwise complementary J_t , go to step 1.

E. Example

$$\begin{aligned} \text{Max} \quad & 3X_1^+ + 2X_2^+ - 5X_3^+ - 2X_4^+ + 3X_5^+ \\ \text{St.} \quad & X_1^+ + X_2^+ + X_3^+ + 2X_4^+ + X_5^+ \leq 4 \\ & 7X_1^+ + 3X_3^+ - 4X_4^+ + 3X_5^+ \leq 8 \\ & 11X_1^+ + 6X_2^+ + 3X_4^+ - 3X_5^+ \geq 3 \\ & X_j^+ = \{0,1\} \forall_j \end{aligned}$$

— Convert to minimization

$$\text{Min} \quad -3X_1^+ - 2X_2^+ + 5X_3^+ + 2X_4^+ - 3X_5^+$$

— All constraint to ≤ 0 constraint

$$-11X_1^+ + 6X_2^+ - 3X_4^+ + 3X_5^+ \leq -3$$

— convert $C_i \leq 0$ to $C_i \geq 0$ by replaced

$$X_j^+ = 1 - X_j \quad \text{for } j=1,2,5$$

$$\begin{aligned} \text{min} \quad & -3(1 - X_1) - 2(1 - X_2) + 5X_3^+ + 2X_4^+ - 3(1 - X_5) \\ \text{st.} \quad & (1 - X_1) + (1 - X_2) + X_3^+ + 2X_4^+ + (1 - X_5) \leq 4 \\ & 7(1 - X_1) + 3X_3^+ - 4X_4^+ + 3(1 - X_5) \leq 8 \\ & -11(1 - X_1) + 6(1 - X_2) - 3X_4^+ + 3(1 - X_5) \leq 3 \\ & X_j^+ = \{0,1\} \forall_j \end{aligned}$$

$$\begin{aligned} \longrightarrow \text{min} \quad & 3X_1 + 2X_2 + 5X_3 + 2X_4 + 3X_5 - 8 \\ \text{st.} \quad & -X_1 - X_2 + X_3 + 2X_4 - X_5 \leq 1 \\ & -7X_1 + 3X_3 - 4X_4 - 3X_5 \leq -2 \\ & 11X_1 - 6X_2 + 3X_4 - 3X_5 \leq -1 \\ & X_j^+ = \{0,1\} \forall_j \end{aligned}$$

— LP simplex tableau

Basis	X_1	X_1	X_1	X_1	X_1	S_1	S_2	S_3	<i>RHS</i>
	3	2	5	2	3	0	0	0	
S_1	-1	-1	1	2	-1	1	0	0	1
S_2	17	0	3	-4	-3	0	1	0	-2
S_3	11	-6	0	-3	-3	0	0	1	-1

t = 0

— $J_0 = \{ \}$, $Z^0 = 0$, $(S_1^0, S_2^0, S_3^0) = (1, -2, -1)$, $Z^* = \infty$

— Test A: $a_{ij} = 3 \geq 0$ and $a_{33} = 0 \geq 0 \Rightarrow X_3$ is discard

— Test B: $\{ \}$

- $X_1 : Z^0 + C_1 = 0 + 3 = 3 < Z^* = \infty$

- $X_2 : Z^0 + C_2 = 0 + 2 = 2 < Z^* = \infty$

- $X_3 : Z^0 + C_3 = 0 + 5 = 5 < Z^* = \infty$

- $X_4 : Z^0 + C_4 = 0 + 2 = 2 < Z^* = \infty$

- $X_5 : Z^0 + C_5 = 0 + 3 = 3 < Z^* = \infty$

— Test C:

- $S_2 = -7 - 4 - 3 = -14 < -2$

- $S_3 = -6 - 3 - 3 = -12 < -1$

- $N_1 = \{1, 2, 4, 5\}$

— Test D:

- $V_1 = \min\{0, 1 - (-1)\} + \min\{0, -2 - (-7)\} + \min\{0, -1 - 11\} = 12$

- $V_2 = \min\{0, 1 - (-1)\} + \min\{0, -2 - 0\} + \min\{0, -1 - (-6)\} = -2$

- $V_4 = \min\{0, 1 - 2\} + \min\{0, -2 - (-4)\} + \min\{0, -1 - (-3)\} = -1$

- $V_5 = \min\{0, 1 - (-1)\} + \min\{0, -2 - (-3)\} + \min\{0, -1 - (-3)\} = 0$

- $k = 5$

t = 1

— $J_1 = \{5\}$, $Z^1 = 3$, $(S_1^1, S_2^1, S_3^1) = (2, 1, 2)$, $Z^* = 3$

— all $S_i \geq 0 \Rightarrow$ Fathomed

t = 2

— $J_2 = \{-5\}$, $Z^2 = 0$, $(S_1^2, S_2^2, S_3^2) = (1, -2, -1)$, $Z^* = 3$

— Test A: $a_{32} = 3 \geq 0$ and $a_{33} = 0 \geq 0 \Rightarrow X_3$ is discard

— Test B: $\{ \}$

- $X_1 : Z^2 + C_1 = 0 + 3 = 3 = Z^* = 3$

X_3 is discard

- $X_2 : Z^2 + C_2 = 0 + 2 = 2 < Z^* = 3$

- $X_3 : Z^2 + C_3 = 0 + 2 = 2 < Z^* = 3$

— Test C:

- $S_2 = 0 - 4 = -4 < -2$

- $S_3 = -6 - 3 = -9 < -1$

- $N_1 = \{2, 4\}$

— Test D:

- $V_2 = \min\{0, 1 - (-1)\} + \min\{0, -2 - 0\} + \min\{0, -1 - (-6)\} = -2$

- $V_4 = \min\{0, 1 - 2\} + \min\{0, -2 - (-4)\} + \min\{0, -1 - (-3)\} = -1$

- $k = 4$

t = 3

— $J_3 = \{-5, 4\}$, $Z^3 = 2$, $(S_1^3, S_2^3, S_3^3) = (-1, 2, 2)$, $Z^* = 3$

— Test A: $a_{31} = 1 \geq 0$ and $\Rightarrow X_3$ is discard

— Test B: $\{ \}$

- $X_1 : Z^3 + C_1 = 2 + 3 = 5 > Z^* = 3$

X_1 is discard

- $X_2 : Z^3 + C_2 = 2 + 2 = 4 > Z^* = 3$

X_2 is discard

- $N_3 = \{ \}$ is fathomed

t = 4

— $J_4 = \{-5, -4\}$, $Z^4 = 0$, $(S_1^4, S_2^4, S_3^4) = (1, -2, -1)$, $Z^* = 3$

— Test B: $\{ \}$

- $X_1 : Z^4 + C_1 = 0 + 3 = 3 = Z^* = 3$

X_1 is discard

- $X_2 : Z^4 + C_2 = 0 + 2 = 2 < Z^* = 3$

— Test C:

- $S_2 = 0 > -2$

- $S_3 = -6 < -1$

- $N_1 = \{ \}$ fathomed

F. Example

$$\text{Max } 2X_1^+ + 2X_2^+ + 3X_3^+ - X_4^+ - 2X_5^+$$

$$\text{St. } -2X_1^+ + 3X_2^+ - 4X_3^+ + 5X_4^+ + 5X_5^+ \geq 5$$

$$X_1^+ + 2X_2^+ - 4X_4^+ - 3X_5^+ \leq 0$$

$$X_j^+ = \{0, 1\} \forall_j$$

— Convert to minimization

$$\text{Min } -2X_1^+ - 2X_2^+ - 3X_3^+ + X_4^+ + 2X_5^+$$

— All constraint to $\leq 0 \Rightarrow$ constraint

$$2X_1^+ - 3X_2^+ + 4X_3^+ - 5X_4^+ - 5X_5^+ \leq -5$$

— convert $C_i \leq 0$ to $C_i \geq 0$ by replaced

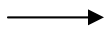
$$X_j^+ = 1 - X_j \text{ for } j=1,2,3$$

$$\text{min. } -2(1 - X_1) - 2(1 - X_2) - 3(1 - X_3) + X_4^+ + 2X_5^+$$

$$\text{st. } 2(1 - X_1) - 3(1 - X_2) + 4(1 - X_3) - 5X_4^+ - 5X_5^+ \leq -5$$

$$-(1 - X_1) + 2(1 - X_2) - X_4^+ - 3X_5^+ \leq 0$$

$$X_j^+ = \{0, 1\} \forall_j$$



$$\min \underline{2X_1 + 2X_2 + 3X_3 + X_4 + 2X_5 - 7}$$

$$\text{st.} \quad -2X_1 + 3X_2 - 4X_3 - 5X_4 - 5X_5 \leq -8$$

$$-X_1 - 2X_2 - X_4 - 3X_5 \leq -3$$

$$X_j^+ = \{0,1\} \forall_j$$

— LP simplex tableau

Basis	X_1	X_1	X_1	X_1	X_1	S_1	S_2	RHS
	2	2	3	1	2	0	0	
S_1	-2	3	-4	-5	-5	1	0	-8
S_2	-1	-2	0	-1	-3	0	1	-3

t = 0

— $J_0 = \{ \}$, $Z^0 = 0$, $(S_1^0, S_2^0) = (-8, -3)$, $Z^* = \infty$

— Test A: $\{ \}$ →

— Test B: $\{ \}$

- $X_1 : Z^0 + C_1 = 0 + 2 = 2 < Z^* = \infty$

- $X_2 : Z^0 + C_2 = 0 + 2 = 2 < Z^* = \infty$

- $X_3 : Z^0 + C_3 = 0 + 3 = 3 < Z^* = \infty$

- $X_4 : Z^0 + C_4 = 0 + 1 = 1 < Z^* = \infty$

- $X_5 : Z^0 + C_5 = 0 + 2 = 2 < Z^* = \infty$

— Test C:

- $S_2 = -2 - 0 - 4 - 5 - 5 = -16 < -2$

- $S_3 = -1 - 2 - 0 - 1 - 3 = -7 < -1$

- $N_1 = \{1, 2, 3, 4, 5\}$

— Test D:

- $V_1 = \min\{0, -8 - (-2)\} + \min\{0, -3 - (-1)\} = -8$

- $V_2 = \min\{0, -8 - (3)\} + \min\{0, -3 - (-2)\} = -12$

- $V_3 = \min\{0, -8 - (-4)\} + \min\{0, -3 - (0)\} = -7$

- $V_4 = \min\{0, -8 - (-5)\} + \min\{0, -3 - (-1)\} = -5$

- $V_5 = \min\{0, -8 - (-5)\} + \min\{0, -3 - (-3)\} = -3$

- k = 5

t = 1

— $J_1 = \{5\}$, $Z^1 = 2$, $(S_1^1, S_2^1) = (-3, 0)$, $Z^* = \infty$

— Test A: $\{ \}$

$3 > 0$ X_2 is discard

— Test B: $\{ \}$

- $X_1 : Z^1 + C_1 = 2 + 2 = 4 < Z^* = \infty$

- $X_3 : Z^1 + C_3 = 2 + 3 = 5 < Z^* = \infty$

- $X_4 : Z^1 + C_4 = 2 + 1 = 3 < Z^* = \infty$

— Test C:

- $S_1 = -2 - 4 - 5 = -11 < -3$

- $S_2 = -1 - 0 - 1 = -2 < 0$

- $N_1 = \{1, 3, 4\}$

— Test D:

- $V_1 = \min\{0, -3 - (-2)\} + \min\{0, -0 - (-1)\} = -1$

- $V_3 = \min\{0, -3 - (-4)\} + \min\{0, 0 - (0)\} = 0$

- $V_4 = \min\{0, -3 - (-5)\} + \min\{0, -0 - (-1)\} = 0$

- $k = 4$

→

t = 2

— $J_2 = \{5, 4\}$, $Z^2 = 2$, $(S_1^2, S_2^2) = (2, 1)$, $Z^* = 3$

— all $S_i \geq 0 \Rightarrow$ Fathomed

t = 3

— $J_3 = \{5, -4\}$, $Z^3 = 2$, $(S_1^3, S_2^3) = (-3, 0)$, $Z^* = 3$

— Test A: $\{ \}$

$3 > 0$ X_2 is discard

— Test B: $\{ \}$

- $X_1 : Z^3 + C_1 = 2 + 2 = 4 > Z^* = 3$

- $X_3 : Z^3 + C_3 = 2 + 3 = 5 > Z^* = 3$

- $N_3 = \{ \}$ is fathomed

t = 4

— $J_4 = \{-5, -4\}$, $Z^4 = 0$, $(S_1^4, S_2^4) = (-8, -3)$, $Z^* = 3$

— Test A: $\{ \}$

— Test B: $\{ \}$

- $X_1 : Z^4 + C_1 = 0 + 2 = 2 < Z^* = 3$

- $X_2 : Z^4 + C_2 = 0 + 2 = 2 < Z^* = 3$

- $X_3 : Z^4 + C_3 = 0 + 3 = 3 = Z^* = 3$

X_3 is discard

- $X_4 : Z^4 + C_2 = 0 + 2 = 2 < Z^* = 3$

— Test C:

- $S_1 = -2 - 0 - 5 = -7 > -8$

- $S_2 = -1 - 2 - 1 = -4 < -1$

- $N_4 = \phi$

例題

考慮5種不可分割的貨物，其單位重量與價值如下：

號列	重量	價值
1	18	27
2	12	9
3	15	30
4	16	16
5	13	6.5

設卡車的總載重限制為45單位重，此問題為如何裝運貨物，使在載重限制內，裝運價值最大的貨物。

解：首先，依照貨物單位重量的價值排列這些貨物：

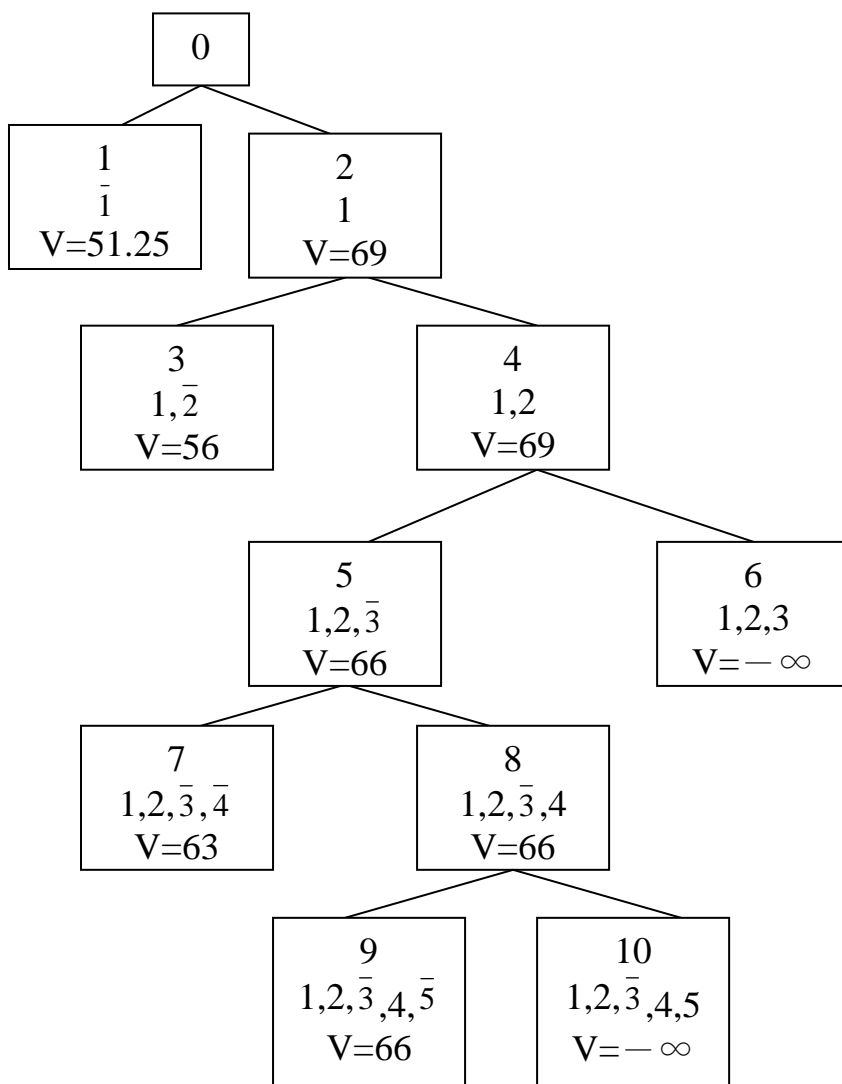
索引號列	貨物號列	重量	價值	價值/重量
1	3	15	30	2
2	1	18	27	1.5
3	4	16	16	1
4	2	12	9	0.75
5	5	13	6.5	0.5

由於貨物不可分割，依排列順序裝運貨物，直至未超過載運限制的裝貨為止。則裝運所有的貨物3與貨物1，總重量33，總價值57；但並非最佳解，因為若裝運所有的貨物3、1與4，則裝運總重量為45，總價值為66。則此問題之樹形解（solution of tree）如下所示，過程如同分枝限定法：

節點1包含了所有貨物的組合，但不包括索引號列1的貨物；節點1之上界計算如下：

索引號列	貨物號列	重量	價值
2	1	18	9
3	4	15	30
4	2	16	16

若裝貨內容包含所有貨物1、貨物4及11/12的貨物2，則裝載總重量為 $18+16+11/12(12) = 45$ 單位，而總價值為 $V = 27+16+11/12(9) = 51.25$ 單位。



其中，節點9為最佳的裝貨組合。