Chapter 12 Linearized Supersonic Flow

$$(1 - M_\infty^2)\, \partial^2 \hat{\phi}/\partial x^2 + \partial^2 \hat{\phi}/\partial y^2 = 0$$

Holds for both subsonic and supersonic flow. For subsonic: $1- M_\infty^2 > 0$, for supersonic: $1- M_\infty^2 < 0$. Mathematically, when $1- M_\infty^2 < 0$, equation becomes a hyperbolic PDE. The purpose of this chapter is to obtain a solution of the equation for supersonic flow and to apply this solution to the calculation of supersonic airfoil properties.

Derivation of the Linearized Supersonic Pressure Coefficient formula

For the case of supersonic flow, the equation is written as

$\lambda^2 \partial^2 \hat{\phi}/\partial x^2 - \partial^2 \hat{\phi}/\partial y^2 = 0$, where $\lambda = (M_\infty^2 - 1)^{1/2}$

A solution to this equation is the functional relation, $\hat{\phi} = f(x - \lambda y)$

Proof: $\partial \hat{\phi}/\partial x = f'(x - \lambda y)[\partial(x - \lambda y)/\partial x] \Rightarrow \partial \hat{\phi}/\partial x = f'$ and $= f''$

The prime denotes differentiation of f with respect to its argument, '$x - \lambda y$'. Also,

$\partial \hat{\phi}/\partial y = f'(x - \lambda y)[\partial(x - \lambda y)/\partial y] \Rightarrow \partial \hat{\phi}/\partial y = f'(-\lambda) = -\lambda f'$ and $\partial^2 \hat{\phi}/\partial y^2 = \lambda^2 f''$

$\therefore$ for $\lambda^2 \partial^2 \hat{\phi}/\partial x^2 - \partial^2 \hat{\phi}/\partial y^2 = 0 \Rightarrow \lambda^2 f'' - \lambda^2 f'' = 0$

f can be any function of '$x - \lambda y$'. But $\hat{\phi}$ is constant along lines of $x - \lambda y$ = constant, since for any

function of $f(A)$, if A is constant, $f(A)$ must be constant! The slope of these lines is obtained from $x - \lambda y$ = constant $\therefore dy/dx = 1/\lambda = 1/(M_\infty^2 - 1)^{1/2}$ and $\tan\mu = 1/(M_\infty^2 - 1)^{1/2}$

where μ is the Mach angle. Thus, a line along which $\hat{\phi}$ is constant is a Mach line.

Mach line:

Consider a beeper, the point of contact with the circles of their common tangent is the location of the source. The disturbance at this point tends to build up into a much stronger disturbance than the one being created by the source; but since the latter is infinitesimal, the disturbance remains vanishingly weak. There is still no change in flow properties across this common tangent, which, however, divides the region which is affected by the disturbance from that which is not. This line is known as a normal Mach line.

In Fig. 12.1:

1. All disturbances created at the wall propagate unchanged away from the wall along Mach waves.
2. All the Mach wave have the same slope, $dy/dx = 1/(M_\infty^2 - 1)^{1/2}$
3. Mach waves slope downstream above the wall. Hence any disturbance at the wall cannot propagate upstream. Disturbances propagate everywhere throughout a subsonic flow, whereas they cannot propagate upstream in a steady supersonic flow.

All the results mentioned till now pertain to linearized supersonic flow because we start the derivation from (12.1). In real case, a shock wave and an expansion wave will be induced by the forward and the rearward parts of the hump, respectively. These are waves of finite strength and are not a part of linearized theory. Linearized theory is approximate; one of the consequences of this approximation is that waves of finite strength are not admitted.

Expression for C_p:

$\hat{u} = \partial\hat{\phi}/\partial x = f', \hat{v} = \partial\hat{\phi}/\partial y = -\lambda f' \Rightarrow \hat{u} = -\hat{v}/\lambda$

From the linearized B.C.: $\hat{v} = \partial\hat{\phi}/\partial y = V_\infty \tan\theta \approx V_\infty\theta \quad \therefore \hat{u} = -V_\infty\theta/\lambda$

Since $C_p = -2\hat{u}/V_\infty \Rightarrow C_p = -(-2V_\infty\theta/\lambda)/V_\infty = 2\theta/\lambda = 2\theta/(M_\infty^2 - 1)^{1/2}$

It is the linearized supersonic C_p, where C_p is directly proportional to the local surface inclination with respect to the freestream. According to θ, C_p is positive on the forward potion of the hump, and negative on the rear portion (pressure increase before the hump, decrease after that). ∴ The pressure is higher on the front section of the hump, and lower on the rear section. As a result, a drag force exists on the hump. This drag is called wave drag and is a characteristic of supersonic flows — without viscous effect.

Also, since $C_p \propto (M_\infty^2 - 1)^{1/2}$, thus C_p decreases as M_∞ increases.

Application to Supersonic Airfoils

Fig. 12.3

When the surface is inclined into the freestream direction, linearized theory predicts a positive C_p.

$C_{p,A} = 2\theta_A/(M_\infty^2 - 1)^{1/2}$, $C_{p,B} = 2\theta_B/(M_\infty^2 - 1)^{1/2}$

When the surface is inclined away from the freestream direction, linearized theory predicts a negative C_p.

$C_{p,C} = -2\theta_C/(M_\infty^2 - 1)^{1/2}$, $C_{p,D} = -2\theta_D/(M_\infty^2 - 1)^{1/2}$

For a flat plate at a small AOA α:

$C_{p,l} = 2\alpha/(M_\infty^2 - 1)^{1/2}$, $C_{p,u} = -2\alpha/(M_\infty^2 - 1)^{1/2}$

The normal force coefficient for the flat plate is $C_N = (1/c)\int_0^c (C_{p,l} - C_{p,u})dx$

Thus, $C_N = (4\alpha/(M_\infty^2 - 1)^{1/2})(1/c)\int_0^c dx = 4\alpha/(M_\infty^2 - 1)^{1/2}$

The axial force coefficient is $C_A = (1/c)\int_{LE}^{TE} (C_{p,l} - C_{p,u})dy$

Since the flat plate has zero thickness, hence $dy = 0$ and thus $C_A = 0$

From $C_l = C_N\cos\alpha - C_A\sin\alpha$ and $C_d = C_N\sin\alpha + C_A\cos\alpha$

For small α and $C_A = 0 \Rightarrow C_l = C_N$, $C_d = C_N\alpha$

∴ $C_l = 4\alpha/(M_\infty^2 - 1)^{1/2}$, $C_d = 4\alpha^2/(M_\infty^2 - 1)^{1/2}$ [Valid for small α]

For an airfoil:

$$\theta_u = (dz_u/dx - \alpha)$$
$$\theta_l = (-dz_u/dx + \alpha)$$

The lift acting on the chordwise segment is:

$$dl = p_l dS_l \cos\theta_l - p_u dS_u \cos\theta_u$$

assume thin airfoil $\rightarrow dl \approx (p_l - p_u)dx, \therefore dC_l \approx (Cp_l - Cp_u)d(x/c)$

for $C_p = 2\theta/(M_\infty^2 - 1)^{1/2} \Rightarrow dC_l = (2/(M_\infty^2 - 1)^{1/2})(2\alpha - dZ_l/dx - dZ_u/dx)d(x/c)$

and $Z_u = Z_l = 0$ @ both the LE and the TE:

$\int_0^1 (dZ_l/dx)d(x/c) = 0$ and $\int_0^1 (dZ_u/dx)d(x/c) = 0$

Thus, $C_l = 4\alpha/(M_\infty^2 - 1)^{1/2}$ note the AOA for zero lift is zero.

$dC_l/d\alpha = 4/(M_\infty^2 - 1)^{1/2}$ is a function of M_∞ only.

Drag:

$$\text{d(drag)} = p_l dS_l \sin\theta_l - p_u dS_u \sin\theta_u \approx p_l \theta_l dx + p_u \theta_u dx$$

Integrate $\Rightarrow c_d = D/q_\infty c = 4\alpha^2/(M_\infty^2 - 1)^{1/2} + (2/(M_\infty^2 - 1)^{1/2})(\underline{\sigma_u}^2 + \underline{\sigma_l}^2)$

= drag due to lift + drag due to thickness

Note: the drag is not zero even though the airfoil is of infinite span and viscous forces have been neglected. This is the wave drag.