Vector Algebra

General Inner Product Space

Let $\mathcal{X}$ be a vector space over a field $F$ (here our vector space $\mathcal{X}$ denotes $\mathbb{R}^n$ or $\mathbb{C}^n$ and $F$ denotes either real field $\mathbb{R}$ or complex field $\mathbb{C}$ throughout this course).

**Definition 1** A semi-inner product is a binary operation $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to F$ such that for all $\alpha, \beta \in F$ and $x, y, z \in \mathcal{X}$, the followings are satisfied:

1. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$
2. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$
3. $\langle x, x \rangle \geq 0$
4. $\langle x, y \rangle = \langle y, x \rangle$

An inner product on $\mathcal{X}$ is a semi-inner product that also satisfies

5. If $\langle x, x \rangle = 0$, then $x = 0$.

**Theorem 1 (Cauchy-Schwarz Inequality)** If $\langle \cdot, \cdot \rangle$ is a semi-inner product on $\mathcal{X}$, then

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

for all $x, y \in \mathcal{X}$.

Moreover, the equality occurs iff $\exists \alpha, \beta \in F$, both not 0, such that $\langle \alpha x + \beta y, \alpha x + \beta y \rangle = 0$.

**Corollary 1** If $\langle \cdot, \cdot \rangle$ is a semi-inner product on $\mathcal{X}$ and $\|x\| \overset{\text{def}}{=} \langle x, x \rangle^{\frac{1}{2}}$ for all $x \in \mathcal{X}$, then

$$\|x + y\| \leq \|x\| + \|y\|$$

for all $x, y \in \mathcal{X}$ (Triangle Inequality),

$$\|\alpha x\| = |\alpha|\|x\|$$

for $\alpha \in F$ and $x \in \mathcal{X}$.

If $\langle \cdot, \cdot \rangle$ is an inner product, then, $\|x\| = 0$ implies $x = 0$.

The quantity $\|x\| \overset{\text{def}}{=} \langle x, x \rangle^{\frac{1}{2}}$ for an inner product is called the norm of $x$, said it’s the norm induced by the inner product.

By definitions of $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, 

$$\langle x + y, x + y \rangle = \langle x, x \rangle + 2 \text{Re}(\langle x, y \rangle) + \langle y, y \rangle,$$

i.e. 

$$\|x + y\|^2 = \|x\|^2 + 2 \text{Re}(\langle x, y \rangle) + \|y\|^2.$$

Abstract vector algebra on Hilbert spaces

**Exercise 1** Look up metric space and complete metric space.

**Remark 1** A Hilbert space is a vector space $\mathcal{H}$ over $F$ together with an inner product $\langle \cdot, \cdot \rangle$ such that relative to the metric $d(x, y) \overset{\text{def}}{=} \|x - y\|$ induced by the norm, $\mathcal{H}$ is a complete metric space. (for the continuity issue).

**Definition 2 (Orthgonality)** If $\mathcal{H}$ is a Hilbert space and $x, y \in \mathcal{H}$, then $x$ and $y$ are orthogonal (perpendicular) to each other if $\langle x, y \rangle = 0$, in symbol, $x \perp y$. If $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{H}$, then $\mathcal{X} \perp \mathcal{Y}$ provided that $x \perp y$ for every $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.
Theorem 2 (The Pythagorean Theorem) If \( x_1, \ldots, x_n \) are orthogonal to one another in \( H \), then
\[
\| x_1 + \cdots + x_n \|^2 = \| x_1 \|^2 + \cdots + \| x_n \|^2.
\]

Theorem 3 (Parallelogram Law) If \( H \) is a Hilbert space and \( x, y \in H \), then
\[
\| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + \| y \|^2).
\]

Theorem 4 If \( M \subseteq H \) is a closed linear subspace and \( h \in H \), let \( P_h \in M \) be the unique point such that \( h - P_h \perp M \). Then
1. \( P \) is a linear transformation on \( H \),
2. \( \| Ph \| \leq \| h \| \) for every \( h \in H \),
3. \( P^2 = P \),
4. \( \ker P = M^\perp \) and \( \ran P = M \).

Such \( P \) is called the orthogonal projection of \( H \) onto subspace \( M \).

Exercise 2 Prove the Cauchy-Schwarz inequality.

Exercise 3 Prove the triangle inequality.

Exercise 4 Prove the Parallelogram Law.

A Hilbert Space \( (\mathbb{R}^n, \langle \cdot, \cdot \rangle) \)

We will focus on \( \mathbb{R}^n \), specially \( \mathbb{R}^3 \) from now on.

It can be shown that \( \mathbb{R}^n \) together with the inner product defined this way
\[
\langle x, y \rangle \overset{\text{def}}{=} \sum_{i=1}^n x_i y_i \text{ for any } x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \text{ in } \mathbb{R}^n
\]
is a Hilbert space.

The projection of vector \( x \) onto vector \( y \) is a vector denoted by \( \proj_y x \overset{\text{def}}{=} \frac{x}{\| y \|} \langle y, \frac{y}{\| y \|} \rangle \frac{y}{\| y \|^2} = \frac{\langle x, y \rangle}{\| y \|^2} \).

Note that usually \( \langle x, y \rangle \) is not equal to the magnitude of the projection of one onto the other.

Let \( \theta \) be the angle between vectors \( x \) and \( y \). By the law of cosine, \( \cos \theta = \frac{\langle x, y \rangle}{\| x \| \| y \|} \).

Exercise 5 State the law of cosine and prove it.

Definition 3 Cross Product (in \( \mathbb{R}^3 \)) is a binary operation between two vectors:
\[
x \times y \overset{\text{def}}{=} \| x \| \| y \| \sin \theta z \in \mathbb{R}^3
\]
where \( z \) is a unit vector in the direction of a right-hand screw as \( x \) rotating toward \( y \) through angle \( \theta \).

The alternative definition of cross product is
\[
x \times y \overset{\text{def}}{=} \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3
\end{vmatrix}
\]
Exercise 6

1. Show that the two definitions of cross product are equivalent.

2. Let \( x, y, z \in \mathbb{R}^3 \), \( \lambda \in \mathbb{R} \). Show that cross product has the following properties:
   (a) \( x \times y = -y \times x \) (skew-symmetry)
   (b) \( x \times (y + z) = x \times y + x \times z \) (distributive law)
   (c) \( \lambda(x \times y) = x \times (\lambda y) = (\lambda x) \times y \)
   (d) \( x \times x = 0 \)

3. Prove the Lagrange’s identity: \( \|x \times y\|^2 = \|x\|^2 \|y\|^2 - (x \cdot y)^2 \).

4. State the law of sines and prove it.

Notice that cross product does not have associativity, i.e. \( x \times (y \times z) \neq (x \times y) \times z \)

Exercise 7

Namely, \( x \times y \times z \) means \( x \times (y \times z) \) and \( x \times y \cdot z \) means \( (x \times y) \cdot z \).

1. Show that \( x \times y \times z = x \times y \cdot z \). This sometimes is called scalar triple product.

2. Let \([xyz] \equiv x \cdot y \times z\). Show that \([xyz] = \det(x, y, z)\) and \([xyz] = [yiz] = [zxy] = -[xyz] = [yzx] = [yxz] \). Geometrically, \( x, y, z \) are co-planar iff \([xyz] = 0\).

3. Show that vector triple products \( x \times (y \times z) = (x \cdot z)y - (x \cdot y)z \), and \( (x \times y) \times z = (x \cdot z)y - (y \cdot z)x \).

4. Show that \( (u \times v) \times (x \times y) = [uv]x - [uv]y = [xy]u - [xy]v \). and implies that any vector can be expressed as a linear combination of any non-co-planar vectors.

5. Show the extended Lagrange identity: \( (u \times v) \cdot (x \times y) = (u \cdot x)(v \cdot y) - (v \cdot x)(u \cdot y) \).

6. Show the Jacobi identity: \( x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0 \).

7. Show that \( (x \times y) \cdot (y \times z) \times (z \times x) = [xyz]^2 \).

Definition 4 (Orthonormal Set) Let \( X = \{x_1, \cdots, x_k\} \subset \mathbb{R}^n \). \( X \) is called orthonormal if \( x_i \cdot x_j = \delta_{ij} \) for any \( i, j = 1, \cdots, k \). If \( k = \#X = n \), then \( X \) is called an orthonormal basis of \( \mathbb{R}^n \).

Definition 5 (Reciprocal Sets of Vectors) Let \( X = \{x_1, \cdots, x_k\}, Y = \{y_1, \cdots, y_k\} \subset \mathbb{R}^n \). \( X \) and \( Y \) are said reciprocal to each other if \( x_i \cdot y_j = \delta_{ij} \) for any \( i, j = 1, \cdots, k \).

Exercise 8

Show that if \( X = \{x_1, x_2, x_3\} \) and \( Y = \{y_1, y_2, y_3\} \) are reciprocal sets in \( \mathbb{R}^3 \), then

1. \( [x_1 x_2 x_3] \neq 0 \) and \( [y_1 y_2 y_3] \neq 0 \).
2. \( x_1 = \frac{y_3 \times y_1}{[x_1 x_2 x_3]}, x_2 = \frac{y_3 \times y_2}{[x_1 x_2 x_3]}, x_3 = \frac{y_3 \times y_3}{[x_1 x_2 x_3]} \), and
   \( y_1 = \frac{y_1 \times x_3}{[x_1 x_2 x_3]}, y_2 = \frac{y_1 \times x_2}{[x_1 x_2 x_3]}, y_3 = \frac{y_1 \times x_1}{[x_1 x_2 x_3]} \).