Chapter 5　　L-P　sensitivity analysis

5.1　Introduction

A. Definition

— To study how changes of coefficients affect the optimal solution.

— Referred as

　　— Parameter analysis

　　— Post-optimal analysis

B. Reasons

— Omitted factors

— Inexactitudes

— Uncertainty

C. Change in

— C_j

— b_i

D. Objective detect changes in

— Objective function value

— Optimal solution

E. Application

— Provide respond information

— Identify critical coefficients

— Determine the worth of additional resource

5.2 Objective function coefficient

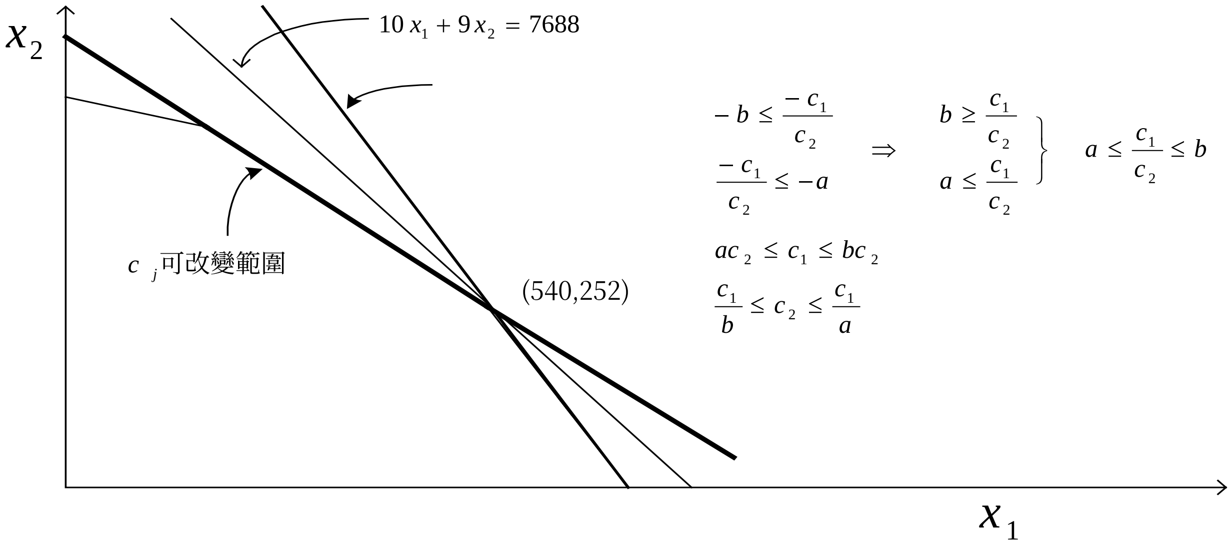
A. Background

— One C_j change at a time

— Range of optimality

— The range a C_j can vary without cause change in the optimal solution

B. Graphical solution procedure



$$-b \le \frac{-c_1}{c_2} \qquad \Rightarrow \qquad b \ge \frac{c_1}{c_2}$$

$$\frac{-c_1}{c_2} \le -a \qquad\qquad a \le \frac{c_1}{c_2}$$

$$\left.\begin{matrix} b \ge \frac{c_1}{c_2} \\ a \le \frac{c_1}{c_2} \end{matrix}\right\} \quad a \le \frac{c_1}{c_2} \le b$$

$$ac_2 \le c_1 \le bc_2$$

$$\frac{c_1}{b} \le c_2 \le \frac{c_1}{a}$$

Ex ： Max $10x_1 + 9x_2$

st

$$\frac{7}{10}x_1 + - x_2 \le 630$$

$$\frac{1}{2}x_1 + \frac{5}{6}x_2 \le 600$$

$$x_1 + \frac{2}{3}x_2 \le 780$$

$$\frac{1}{10}x_1 + \frac{1}{4}x_2 \le 135$$

$$x_1 \ge 0$$

$$x_2 \ge 0 \qquad \Rightarrow \begin{cases} x_1 = 540 \\ x_2 = 252 \end{cases}$$

$$\begin{matrix} \frac{7}{10}x_1 + - x_2 = 630 \\ x_1 + \frac{2}{3}x_2 = 780 \end{matrix} \quad \Rightarrow \quad \begin{matrix} -\frac{7}{10} = -a \\ -\frac{3}{2} = -b \end{matrix} \quad \Rightarrow \quad \begin{matrix} -\frac{3}{2} \le -\frac{c_1}{c_2} \le \frac{7}{10} \\ \frac{3}{2} \ge \frac{c_1}{c_2} \ge \frac{7}{10} \end{matrix}$$

$$\Rightarrow \frac{3}{2} \times 9 \ge c_1 \ge \frac{7}{10} \times 9 \qquad [6.3, 13.5]$$

$$\Rightarrow \left(10 \Big/ \tfrac{3}{2}\right) \frac{c_1}{b} \ge c_2 \ge \frac{c_1}{a} \left(10 \Big/ \tfrac{7}{10}\right) \qquad [6.8, 13.7]$$

C. Simplex solution procedure

— The range of optimality is determine by $c_j - z_j \le 0 \quad \forall x_j$ in the final tableau

— Procedure

— Replace the value of c_j by c_j variable in the final tableau

— If x_j is basic variable ：recompute $(c_j - z_j)$ for all nonbasic

— If x_j is non-basic variable ：recompute $(c_j - z_j)$ for x_j only

— Set all associate $c_j - z_j \le 0$ under consideration

— Identify the 【LB,UB】for c_j

Ex：

		X_1	X_2	S_1	S_2	S_3	RHS
basis	C_B	3	5	0	0	0	
S_1	0	0	0	1	$1/3$	$-1/3$	2
X_2	5	0	1	0	$1/2$	0	6
X_1	3	1	0	0	$-1/3$	$1/3$	2
Z_j		3	5	0	$3/2$	1	
$C_j - Z_j$		0	0	0	$-3/2$	-1	

For C_1
$$\frac{2c_1 - 15}{6} \le 0 \Rightarrow c_1 \le 15/2$$
$$\frac{-c_1}{3} \le 0 \Rightarrow c_1 \ge 0 \qquad \left[0, \tfrac{15}{2}\right]$$

For C_2
$$c_{s_2} - 3/2 \le 0 \Rightarrow c_{s_2} \le 3/2$$

5.3 Right hand side value

A. Shadow price

— The change in the O.F.V. if a unit of b_i increase

— Applicable to small change

— Simplex method

— "<" constraint： z_j value of s_i

— ">" constraint： - z_j value of s_i

— "=" constraint： z_j value of a_j

EX. Shadow price for resource 1：0 $\rightarrow s_1 > 0$

2：$3/2$ $\rightarrow s_2 = 0$

3：1 $\rightarrow s_3 = 0$

basis	C_B	X_1 3	X_2 5	S_1 0	S_2 0	S_3 0	RHS
S_1	0	0	0	1	$1/3$	$-1/3$	2
X_2	5	0	1	0	$1/2$	0	6
X_1	3	1	0	0	$-1/3$	$1/3$	2
Z_j		3	5	0	$3/2$	1	36
$C_j - Z_j$		0	0	0	$-3/2$	-1	

B. Range of feasibility

— the range of value b_i may vary without causing current basic variables become infeasible.

— Simplex approach

— "≤"constraint

$$\begin{bmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_m \end{bmatrix} + \Delta b_i \begin{bmatrix} \bar{a}_{1j} \\ \vdots \\ \bar{a}_{mj} \end{bmatrix} \geq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \bar{b}_1 + \Delta b_i \bar{a}_{1j} \geq 0 \\ \vdots \\ \bar{b}_m + \Delta b_i \bar{a}_{mj} \geq 0 \end{cases}$$

— ">"constraint

$$\begin{bmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_m \end{bmatrix} - \Delta b_i \begin{bmatrix} \bar{a}_{1j} \\ \vdots \\ \bar{a}_{mj} \end{bmatrix} \geq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Ex Δb_2

$$\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} + \Delta b_2 \begin{bmatrix} 1/3 \\ 1/2 \\ -1/3 \end{bmatrix} \geq 0$$

$$\begin{aligned} & 2 + \tfrac{1}{3}\Delta b_2 \geq 0 & & \Delta b_2 \geq -6 \\ \Rightarrow\ & 6 + \tfrac{1}{2}\Delta b_2 \geq 0 & \Rightarrow\ & \Delta b_2 \geq -12 \\ & 2 - \tfrac{1}{3}\Delta b_2 \geq 0 & & \Delta b_2 \leq 6 \end{aligned}$$

$$\Rightarrow -6 \leq \Delta b_2 \leq 6$$

$$\Rightarrow 6 \leq b_2 \leq 12$$

C. Simultaneous change

— if there are two or more coefficients change

— use 100 % rule for acceptance

— compute the ratio of the net change to allowable change

— if total change ratio do not exceed 100 % .we assume it has no impact on solution

Ex
$$\begin{aligned} -5 \leq \Delta b_1 \leq 10 \\ -30 \leq \Delta b_2 \leq 20 \end{aligned}$$

if $\left.\begin{aligned} \Delta b_1 &= -4 \\ \Delta b_2 &= 2 \end{aligned}\right\} \Rightarrow \dfrac{-4}{-5} + \dfrac{2}{20} = \dfrac{18}{20} < 100\%$

Ex.

Max. 15 x_1 + 5 x_2 (1)

s.t. 2 x_1 + 3 x_2 ≦ 54 (2)

4 x_1 +2 x_2 ≦ 40 (3)

$x_1, x_2 \geq 0$

此問題的解為 $x_1^* = 10$，$x_2^* = 0$, 且 $f^* = 150$

（a）試就目標函數中的 x_2 之係數（c_2）的變動範圍，討論最優解的變化情形。

（b）假設方程式(2)變為

$$3.5x_1 + 2x_2 \leq 40$$

（或許由於某些方法或設備改良的結果）此問題之結果為何？若 a_{11}=2 有變動，則此問題的結果為何？

（c）(2)式右端值的變動，b_1 從54變為60；然後 b_2 從40變為44。各情形對問題的影響為何？

（d）假設限制式

$$4x_1 + 1.5x_2 = 36$$

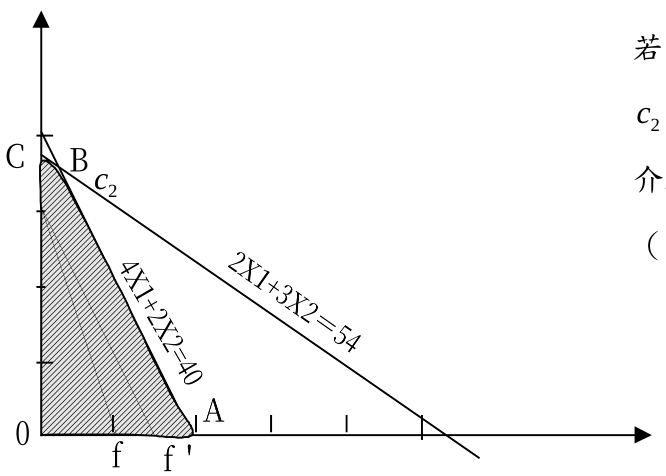
加入此問題，是否會影響最佳解？

Sol：（a）假設變數 x_2 的利潤增加＄2.50

由於只有目標函數變動，很顯然圖中可行解集OABC並未改變。

然而，等利潤線的斜率已改變，虛線f'＝＄75為新的等利潤線。

若目標函數係數中的 x_2 進一步地增加，則B將為唯一的最優解，此時的 c_2 ＝＄22.5。在此點時，點B與C皆是最優的。



目標函數係數之變動

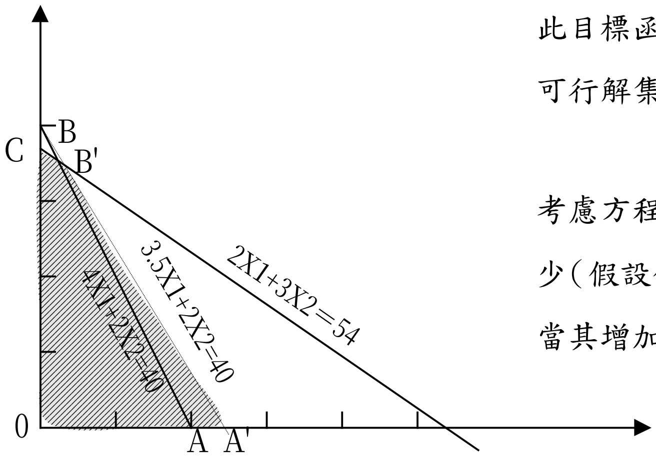
若 c_2 ＞＄22.5，則C點最優。

c_2 ＜＄7.5，則A點最優。

介於＄7.5與＄22.5之間則B點最優。

（當 c_2 ＝＄7.5，點A、B；c_2 ＝＄22.5，點B、C）

（b）此時新的最佳解發生在A’＝（11.43 , 0）而不是點A（10 , 0）

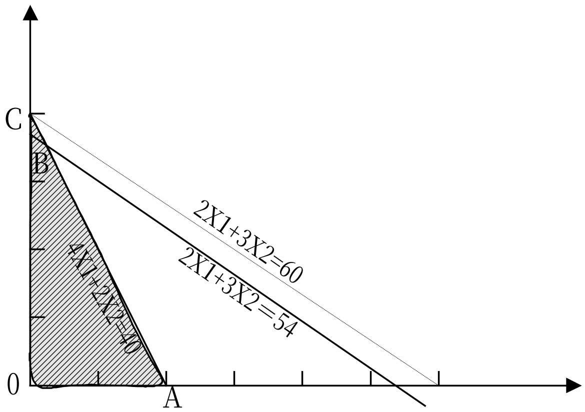


此目標函數之新的最佳解為＄15×11.43＝＄171.45。

可行解集從OABC變為OA’B’C。

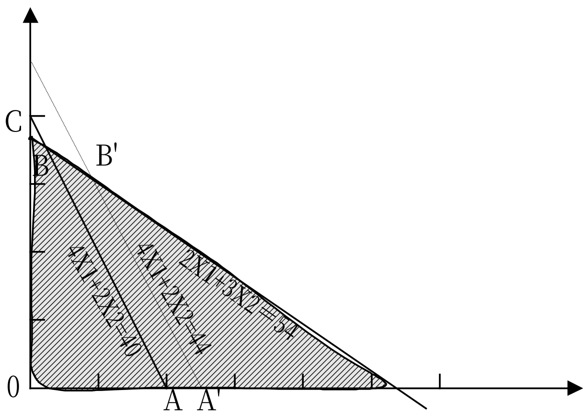
考慮方程式中 x_1 係數（ a_{11}=2 ）的變動情形，若 a_{11} 減少（假設仍為正的），則不論減少為何皆不會有影響，當其增加至5.4時，方有影響。

（c）限制式右端的常數項變動與原來的限制式直線平行。



b1從54改變為60，

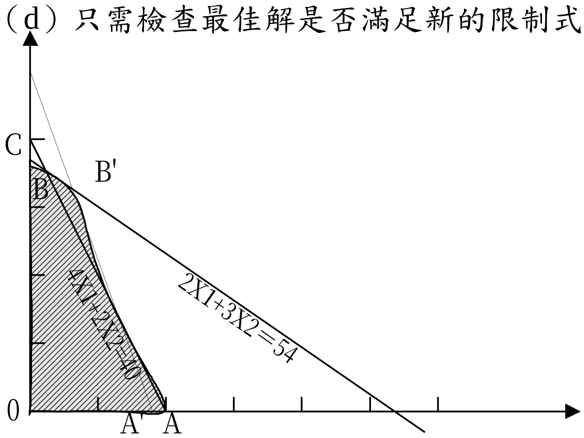
不影響最佳解。



b2從40改為44

最佳解變為 $X_1{*}=11$ ， $X_2{*}=0$ 且 $f{*}=165$

（d）只需檢查最佳解是否滿足新的限制式。若是的話，則沒有任何改變。



此新的最佳解為 $X_1{*}=9$，$X_2{*}=0$ 且 $f{*}=135$

5.4 Duality

A. Purpose

- In theory

  (1) Active or inactive constraint

  (2) Complementary slackness principle

- In economics：shadow price
- In computation：efficiency

B. Transformation rules

Primal	Dual
Max	Min
n	m
c_j	b_i
b_i	c_j
a_{ij}	a_{ji}
$A_{ij}X_j > b_i$	$Y_i \le 0$
$A_{ij}X_j = b_i$	Y_i unrestricted
$A_{ij}X_j \le b_i$	$Y_i \ge 0$
$X_j < 0$	$A_{ji}Y_i < c_j$
X_j unrestricted	$A_{ji}Y_i = c_j$
$X_j \ge 0$	$A_{ji}Y_i \ge c_j$

C. Summary

- Canonical form ：converted problem to canonical form

Primal	Dual
Max cx	Min uB
St $Ax \le B$	St $uA^T \ge c$
$x \ge 0$	$u \ge 0$

D. Specially case

Max cx	Min uB
St $A_i X_j \le b_i$	St $u \ge 0$
$A_i X_j = b_i$	u_i unrestricted
$A_i X_j \ge b_i$	$u_i \le 0$
$X_j \ge 0$	$u_i A_j \ge c_j$
$X_j \le 0$	$u_i A_j \le c_j$
X_j unrestricted	$u_i A_j = c_j$

Ex：

Max	$2x_1 + 4x_2$	Min	$7u_1 + 70u_2 + 10u_3$
St	$x_1 + 3x_2 \le 7$	St	$u_1 + 4u_2 + u_3 \ge 2$
	$4x_1 + 5x_2 \le 70$		$3u_1 + 5u_2 - u_3 \ge 4$
	$x_1 - x_2 \le 10$		
	$x_1, x_2 \ge 0$		$u_1, u_2, u_3 \ge 0$

Ex：

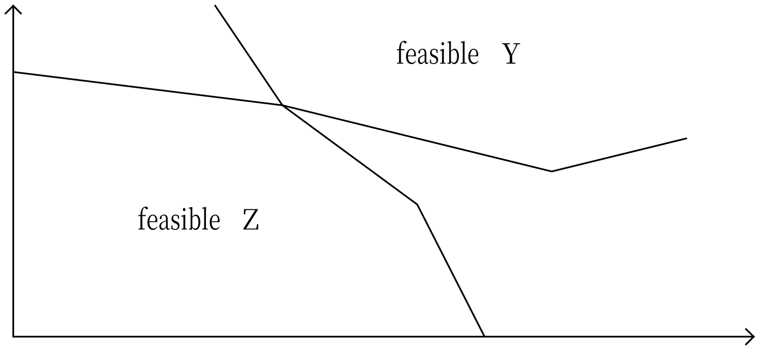
Max	$x_1 - 2x_2 - 3x_3$	Min	$10u_1 - 6u_2$
St	$3x_1 + 5x_2 + 10x_3 \le 10$	St	$3u_1 + 2u_2 \le 1$
	$2x_1 - x_2 + 11x_3 \ge -6$		$5u_1 - u_2 = -2$
			$10u_1 + 11u_2 \ge -3$
	$x_1 \le 0$		
	x_2 unconstrained		$u_1 \ge 0$
	$x_3 \ge 0$		$u_2 \le 0$

E. Primal-Dual properties

— If both the primal（P）and dual（D）have feasible solution then they have optimal solution and Max Z = min Y

— For any feasible solution to P and any feasible solution to D the value of corresponding objective function will always be relative by $Z \le Y$

— For any feasible solution to P and any feasible solution to D. If $Z^* = Y^*$ then they are both optimal for each respective problem.



$$Z < Z^* = Y^* < Y$$

Primal	Dual
Suboptimal	Over optimal
Optimal	Optimal
Over optimal	Suboptimal
Unbounded	Infeasible
Infeasible	Unbounded

— If either problem had an unbounded feasible solution then the other problem has no feasible solution.

— The primal = the dual of the dual

— U（dual variable）to primal = price vector， dual price， shadow price， imputed price

F. Complementary slackness principle

Primal	Dual
$u^*(a_i x - b_i) = 0$ $\Downarrow$ $u^* = 0$ or $a_i x - b_i = 0$	$x^*(ua_j - c_j) = 0$ $\Downarrow$ $Ya_j - c_j = 0$ or $x^* = 0$

Primal value		**Dual value**	
Original	X_j	$Z_j - C_j$	Surplus variable
Slack	X_{n+j}	U_i	Original variable

Ex：

Min $\quad 4u_1 + 12u_2 + 18u_3$ $\qquad$ Max $\quad 3x_1 + 2x_2$

St $\quad u_1 \qquad + 3u_3 \geq 3$ $\qquad$ St $\quad x_1 \qquad \leq 4$

$2u_2 + 2u_3 \geq 2$ $\qquad$ $2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

$u_1, u_2, u_3 \geq 0$ $\qquad$ $x_1, x_2 \geq 0$

$$S_1^* = 0 \quad S_2^* = 0 \quad U_1^* = 0 \quad U_2^* = 0 \quad U_3^* = 1$$

know：

$U_1^* = 0 \quad \Rightarrow \quad x_1 \leq 4$ .... (1)

$U_1^* = 0 \quad \Rightarrow \quad 2x_2 \leq 12$ .... (2)

$U_1^* = 0 \quad \Rightarrow \quad 3x_1 + 2x_2 = 18$ .... (3)

（1）（3）$x_1 = 4 \quad x_2 = 3 \quad s_1 = 0 \quad s_2 = 6 \quad s_3 = 0$

（2）（3）$x_1 = 2 \quad x_2 = 6 \quad s_1 = 2 \quad s_2 = 0 \quad s_3 = 0$

$$(x_1, x_2, s_1, s_2, s_3) = (4,3,0,6,0)$$

$$(2,6,2,0,0)$$

Ex.

Max. Z＝15X$_1$＋ 8X$_2$

s.t. $\quad$ X$_1$＋ 2X$_2$≧5

3 X$_1$＋ 2X$_2$＝12

5 X$_1$＋ 4X$_2$≦15

X$_1$≧0 , X$_2$≧0

→

Max. Z＝15X$_1$＋ 8X$_2$

s.t. $\quad$ －X$_1$－ 2X$_2$ ≦－5

3 X$_1$＋ 2X$_2$ ≦12

－3 X$_1$－ 2X$_2$ ≦－12

5 X$_1$＋ 4X$_2$ ≦15

X$_1$≧0 , X$_2$≧0

原始問題

		X_1	X_2	
對偶問題	W_1	-1	-2	-5
	W_2	3	2	12
	W_3	-3	-2	-12
	W_4	5	4	15
		15	8	

Min. Z＝5u$_1$＋12 u$_2$＋15 u$_3$

s.t. $\quad$ u$_1$＋ 3 u$_2$＋ 5 u$_3$ ≧15

2u$_1$＋ 2 u$_2$＋ 4 u$_3$ ≧8

u$_1$ ≦ 0 , u$_2$無符號限制 , u$_3$≧0

5.5 Computer analysis

A. Standard equality constraint form

— background：the optimal solution is a corner point

— procedure：adding slack（surplus）variables

B General form

$$\begin{aligned} \text{Max} \quad & CX \\ \text{st} \quad & AX = B \\ & X \geq 0 \end{aligned} \qquad \begin{aligned} C &: 1 \times n \\ A &: m \times n \\ B &: m \times 1 \\ X &: n \times 1 \end{aligned}$$

$$C = [C_1 \cdots C_n] \qquad A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \qquad X = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix}$$

Ex.

Max. $Z = 3X_1 + 5X_2$

s.t.

$$\begin{aligned} X_1 \quad & \leqq 4 \\ 2X_2 & \leqq 12 \\ 3X_1 + 2X_2 & \leqq 18 \\ x_1, x_2 & \geq 0 \end{aligned}$$

Max. $Z = [3,5] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

s.t. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leqq \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geqq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

C If there are feasible solutions

— m=n：unique solution

— m>n：at least (m-n) redundant constraints

— m<n：infinite number of solutions

D　　X can be partitioned into

X_B　　：basic solution , m variables

X_B　　：non-basic solution , n-m variables

set： $D = m \times n$　　　　　　　matrix

$N = m \times (n-m)$

$$AX = B \Rightarrow (D, N)\begin{pmatrix} X_B \\ X_N \end{pmatrix} = DX_B + NX_N = B$$

$$\Rightarrow DX_B = B$$

$$\Rightarrow X_B = D^{-1}B$$

Ex.

某公司擁有二種礦產A與B。A礦每天可生產1噸高等級的礦石、4噸中等級的礦石與6噸低等級的礦石。B礦每天可生產各種等級的礦石各2噸。設該公司至少需要60噸高等級礦石、120噸中等級礦石與150噸低等級礦石。若開採A礦，每天之成本為＄200，而開採B礦為＄300。若該公司希望成本達到最低，則應該開採A、B礦各幾天？

Sol:

原始題

min　$w = 200y_1 + 300y_2$

s.t.
$$\begin{aligned} &y_1 + 2y_2 \geq 60 \\ &4y_1 + 2y_2 \geq 120 \\ &6y_1 + 2y_2 \geq 150 \\ &y_1, y_2 \geq 0 \end{aligned}$$

其對應之對偶題為

Max　$u = 60x_1 + 120x_2 + 150x_3$

s.t.
$$\begin{aligned} &x_1 + 4x_2 + 6x_3 \leq 200 \\ &2x_1 + 2x_2 + 2x_3 \leq 300 \end{aligned}$$

求解得 $x_1 = \frac{400}{3}$，$x_2 = \frac{50}{3}$，$x_3 = 0$，Max　u＝$10000