Chapter 4 L-P simplex method

4.1 Introduction

A. Algorithm

— Structure

  — Initialization

  — Iterative steps

  — Stopping rule step

— Iteration

B. Characteristics

— An algebraie procedure which each iteration involves solving a system of obtain a new trial solution for optimality test

C. Properties of corner point solution

— If there is exactly one optimal solution it must be a corner solution.

— If there are multipal optimal solutions, then at least two must be adjacent corner points.

— There are finite corner points

— If a corner point has no adjacent corner point with better O.F.V. it must be optimal.

D. Outlines of simplex method

— Initalization step： start at a corner point

— Iterative steps： move to a better adjacent corner point

— Stopping rule step： no adjacent corner point with better O.F.V.

4.2 Special terms

— Degree of freedom

— Basic variables

— Nonbasic variables

— Basic solution：m basic variables, and （n-m）nonbasic variables.

— Basic feasible solution

— Degeneracy

— Simplex tableau

basis	C_k	X_1 C_1	X_2 C_2	X_3 C_3	……..	X_n C_n	RHS
X_1	C_1	a_{11}				a_{1n}	b_1
$\vdots$	$\vdots$	$\vdots$				$\vdots$	$\vdots$
X_m	C_m	a_{m1}				a_{mn}	b_m
Z_j							
$C_j - Z_j$							

4.3 Setting Up the Simplex Method

A Formulation the standard form

— By adding slack / surplus variables to form an equal equation.

EX.

$$\begin{aligned} \text{Max} \quad & 5X_1 + 4X_2 \\ \text{st} \quad & 4X_1 + X_2 \le 60 \\ & 2X_1 + 2X_2 \le 48 \\ & X_1, X_2 \ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \text{Max} \quad & 5X_1 + 4X_2 + 0S_1 + 0S_2 \\ \text{st} \quad & 4X_1 + X_2 + S_1 \qquad = 60 \\ & 2X_1 + 2X_2 + \qquad S_2 = 48 \\ & X_1, X_2, S_1, S_2 \ge 0 \end{aligned}$$

B Tableau form

— For each constrain , there is exactly one basic variable with coefficient of 1 , the rest of basic variable with coefficient of 0.

— For each basic variable , only one constrain with coefficient of 1.

		X_1	X_2	S_1	S_2	RHS
basis	C_B	5	4	0	0	
S_1	0	4	1	1	0	60
S_2	0	2	2	0	1	48
Z_j						
$C_j - Z_j$						

4.4 Improve the solution

A. To improve a solution

- — Select one basic variable become nonbasic
- — Select one nonbasic variable become basic
- — New basic solution yield to a better O.F.V.

B. Procedure

— Z_j ：decrease in O.F.V. if one unit of X_j become basic in next iteration.

$$= \sum_{i=1}^{m} C_i a_{ij}$$

— $C_j - Z_j$ ：the net-change in O.F.V. if one unit of X_j become basic in next iteration.

— Criterion for selecting entering NB

— X_j^* with the largest value of $C_j - Z_j$

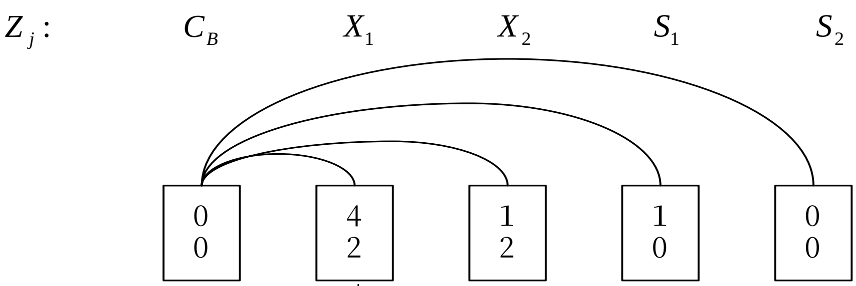
— Criterion for selecting leaving basis

— Select the one with min. value of $\dfrac{b_j}{a_{ij}^*}$, if $a_{ij}^* > 0$

— Criterion for optimality test

— If all $C_j - Z_j \le 0$, stop

— Iteration0:



Basis	C_B	X_1 5	X_2 4	S_1 0	S_2 0	RHS	b_i / a_{ij}	
S_1	0	4	1	1	0	60	60/4＝15	✓
S_2	0	2	2	0	1	48	48/2＝24	
Z_j		0	0	0	0			
$C_j - Z_j$		5	4	0	0			
		↑						

— Iteration1:

Basis	C_B	X_1 5	X_2 4	S_1 0	S_2 0	RHS	b_i / a_{ij}	
X_1	5	1	1/4	1/4	0	15	$15/\frac{1}{4}=60$	
S_2	0	0	3/2	-1/2	1	18	$18/\frac{3}{2}=12$	✓
Z_j		5	5/4	5/4	0	75		
$C_j - Z_j$		0	11/4	-5/4	0			
			↑					

— Calculating the next tableau

— To update the a_{ij} such that a_{ij} of basic variables are unit matrix

— By row operation - won't affect the solution

— Multiply any row by a non-zero number

— Add one row to another row

— Iteration2:

Basis	C_B	X_1 5	X_2 4	S_1 0	S_2 0	RHS
X_1	5	1	0	1/3	-1/6	12
X_2	4	0	1	-1/3	2/3	12
Z_j		5	4	1/3	11/6	108
$C_j - Z_j$		0	0	-1/3	-11/6	

(X_1, X_2, S_1, S_2)＝(12,12,0,0)

Max. Z＝108

4.5 General cases

A. "≥" constrain

— Step one：introducing a surplus variable to form equal equation.（$S_i, C_{S_i} = 0$）

Step two：add a artificial variable to start with feasible solution.（$a_i, C_{a_i} = -M$）

— Characteristics of artificial variable

- — With no physical meaning
- — To start with a feasible solution
- — Assign with big M cofficient to ensure its early removl
- — If a_i is not basic ,it will be eliminated from the simplex tableau

EX.

$$\begin{aligned} \text{Max} \quad & 3X_1 + 4X_2 \\ \text{s.t.} \quad & X_1 + 2X_2 \le 10 \\ & 3X_1 + 5X_2 \ge 18 \\ & 2X_1 + 3X_2 = 12 \\ & X_1, X_2 \ge 0 \end{aligned}$$

Sol:

$$\begin{aligned} \text{Max} \quad & 3X_1 + 4X_2 + 0S_1 + 0S_2 - Ma_2 - Ma_3 \\ \text{s.t.} \quad & X_1 + 2X_2 + S_1 = 10 \\ & 3X_1 + 5X_2 - S_2 + a_2 = 18 \\ & 2X_1 + 3X_2 + a_3 = 12 \\ & X_1, X_2, S_1, S_2, a_2, a_3 \ge 0 \end{aligned}$$

		X_1	X_2	S_1	S_2	a_2	a_3	RHS
Basis	C_B	3	4	0	0	-M	-M	
S_1	0	1	2	1	0	0	0	10
a_2	-M	3	5	0	-1	1	0	18
a_3	-M	2	3	0	0	0	1	12
Z_j		-5M	-8M	0	M	-M	-M	-30M
$C_j - Z_j$		3+5M	4+8M	0	-M	0	0	

		X_1	X_2	S_1	S_2	a_2	a_3	RHS
Basis	C_B	3	4	0	0	-M	-M	
S_1	0	-1/5	0	1	2/5	-2/5	0	14/5
X_2	4	3/5	1	0	-1/5	1/5	0	18/5
a_3	-M	1/5	0	0	3/5	-3/5	1	6/5
Z_j		12/5-1/5M	4	0	-4/5+3/5M	4/5+3/5M	-M	72/5-6/5M
$C_j - Z_j$		3/5+1/5M	0	0	4/5+3/5M	-4/5-2/5M	0	

Basis	C_B	X_1 3	X_2 4	S_1 0	S_2 0	a_2 -M	a_3 -M	RHS
S_1	0	-1/3	0	1	0	0	-2/3	2
X_2	4	2/3	1	0	0	0	1/3	4
S_2	0	1/3	0	0	1	-1	5/3	2
Z_j		8/3	4	0	0	0	4/3	16
$C_j - Z_j$		1/3	0	0	0	-M	-M	

Basis	C_B	X_1 3	X_2 4	S_1 0	S_2 0	a_2 -M	a_3 -M	RHS
S_1	0	0	1/2	1	0	0	-1/2	4
X_1	3	1	3/2	0	0	0	1/2	6
S_2	0	0	-1/2	0	1	-1	3/2	0
Z_j		3	9/2	0	0	0	3/2	18
$C_j - Z_j$		0	-1/2	0	0	-M	-M-3/2	

Basis	C_B	X_1 3	X_2 4	S_1 0	S_2 0	a_2 -M	a_3 -M	RHS
S_1	0	0	0	1	1	-1	1	4
X_2	4	0	1	0	-2	2	-3	0
X_1	3	1	0	0	3	-3	5	6
Z_j		3	4	0	1	-11	3	18
$C_j - Z_j$		0	0	0	-1	-M-3	-M-3	

$(X_1, X_2, S_1, S_2, s_3, s_4) = (6,0,4,0,0,0)$ Max Z＝18

— Two phase method for LP with a_i

— Phase 1：

— Iterations associated with elimination of a_i

— Phase 2：

— Iterations after all a_i are eliminated

EX. Max $50X_1 + 40X_2 + 0S_1 + 0S_2 + 0S_3 - 0S_4 - Ma_4$

st

$$\begin{aligned} 3X_1 + 5X_2 + S_1 &= 150 \\ X_2 + S_2 &= 20 \\ 8X_1 + 5X_2 + S_3 &= 300 \\ X_1 + X_2 - S_4 + a_4 &= 25 \\ X_1, X_2 \geq 0 \quad & S_1, S_2, S_3, S_4 \geq 0 \end{aligned}$$

Basis	C_B	X_1 50	X_2 40	S_1 0	S_2 0	S_3 0	S_4 0	a_4 -M	RHS
S_1	0	3	5	1	0	0	0	0	150
S_2	0	0	1	0	1	0	0	0	20
S_3	0	8	5	0	0	1	0	0	300
a_4	-M	1	1	0	0	0	-1	1	25
Z_j		-M	-M	0	0	0	+M	-M	
$C_j - Z_j$		M+50	M+40	0	0	0	-M	0	

$(X_1, X_2, S_1, S_2, S_3, S_4)$=(0,0,150,20,300,0,25)

Iteration 1：

basis	C_B	X_1 50	X_2 40	S_1 0	S_2 0	S_3 0	S_4 0	RHS
S_1	0	0	2	1	0	0	3	75
S_2	0	0	1	0	1	0	0	20
S_3	0	0	-3	0	0	1	8	100
X_1	50	1	1	0	0	0	-1	25
Z_j		50	50	0	0	0	-50	1250
$C_j - Z_j$		0	-10	0	0	0	-50	

$(X_1, X_2, S_1, S_2, S_3, S_4)$=(25,0,75,20,100,0)

Iteration 2：

S_1	0	0	$25/8$	1	0	$-3/8$	0	37.5
S_2	0	0	1	0	1	0	0	20
S_4	0	0	$-3/8$	0	0	$1/8$	1	12.5
X_1	50	1	$5/8$	0	0	$1/8$	0	37.5
Z_j		50	$250/8$	0	0	$50/8$	0	1875
$C_j - Z_j$		0	$70/8$	0	0	$-50/8$	0	

$(X_1, X_2, S_1, S_2, S_3, S_4)$=(37.5,0,37.5,20,125,0)

X_2	40	0	1	$8/25$	0	$-3/25$	0	12
S_2	0	0	0	$-8/25$	0	$3/25$	0	8
S_4	0	0	0	$3/25$	0	$2/25$	1	17
X_1	50	1	0	$-5/25$	0	$5/25$	0	30
Z_j		50	40	$14/5$	0	$26/5$	0	1980
$C_j - Z_j$		0	0	$-14/5$	0	$-26/5$	0	

$(X_1, X_2, S_1, S_2, S_3, S_4)$=(30,12,0,8,0,17)

B. "=" constraint

— Add an artificial variable to start a feasible solution.

EX. Max $50X_1 + 40X_2 + 0S_1 + 0S_2 + 0S_3 \; -Ma_4$

st

$$
\begin{aligned}
3X_1 + 5X_2 + S_1 \qquad\qquad\qquad\qquad &= 150 \\
X_2 \qquad + S_2 \qquad\qquad\qquad &= 20 \\
8X_1 + 5X_2 \qquad\qquad + S_3 \qquad\quad &= 300 \\
X_1 + X_2 \qquad\qquad\qquad\qquad + a_4 &= 25 \\
X_1, X_2 \ge 0 \qquad\qquad S_1, S_2, S_3 \ge 0 &
\end{aligned}
$$

C. Negative value of RHS （$b_i < 0$）

— If $b_i < 0$ multiply （-1） to the constraint

EX. $3X_1 - 2X_2 \le -5 \quad \Rightarrow \quad -3X_1 + 2X_2 \ge 5$

D. Minimization problem

(1) Approach 1

— revese the rule

— largest negative value of （$C_j - Z_j$） for entering nonbasic

— if all （$C_j - Z_j$） $\ge 0 \quad \Rightarrow$ stop

(2) Approach 2

— convert into maximization problem

EX. Max $2X_1 + 3X_2 \quad \Leftrightarrow \quad$ min $-2X_1 - 3X_2$

4.6 Special cases

A. Infeasible

— If a Max problem with all “$\le$”constraint .It must be feasible.

— The infeasibility exists when at least one artificial variables remain basic in the final tableau.

EX.

$$
\begin{aligned}
\text{Min} \quad & 3X_1 + 5X_2 \\
\text{s.t.} \quad & X_1 - X_2 \ge 2 \\
& X_1 + X_2 \le 5 \\
& -X_1 + 2X_2 \ge 4 \\
& X_1 \ge 0, X_2 \ge 0
\end{aligned}
$$

Basis	C_B	X_1 3	X_2 5	S_1 0	a_1 M	S_2 0	S_3 0	a_3 M	RHS
a_1	M	1	-1	-1	1	0	0	0	2
S_2	0	1	1	0	0	1	0	0	5
a_3	M	-1	2	0	0	0	-1	1	4
Z_j		0	M	-M	M	0	-M	M	6M
$C_j - Z_j$		3	5-M	M	0	0	M	0	

Basis	C_B	X_1 3	X_2 5	S_1 0	a_1 M	S_2 0	S_3 0	a_3 M	RHS
a_1	M	1/2	0	-1	1	0	-1/2	1/2	4
S_2	0	3/2	0	0	0	1	1/2	-1/2	3
X_2	5	-1/2	1	0	0	0	-1/2	1/2	2
Z_j		-5/2+1/2M	5	-M	M	0	-5/2-1/2M	5/2+1/2M	10+4M
$C_j - Z_j$		11/2-1/2M	0	M	0	0	5/2+1/2M	-5/2+1/2M	

Basis	C_B	X_1 3	X_2 5	S_1 0	a_1 M	S_2 0	S_3 0	a_3 M	RHS
a_1	M	0	0	-1	1	-1/3	-2/3	2/3	3
X_1	3	1	0	0	0	2/3	1/3	-1/3	2
X_2	5	0	1	0	0	1/3	-1/3	1/3	3
Z_j		3	5	-M	M	11/3-1/3M	-2/3-2/3	2/3＋2/3M	21＋3M
$C_j - Z_j$		0	0	M	0	-11/3＋1/3M	2/3＋2/3M	-2/3＋1/3M	≧0

B. Unboundness

— If at some iteration with all $a_{ij} \le 0$ in the pivot column then it is unbounded.

EX.

$$\begin{aligned}
\text{Min} \quad & 3X_1 + 4X_2 \\
\text{s.t.} \quad & X_1 + X_2 \geq 4 \\
& X_1 - 2X_2 \leq 5 \\
& X_1 \geq 0, X_2 \geq 0
\end{aligned}$$

		X_1	X_2	S_1	a1	S_2	RHS
Basis	C_B	3	4	0	-M	0	
a1	-M	1	1	-1	1	0	4
S_2	0	1	-2	0	0	1	5
Z_j		-M	-M	M	-M	0	-4M
$C_j - Z_j$		3+M	4+M	-M	0	0	

		X_1	X_2	S_1	a1	S_2	RHS
Basis	C_B	3	4	0	-M	0	
X_2	4	1	1	-1	1	0	4
S_2	0	3	0	-2	2	1	13
Z_j		4	4	-4	4	0	16
$C_j - Z_j$		-1	0	4	-4—M	0	

C. Alternate optimal

— If there are one or more nonbasic variable in the final tableau with $C_j - Z_j = 0$

EX.

$$\begin{aligned}
\text{Max} \quad & X_1 + 2X_2 \\
\text{s.t.} \quad & X_1 + 2X_2 \leq 10 \\
& X_1 + X_2 \geq 1 \\
& X_2 \leq 4 \\
& X_1 \geq 0, X_2 \geq 0
\end{aligned}$$

		X_1	X_2	S_1	S_2	a2	S_3	RHS
Basis	C_B	1	2	0	0	-M	0	
S_1	0	1	2	1	0	0	0	10
a2	-M	1	1	0	-1	1	0	1
S_3	0	0	1	0	0	0	1	4
Z_j		-M	-M	0	M	-M	0	-M
$C_j - Z_j$		1+M	2+M	0	-M	0	0	

		X_1	X_2	S_1	S_2	a2	S_3	RHS
Basis	C_B	1	2	0	0	-M	0	
S_1	0	-1	0	1	2	-2	0	8
X_2	2	1	1	0	-1	1	0	1
S_3	0	-1	0	0	1	-1	1	3
Z_j		2	2	0	-2	2	0	2
$C_j - Z_j$		-1	0	0	2	-2-M	0	

Basis	C_B	X_1 1	X_2 2	S_1 0	S_2 0	a2 -M	S_3 0	RHS
S_1	0	1	0	1	0	0	2	2
X_2	2	0	1	0	0	0	1	4
S_2	0	-1	0	0	1	-1	1	3
Z_j		0	2	0	0	0	2	8
$C_j - Z_j$		1	0	0	0	-M	-2	

Basis	C_B	X_1 1	X_2 2	S_1 0	S_2 0	a2 -M	S_3 0	RHS
X_1	1	1	0	1	0	0	-2	2
X_2	2	0	1	0	0	0	1	4
S_2	0	0	0	1	1	-1	-1	5
Z_j		1	2	1	0	0	0	10
$C_j - Z_j$		0	0	-1	0	-M	0	

$(X_1, X_2, S_1, S_2, a_2, S_3) = (2,4,0,5,0,0)$ Max. Z = 10

Basis	C_B	X_1 1	X_2 2	S_1 0	S_2 0	a2 -M	S_3 0	RHS
X_1	1	1	2	0	0	0	0	10
S_3	0	0	1	0	0	0	1	4
S_2	0	0	1	1	1	-1	0	9
Z_j		1	2	0	0	0	0	10
$C_j - Z_j$		0	0	0	0	-M	0	

$(X_1, X_2, S_1, S_2, a_2, S_3) = (10,0,0,9,0,4)$ Max. Z = 10

D. Degeneracy

— If in some iteration, the min. of $\dfrac{b_i}{a_{ij}}$ has a tie , then there is a basic variable become zero.

EX.

Max $6X_1 + 10X_2$

s.t. $X_1 \le 5$

$X_2 \le 6$

$3X_1 + 2X_2 \le 12$

$X_1 \ge 0, X_2 \ge 0$

Basis	C_B	X_1 6	X_2 10	S_1 0	S_2 0	S_3 0	RHS
S_1	0	1	0	1	0	0	5
S_2	0	0	[1]	0	1	0	6
S_3	0	3	0	0	0	1	12
Z_j		0	0	0	0	0	0
$C_j - Z_j$		6	10	0	0	0	

Basis	C_B	X_1 6	X_2 10	S_1 0	S_2 0	S_3 0	RHS
S_1	0	1	0	1	0	0	5
X_1	10	0	1	0	1	0	6
S_3	0	3	0	0	-2	1	0
Z_j		0	10	0	10	0	60
$C_j - Z_j$		6	0	0	-10	0	

Basis	C_B	X_1 6	X_2 10	S_1 0	S_2 0	S_3 0	RHS
S_1	0	0	0	1	2/3	-1/3	5
X_2	10	0	0	0	1	0	6
X_1	6	1	0	0	-2/3	1/3	0
Z_j		6	10	0	6	2	60
$C_j - Z_j$		0	0	0	-6	-2	

E. Variable with negative bound

— If $X_j \ge -L_j$

— Replace $\begin{matrix} X_j' = X_j + L_j \\ X_j = X_j' - L_j \end{matrix}$ into the model

EX.

$$\begin{array}{lll} \text{Max} & 3X_1 + 5X_2 \\ \text{st} & X_1 \le 4 \\ & 2X_2 \le 12 \\ & 3X_1 + 2X_2 \le 18 \\ & X_1 \ge -10 \end{array} \quad \Rightarrow \quad \begin{array}{lll} \text{Max} & 3(X_1' - 10) + 5X_2 \\ \text{st} & X_1' - 10 \le 4 \\ & 2X_2 \le 12 \\ & 3(X_1' - 10) + 2X_2 \le 18 \\ & (X_1' - 10) \ge -10 \end{array}$$

$X_2 \geq 0$ $X_2 \geq 0$

$$\begin{aligned} \text{Max} \quad & 3X_1' + 5X_2 - 30 \\ \text{st} \quad & X_1' \leq 14 \\ & 2X_2 \leq 12 \\ & 3X_1' + 2X_2 \leq 48 \\ & X_1' \geq 0 \\ & X_2 \geq 0 \end{aligned}$$

F. Variable unrestricted

— Replace X_j by $(X_j^+ - X_j^-)$

EX.

$$\begin{aligned} \text{Max} \quad & 3X_1 + 5X_2 \\ \text{st} \quad & X_1 \leq 4 \\ & 2X_2 \leq 12 \\ & 3X_1 + 2X_2 \leq 18 \\ & X_1 \ \text{restricted} \\ & X_2 \geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \text{Max} \quad & 3X_1^+ - 3X_1^- + 5X_2 \\ \text{st} \quad & X_1^+ - X_1^- \leq 4 \\ & 2X_2 \leq 12 \\ & 3X_1^+ - 3X_1^- + 2X_2 \leq 18 \\ & X_1^+ \geq 0 \\ & X_1^- \geq 0 \\ & X_2 \geq 0 \end{aligned}$$

— Another simplex tableau

	Z_j	X_1	X_2	S_1	S_2	S_3	
Z_j	1	-3	-5	0	0	0	0
S_1	0	1	0	1	0	0	4
S_2	0	0	2	0	1	0	12
S_3	0	3	2	0	0	1	18
Z_j	1	-3	0	0	$3/2$	0	30
S_1	0	1	0	1	0	0	4
X_2	0	0	1	0	$1/2$	0	6
S_3	0	3	0	0	-1	1	6
Z_j	1	0	0	0	$3/2$	1	36
S_1	0	0	0	1	$1/3$	$-1/3$	2
X_2	0	0	1	0	$1/2$	0	6
X_3	0	1	0	0	$-1/3$	$1/3$	2

		X_1	X_2	S_1	S_2	S_3	RHS
basis	C_B	3	5	0	0	0	
S_1	0	1	0	1	0	0	4
S_2	0	0	2	0	1	0	12
S_3	0	3	2	0	0	1	18
Z_j		0	0	0	0	0	0
$C_j - Z_j$		3	50	0	0	0	
S_1	0	1	0	1	0	0	4
X_2	5	0	1	0	$1/2$	0	6
S_3	0	3	0	0	-1	1	6
Z_j		0	5	0	$5/2$	0	30
$C_j - Z_j$		3	0	0	$-5/2$	0	
S_1	0	0	0	1	$1/3$	$-1/3$	2
X_2	5	0	1	0	$1/2$	0	6
X_1	3	1	0	0	$-1/3$	$1/3$	2
Z_j		3	5	0	$2/3$	1	36
$C_j - Z_j$		0	0	0	$-2/3$	-1	