Barenberg

Medium Thick Plate Theory

Assumptions

1. Only Forces on the Surfaces of the Slab are normal to the surface
2. Slab is of Uniform thickness
3. No in-plane forces
4. X-Y Plane is at the mid-depth of the slab
5. Deformation within the elements normal to the slab surfaces can be ignored (i.e., $\epsilon_z \Rightarrow 0$)
6. Shear deformations are small and can be ignored. (See Compatability Conditions)

Ref: Theory of Plates and Shells, Timoshenko, S and Woinowsky-Krieger, Art 21, P79-83

Theory of Elasticity, Timoshenko, S Chapter 1.

CE 497

Pavement Analysis

The following relationships from elastic theory are needed to develop the basic equations for the elastic slab theory

1. (a) $\frac{\partial u}{\partial x} = \epsilon_x$; (b) $\frac{\partial v}{\partial y} = \epsilon_y$; (C) $\frac{\partial w}{\partial z} = \epsilon_z$

   (d) $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$; (e) $\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$; (f) $\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$

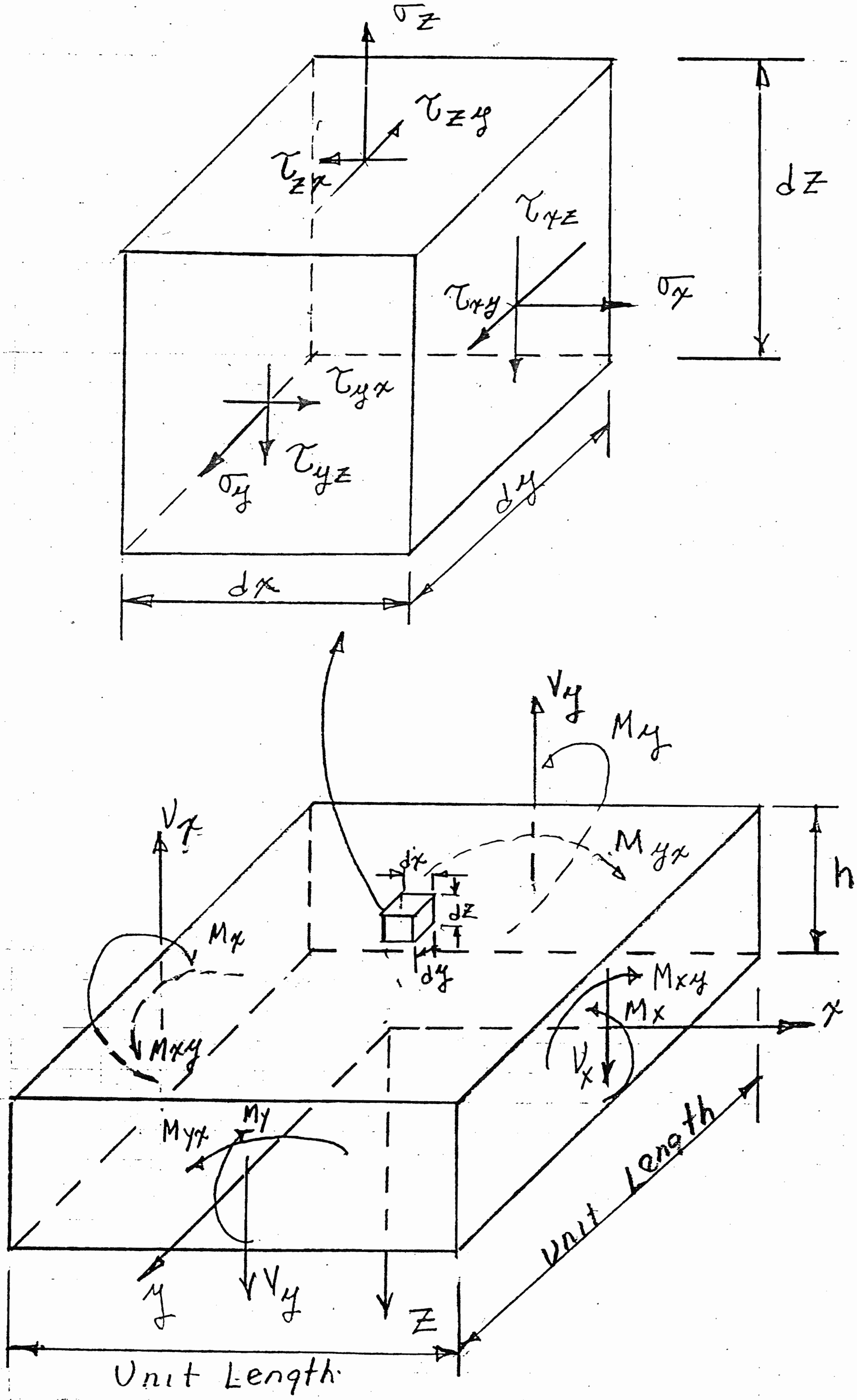
From Hooke's law we can also write

(g) $\epsilon_x = \frac{1}{E}[\sigma_x - \mu(\sigma_y + \sigma_z)]$ (j) $\gamma_{xy} = \frac{\tau_{xy}}{G}$

(h) $\epsilon_y = \frac{1}{E}[\sigma_y - \mu(\sigma_x + \sigma_z)]$ (k) $\gamma_{xz} = \frac{\tau_{xz}}{G}$

(i) $\epsilon_z = \frac{1}{E}[\sigma_z - \mu(\sigma_x + \sigma_y)]$ (l) $\gamma_{yz} = \frac{\tau_{yz}}{G}$

2. $G = \frac{E}{2(1+\mu)}$


σz
τzy
τzx
dZ
τxz
σx
τxy
τyx
τyz
σy
dy
dx
Vy
My
Vx
Myx
dx
dZ
h
Mx
dy
Mxy
Mx
x
Mxy
Vx
Myx
My
Unit Length
y
Vy
Z
Unit Length

a) $0 = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_x \, dz = N_x =$ Normal Force Per Unit Width

b) $0 = \int_{-h/2}^{+h/2} \sigma_y \, dz = N_y =$ Normal Force Per Unit Width

c) $0 = \int_{-h/2}^{+h/2} \tau_{xy} \, dz = N_{xy}$

d) $\int_{-h/2}^{h/2} \sigma_x \, z dz = M_x =$ Bending Moment/unit Length

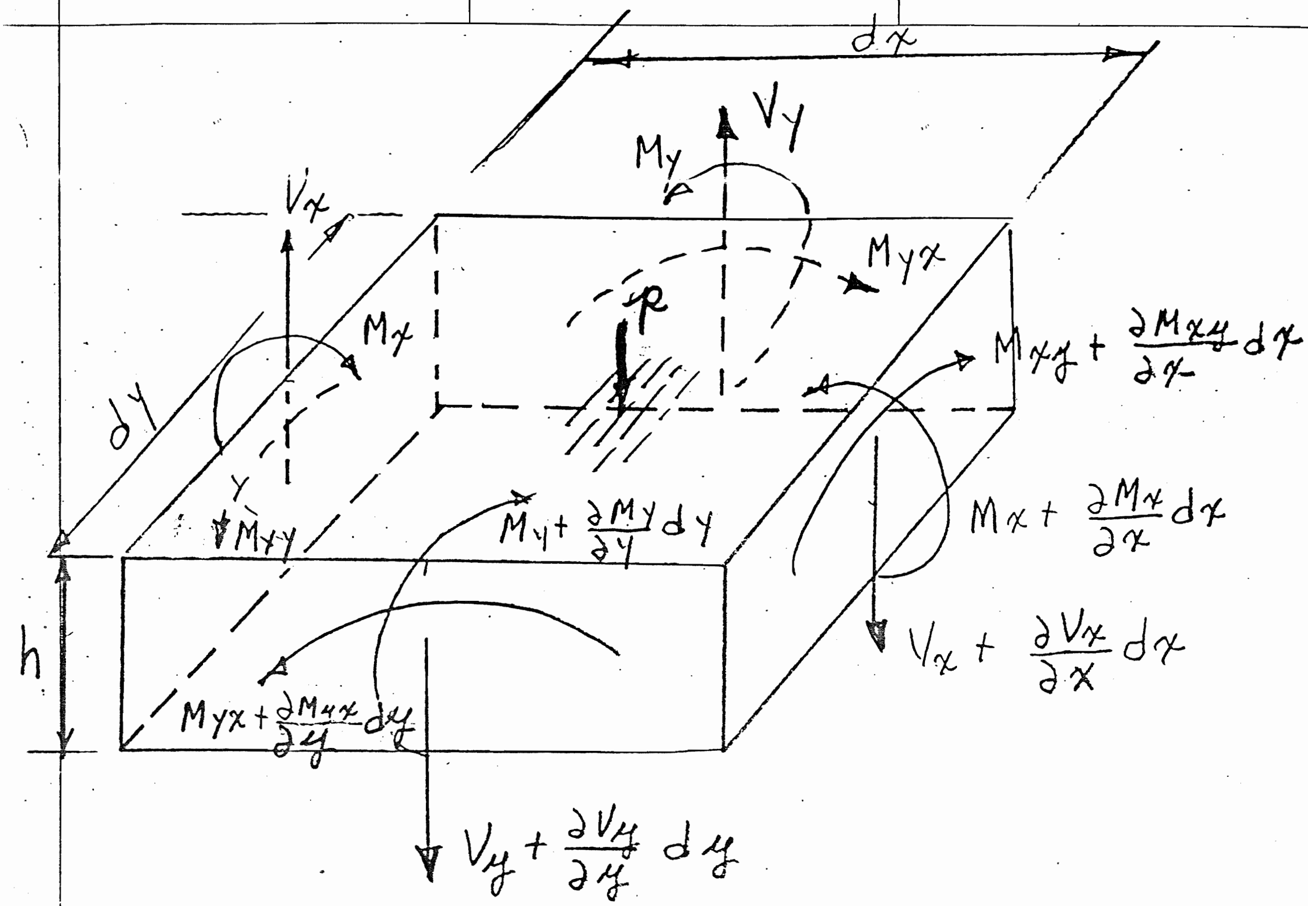
(M_x is Positive when Comp on Top of Slab)

e) $\int_{-h/2}^{h/2} \sigma_y \, z dz = M_y$

f) $\int_{-h/2}^{h/2} \tau_{xy} \, z dz = M_{xy} =$ Twisting moment

g) $\int_{-\frac{h}{2}}^{h/2} \tau_{xz} \, dz = V_x \neq 0$ Shearing force per unit length.

g_1) $\int_{-h/2}^{h/2} \tau_{yz} \, dz = V_y \neq 0$



Equilibrium Equations

$$\sum F_v = 0 \quad \downarrow +$$

$$p\,dx\,dy - \cancel{V_x dy} + \left[\cancel{V_x} + \frac{\partial V_x}{\partial x} dx\right] dy - \cancel{V_y dx}$$

$$+ \left[\cancel{V_y} + \frac{\partial V_y}{\partial y} dy\right] dx = 0$$

$$p\,dx\,dy + \frac{\partial V_x}{\partial x} dx\,dy + \frac{\partial V_y}{\partial y} dx\,dy = 0$$

or

$$\boxed{\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = -p} \qquad (1)$$

$$\Sigma M_y = 0 \quad \overset{+}{\curvearrowright}$$

$$+ \cancel{M_x} dy - \left[\cancel{M_x} + \frac{\partial M_x}{\partial x} dx\right] dy + \cancel{M_{xy}} dx$$

$$- \left[\cancel{M_{xy}} + \frac{\partial M_{xy}}{\partial y} dy\right] dx$$

$$+ \left[V_x + \frac{\partial V_x}{\partial x} dx\right] dy \frac{dx}{2} + V_x dy \frac{dx}{2} = 0$$

$$- \frac{\partial M_x}{\partial x} dx\,dy - \frac{\partial M_{xy}}{\partial y} dx\,dy + \underbrace{\frac{\partial V_x}{\partial x} dx\,dy \frac{dx}{2}}_{\text{Second order Term} \to 0} = 0$$

$$+ V_x\, dx\,dy = 0$$

$$\boxed{\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = V_x} \qquad (2)$$

By $\Sigma M_x = 0$

$$\boxed{\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = V_y} \qquad (3)$$

Substituting Eq 2+3 into Eq 1 gives

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} = -p$$

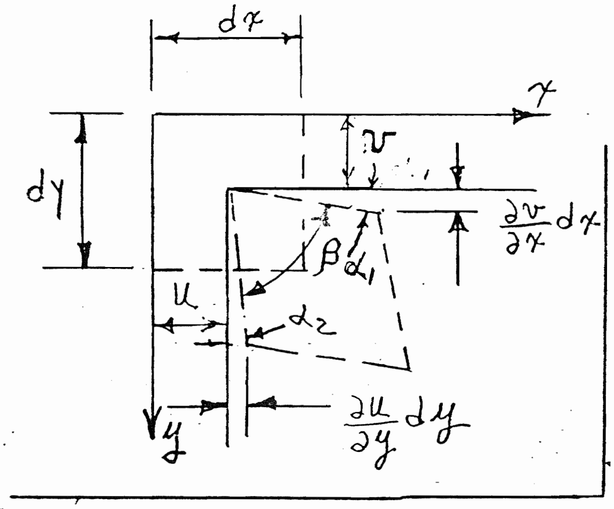
$$\boxed{\frac{\partial^2 M_x}{\partial x^2} + \frac{2\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p} \qquad (4)$$

For Beams all $\frac{\partial}{\partial y} \equiv 0$

To establish the compatability relationships in terms of strains and deformations we need to establish some basic stress strain relationships.

Displacements in the x, y and z directions are designated by the symbols u, v and w respectively. Also, strains in the x, y and directions are designated by the symbols ϵ_x, ϵ_y and ϵ_z. By definition the following relationships hold

$$\epsilon_x = \frac{\partial u}{\partial x} \; ; \quad \epsilon_y = \frac{\partial v}{\partial y} \; ; \quad \epsilon_z = \frac{\partial w}{\partial z}$$



Shear strain γ_{ij} is defined as the angle change or distortion occuring within an element. Thus in the figure shown

$$\gamma_{xy} = 90 - \beta = \alpha_1 + \alpha_2$$

$$\gamma_{xy} = \alpha_1 + \alpha_2 = \frac{\frac{\partial v}{\partial x} dx}{dx} + \frac{\frac{\partial u}{\partial y} dy}{dy} \equiv \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Similarly

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

The relationship between stress and strain for an "elastic" material is given by Hooke's Law as follows

$$\left.\begin{aligned}\epsilon_x &= \frac{1}{E}\left[\sigma_x - \mu(\sigma_y+\sigma_z)\right]\\ \epsilon_y &= \frac{1}{E}\left[\sigma_y - \mu(\sigma_x+\sigma_z)\right]\\ \epsilon_z &= \frac{1}{E}\left[\sigma_z - \mu(\sigma_x+\sigma_y)\right]\end{aligned}\right\}$$

where E is the modulus of elasticity of the material and μ is Poisson's ratio.

Also

$$\gamma_{xy} = \frac{\tau_{xy}}{G}; \quad \gamma_{xz} = \frac{\tau_{xz}}{G}; \quad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

where

$$G = \frac{E}{2(1+\mu)}$$

The equations above can be rearranged as follows:

$$\left.\begin{aligned}\sigma_x &= \lambda e + 2G\epsilon_x\\ \sigma_y &= \lambda e + 2G\epsilon_y\\ \sigma_z &= \lambda e + 2G\epsilon_z\end{aligned}\right\} \quad h.$$

where

$$e = (\epsilon_x + \epsilon_y + \epsilon_z); \quad \lambda = \frac{\mu E}{(1+\mu)(1-2\mu)}$$

If ϵ_z is small and can be ignored then the above equations(h) can be simplified to

$$\sigma_x = \frac{E}{1-\mu^2}\left[\epsilon_x + \mu\epsilon_y\right]$$

$$\sigma_y = \frac{E}{1-\mu^2}\left[\epsilon_y + \mu\epsilon_x\right]$$

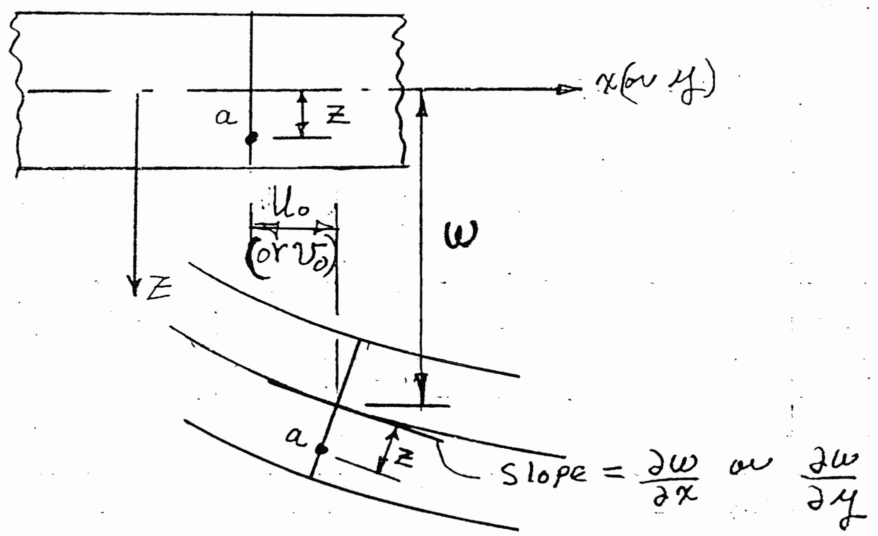
Substituting the basic relationships of $\epsilon_x = \frac{\partial u}{\partial x}$ and $\epsilon_y = \frac{\partial v}{\partial y}$ gives the stresses in terms of strain as

$$\sigma_x = \frac{E}{1-\mu^2}\left[\frac{\partial u}{\partial x} + \mu\frac{\partial v}{\partial y}\right] \qquad (i)$$

$$\sigma_y = \frac{E}{1-\mu^2}\left[\frac{\partial v}{\partial y} + \mu\frac{\partial u}{\partial x}\right] \qquad (j)$$

$$\tau_{xy} = \frac{E}{2(1+\mu)}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] \qquad (k)$$

Relationships between $\frac{\partial u}{\partial x}$, $\frac{\partial w}{\partial x \text{ or } \partial y}$ etc are called compatability conditions and are determined from the figure below provided lines ⊥ to planes before bending remain ⊥ during bending.



Displacement of point (a) in "x" direction is given by

$$u = u_o - z\frac{\partial w}{\partial x}$$

$$\therefore \frac{\partial u}{\partial x} = 0 - z\frac{\partial^2 w}{\partial x^2} \quad \text{or, Similarly, } \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}$$

$$\text{Also} \quad \frac{\partial u}{\partial y} = -z\frac{\partial^2 w}{\partial x \partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -z\frac{\partial^2 w}{\partial x \partial y}$$

Substituting the compatability relationships into equations i, j, k gives

$$\sigma_x = \frac{E}{1-\mu^2}\left[-z\frac{\partial^2 w}{\partial x^2} - \mu z\frac{\partial^2 w}{\partial y^2}\right] = -\frac{Ez}{1-\mu^2}\left[\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right] \quad (l)$$

$$\sigma_y = \frac{E}{1-\mu^2}\left[\frac{-z\partial^2 w}{\partial y^2} - \mu z\frac{\partial^2 w}{\partial x^2}\right] = \frac{-Ez}{1-\mu^2}\left[\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right] \quad (m)$$

$$\tau_{xy} = \frac{E}{2(1+\mu)}\left[\frac{-z\partial^2 w}{\partial x\partial y} - \frac{z\,\partial^2 w}{\partial x\partial y}\right] = \frac{-Ez}{1+\mu}\left[\frac{\partial^2 w}{\partial x\partial y}\right] \quad (n)$$

According to equation (d)

$$M_x = \int_{-h/2}^{+h/2} \sigma_x\, z\, dz \quad (d)$$

substituting eq. l for σ_z in eq(d) gives

$$M_x = \frac{-E}{1-\mu^2}\int_{-h/2}^{h/2}\left[\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right] z^2 dz = \frac{-E[\;\;]}{1-\mu^2}\left[\frac{z^3}{3}\right]_{-h/2}^{h/2}$$

$$M_x = \frac{-E}{1-\mu^2}[\;\;]\frac{\left[\frac{h^3}{8} + \frac{h^3}{8}\right]}{3} = \frac{-Eh^3}{12(1-\mu^2)}\left[\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right]$$

Let $D = \dfrac{Eh^3}{12(1-\mu^2)}$, Then

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right] \quad (o)$$

Similarly

$$M_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right] \quad (p)$$

$$M_{xy} = \frac{-E}{12(1+\mu)}\cdot\frac{(1-\mu)}{(1-\mu)}\, h^3\frac{\partial^2 w}{\partial x\partial y} = -D(1-\mu)\frac{\partial^2 w}{\partial x\partial y} \quad (q)$$

Also, after differentiation as indicated

$$\frac{\partial^2 M_x}{\partial x} = -D\left[\frac{\partial^4 w}{\partial x^4} + \mu \frac{\partial^4 w}{\partial x^2 \partial y^2}\right] \qquad (r)$$

$$\frac{\partial^2 M_y}{\partial y^2} = -D\left[\frac{\partial^4 w}{\partial y^4} + \mu \frac{\partial^4 w}{\partial x^2 \partial y^2}\right] \qquad (s)$$

$$\frac{\partial^2 M_{xy}}{\partial x \partial y} = -D(1-\mu)\frac{\partial^4 w}{\partial x^2 \partial y^2} \qquad (t)$$

Substituting equations r, s and t into equation (4) yields

$$-D\left[\frac{\partial^4 w}{\partial x^4} + \overset{\text{cancels}}{\mu \frac{\partial^4 w}{\partial x^2 \partial y^2}} + \overset{\text{cancels}}{2(1-\mu)\frac{\partial^4 w}{\partial x^2 \partial y^2}} + \frac{\partial^4 w}{\partial y^4} + \overset{\text{cancels}}{\mu \frac{\partial^4 w}{\partial x^2 \partial y^2}}\right] = -p$$

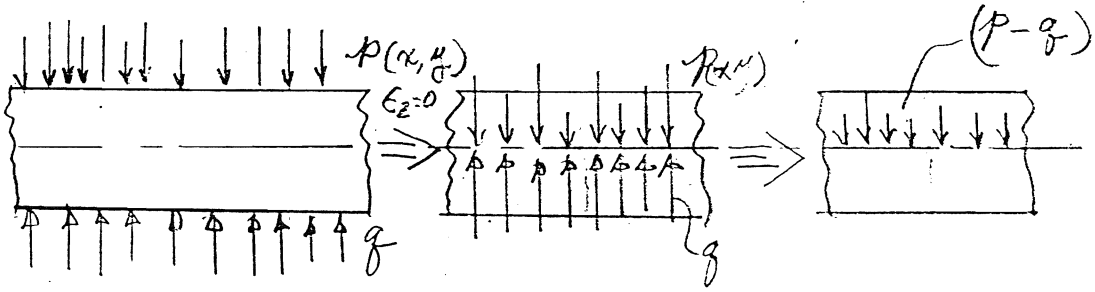
The terms with μ cancel leaving

$$\boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}} \quad - (5)$$

Equation 5 is LaGrange Equation for Elastic Slabs with unspecfied boundary and Support Conditions. This equation is the basis for both the Westergaard equation of a slab on a dense liquid (Winkler) subgrade and the slab on an elastic solid subgrade.

Ref <u>Theory of Plates and Shells</u>, Timoshenko, S and - Art 57, 58, 59, Chapter 8

In the development of equation 5 two important assumptions were made with respect to its application to pavement slabs. It was assumed that there was no in-plane forces, and that all loads are normal to the slab surface. It follows, therefore, that there can be no shear stress at the bottom of the slab, i.e., between the slab and the supporting subgrade. Also, since $\epsilon_z \Rightarrow 0$, it can be assumed that all forces normal to the slab surfaces can be assumed to be acting at the midplane of the slab as shown in the sketch.



Thus Eq 5 can be written as

$$\frac{\partial^4 w}{\partial x^2} + \frac{2\,\partial^4 w}{\partial^2 x\,\partial^2 y} + \frac{\partial^4 w}{\partial y^4} = \frac{p-q}{D} = \frac{p}{D} - \frac{q}{D}$$

The problem now is to express "q" as a function of the deflection "w"

Westergaard assumed that at any point under the slab, the value for "q" is directly proportional to the deflection "w" at that point. Thus, if "k" is the proportionality constant, then $q = kw$ where "k" is the modulus of subgrade reaction. Also, since "q" is a function of "w", it should be placed on the L.H side of eq 5 so the equation becomes

$$\frac{\partial^4 w}{\partial x^4} + \frac{2\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{kw}{D} = \frac{P}{D} \qquad [6]$$

To solve this eq, the coefficient for w should be of the same order as the differentials.

Substuting $l^4 = \frac{D}{k}$ into Equation 6 gives

$$\frac{\partial^4 w}{\partial x^4} + \frac{2\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{w}{l^4} = \frac{P}{k l^4} \qquad (7)$$

or

$$l^4\left[\frac{\partial^4 w}{\partial x^4} + \frac{2\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] + w = \frac{P}{k} \qquad (7a)$$

see scale Fig 5.96 p150

Equation 7 or 7a is the basic equation for solution of slabs resting on a dense liquid (Winkler) subgrade.

In Equations 7 and 7a

k represents the subgrade strength and is referred to as "the Modulus of Subgrade Reaction". It has the dimension of $\frac{\#/in^2}{in}$ or pounds per cubic inch, hence the term "Dense Liquid" applied to the subgrade.

$$l^4 = \frac{D}{k} \quad \text{or} \quad l = \sqrt[4]{\frac{Eh^3}{12(1-\mu^2)k}}$$

The dimensions for l is one of length and in English units is almost always given in inches (cm or m in SI units). l is referred to in the literature as "the radius of relative stiffness" of the slab.

Solutions to Equation 7 or 7a depend upon the location of the load and boundary conditions. (Mechanics of the solutions will not be discussed.)

If the slab is assumed to be infinitely large, and if the applied load is a concentrated force, then the basic solution to Equation 7 is

Set $P = (kw = F)$ over total area

$$w = -\frac{P\,l^2}{2\pi D}\ \mathrm{kei}\ \bar{x} \qquad (8)$$

where $\bar{x} = \dfrac{r}{l}$

$\mathrm{kei}\ \bar{x}$ is a Bessel function for a complex argument. (Tabulated for specific values of $\bar{x}$).

For small values of $\bar{x}$

$$\mathrm{kei}(\bar{x}) = -(\bar{x}^2/4)\log\bar{x} - \pi/4 + (1+\log 2 - \gamma)\,\bar{x}^2/4 + \cdots$$

where $\gamma = 0.5772157\cdots$ (Euler's Constant), and

$\log 2 - \gamma = 0.11593\cdots$

Thus, for small $\bar{x}$

$$\mathrm{kei}\ \bar{x} = -\left(\frac{\bar{x}^2}{4}\right)\log\bar{x} - \frac{\pi}{4} + 1.116\left(\frac{\bar{x}^2}{4}\right) + \cdots \quad (u)$$

For large values of $\bar{x}$

$$\mathrm{kei}(\bar{x}) \approx -\frac{e^{-\sigma}}{\sqrt{\frac{2\bar{x}}{\pi}}}\sin\left(\sigma + \frac{\pi}{8}\right) \qquad (v)$$

where $\sigma = \dfrac{\bar{x}}{\sqrt{2}}$

Since the deflection is maximum at the point of contact, W_{max} at $\bar{x} = 0$.

From equation (u),

$$\mathrm{kei}(\bar{x}) = -\frac{\pi}{4} \quad \text{at } \bar{x} = 0$$

Thus, the maximum deflection at

an interior point of a large slab Under a Concentrated load is, by Equations 8 and u

$$W_o = -\frac{Pl^2}{2\pi D}\cdot\left(-\frac{\pi}{4}\right) = \frac{Pl^2}{8D} \qquad (w)$$

which can be rewritten as

$$w_o = \frac{Pl^2}{8D} = \frac{Pl^2}{8k\left(\frac{D}{k}\right)} = \frac{Pl^2}{8k\,l^4} = \frac{P}{8kl^2} \qquad (w)$$

Equations (O), (p) and (q) give the expression for moments in the slab in terms of the deflections "w". Differentiating the equations such as u an v and substituting into equations o, p and q give expressions for moments, which can be expressed in terms of stress by observing

$$\sigma_x = \frac{Ez}{1-\mu^2}\,\frac{M_x}{\frac{Eh^3}{12(1-\mu^2)}} = 12\frac{M_x Z}{h^3}$$

Maximum values for σ_x occur at the top and bottom faces of the slab where $z = \pm\frac{h}{2}$. substituting into the above equation gives

$$(\sigma_x)_{max} = \frac{6M_x}{h^2}$$

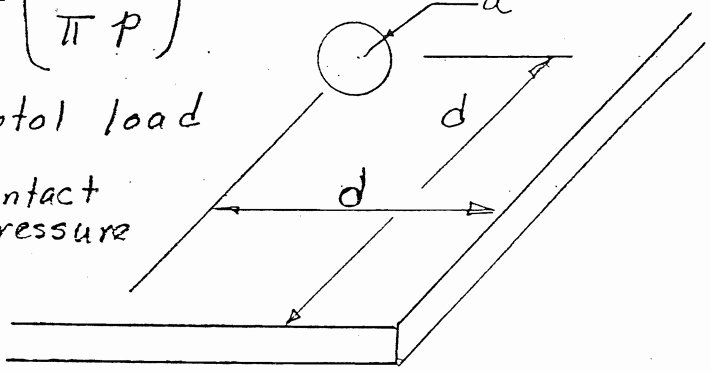
Substituting for $M_{x,y}$ the equations for stresses and deflections shown on pages 17 and 18 can be obtained for the following loading conditations and limitations

INTERIOR Loading

$$a = \left(\frac{P}{\pi p}\right)^{1/2}$$

P = Totol load

p = Contact Pressure

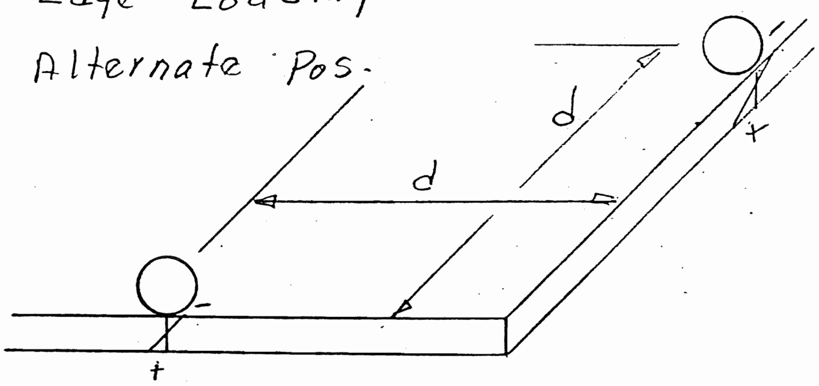


$d_{min} = 5\frac{1}{2}\,\ell$ desired

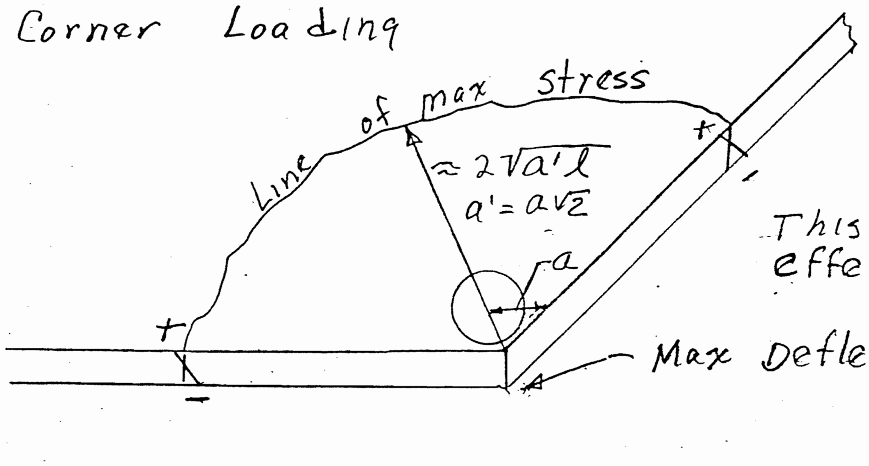
$d_{min} = 3\,\ell$ practical min

Edge Loading

Alternate Pos.



Corner Loading



This is a Cantilever effect

Max Deflection @ Corner

Original

SOLUTIONS FOR SLAB EQUATIONS

INTERIOR (SPECIAL THEORY)

$$\sigma_{max} = \frac{3P(1+\mu)}{2\pi h^2}\left[\ln\left(\frac{2\ell}{b}\right) + .5 - \gamma\right] + \frac{3P(1+\mu)}{64h^2}\left[\left(\frac{b}{\ell}\right)^2\right]$$

$$w_{max} = \frac{P}{8k\ell^2}\left[1 + \frac{1}{2\pi}\left\{\ln\left(\frac{a}{2\ell}\right) + \gamma - 1.25\right\}\left(\frac{a}{\ell}\right)^2\right]$$

EDGE ('NEW' FORMULA FOR CIRCLE, SIMPLIFIED BY LOSBERG)

See WESTER

$$\sigma_{max} = \frac{6P}{h^2}\left(1+0.5\mu\right)\left[0.489\log\left(\frac{a}{\ell}\right) - 0.012 - 0.063\left(\frac{a}{\ell}\right)\right]$$

$$w_{max} = \frac{1}{\sqrt{6}}\left(1+0.4\mu\right)\frac{P}{k\ell^2}\left[1 - 0.760\left(1+0.5\mu\right)\left(\frac{a}{\ell}\right)\right]$$

CORNER (WESTERGAARD, 1926)

$$\sigma_{max} = \frac{3P}{h^2}\left[1 - \left(\frac{a_1}{\ell}\right)^{0.6}\right] \quad \text{at } x_1 = 2\sqrt{a_1 \ell}$$

$$w_{max} = \frac{P}{k\ell^2}\left(1.1 - 0.88\left(\frac{a_1}{\ell}\right)\right)$$

ORDINARY THEORY: a = radius of loaded area

SPECIAL THEORY: $b = \sqrt{1.6a^2 + h^2} - 0.675h$ for $a < 1.724h$

$b = a$ for $a > 1.724h$

σ = bending stress (psi)

w = deflection (inches)

γ = Euler's Constant = 0.577215

μ = Poisson's Ratio for Concrete (0.15) Typ

$\ln$ = Natural log

$\log$ = log to base 10

P = Total load (pounds)

h = Slab Thickness (inches)

ℓ = Radius of Relative Stiffness

$$= \sqrt[4]{\frac{E h^3}{12(1-\mu) k}} \quad \text{for } \mu = .15 \quad \ell = \sqrt[4]{\frac{E h^3}{11.23 k}}$$

k = Modulus of Subgrade Reaction

$$= P/\omega = psi/in = pci$$

E = Modulus of Concrete

$$E = 57{,}000 \sqrt{f'_c} \approx 4{,}000{,}000 \text{ psi (Typ)}$$

Note: Max subgrade stress = $k \times \omega_{max}$

REFERENCES ON PLATE THEORY FOR PAVEMENTS

1. Hertz, H., "Über das Gleichgewict Schwimmender Elastischer Platten," Ann. d. Physik u Chemie, Vol. 22, 1884.

2. Westergaard, H. M., "Stresses in Concrete Pavements Computed by Theoretical Analysis," Public Roads, Vol. 7, No. 2, 1926-27.

3. Westergaard, H. M., "Mechanics of Progressive Cracking in Concrete Pavements," Public Roads, Vol. 10, No. 4, 1929-30.

4. Westergaard, H. M., "Stresses in Concrete Runways of Airports," Proceedings, Highway Research Board, Vol. 19, 1939.

5. Westergaard, H. M., "New Formulas for Stresses in Concrete Pavements of Airfields," Proceedings, American Society of Civil Engineers, Vol. 113, 1947.

6. Westergaard, H. M., "Stress Concentrations in Plates Loaded Over Small Areas," Transactions, American Society of Civil Engineers, Vol. 108, 1943.

7. Westergaard, H. M., "Analytical Tools for Judging Results of Structural Tests of Concrete Pavements," Public Roads, Vol. 14, No. 10, 1933-34.

8. Timoshinko, S., and Woinowsky-Krieger, S., "Theory of Plates and Shells," McGraw-Hill Book Co., Inc., Second Edition, New York, 1959.

9. Hogg, A. H. A., "Equilibrium of a Thin Plate Symmetrically Loaded, Resting on an Elastic Subgrade of Infinite Depth," Philosophical Mag., Series 7, Vol. 25, March 1938.

10. Holl, D. L., "Thin Plates on Elastic Foundations," Proceedings, 5th International Congress of Applied Mechanics, Cambridge, Mass., 1938.

11. Holl, D. L., "Equilibrium of a Thin Plate, Symmetrically Loaded on a Flexible Subgrade," Journal of Science, Iowa State College, Vol. 12, No. 4, July 1938.