CHAPTER 1

BENDING OF LONG RECTANGULAR PLATES TO A CYLINDRICAL SURFACE

1. Differential Equation for Cylindrical Bending of Plates. We shall begin the theory of bending of plates with the simple problem of the bending of a long rectangular plate that is subjected to a transverse load that does not vary along the length of the plate. The deflected surface of a portion of such a plate at a considerable distance from the ends[1] can be assumed cylindrical, with the axis of the cylinder parallel to the length of the plate. We can therefore restrict ourselves to the investigation of the bending of an elemental strip cut from the plate by two planes perpendicular to the length of the plate and a unit distance (say 1 in.) apart. The deflection of this strip is given by a differential equation which is similar to the deflection equation of a bent beam.

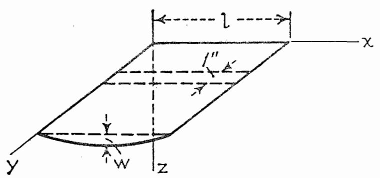

FIG. 1

To obtain the equation for the deflection, we consider a plate of uniform thickness, equal to h, and take the xy plane as the middle plane of the plate before loading, *i.e.*, as the plane midway between the faces of the plate. Let the y axis coincide with one of the longitudinal edges of the plate and let the positive direction of the z axis be downward, as shown in Fig. 1. Then if the width of the plate is denoted by l, the elemental strip may be considered as a bar of rectangular cross section which has a length of l and a depth of h. In calculating the bending stresses in such a bar we assume, as in the ordinary theory of beams, that cross sections of the bar remain plane during bending, so that they undergo only a rotation with respect to their neutral axes. If no normal forces are applied to the end sections of the bar, the neutral surface of the bar coincides with the middle surface of the plate, and the unit elongation of a fiber parallel to the x axis is proportional to its distance z

[1] The relation between the length and the width of a plate in order that the maximum stress may approximate that in an infinitely long plate is discussed later; see pp. 118 and 125.

from the middle surface. The curvature of the deflection curve can be taken equal to $-d^2w/dx^2$, where w, the deflection of the bar in the z direction, is assumed to be small compared with the length of the bar l. The unit elongation ϵ_x of a fiber at a distance z from the middle surface (Fig. 2) is then $-z\, d^2w/dx^2$.

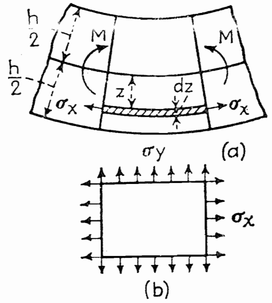

FIG. 2

Making use of Hooke's law, the unit elongations ϵ_x and ϵ_y in terms of the normal stresses σ_x and σ_y acting on the element shown shaded in Fig. 2*a* are

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E}$$
$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E} = 0 \qquad (1)$$

where E is the modulus of elasticity of the material and ν is Poisson's ratio. The lateral strain in the y direction must be zero in order to maintain continuity in the plate during bending, from which it follows by the second of the equations (1) that $\sigma_y = \nu\sigma_x$. Substituting this value in the first of the equations (1), we obtain

$$\epsilon_x = \frac{(1-\nu^2)\sigma_x}{E}$$

and

$$\sigma_x = \frac{E\epsilon_x}{1-\nu^2} = -\frac{Ez}{1-\nu^2}\frac{d^2w}{dx^2} \qquad (2)$$

If the plate is submitted to the action of tensile or compressive forces acting in the x direction and uniformly distributed along the longitudinal sides of the plate, the corresponding direct stress must be added to the stress (2) due to bending.

Having the expression for bending stress σ_x, we obtain by integration the bending moment in the elemental strip:

$$M = \int_{-h/2}^{h/2} \sigma_x z\, dz = -\int_{-h/2}^{h/2} \frac{Ez^2}{1-\nu^2}\frac{d^2w}{dx^2}\, dz = -\frac{Eh^3}{12(1-\nu^2)}\frac{d^2w}{dx^2}$$

Introducing the notation

$$\frac{Eh^3}{12(1-\nu^2)} = D \qquad (3)$$

we represent the equation for the deflection curve of the elemental strip in the following form:

$$D\frac{d^2w}{dx^2} = -M \qquad (4)$$

in which the quantity D, taking the place of the quantity EI in the case due to the pure bending by radial moments $P/4\pi$ applied along the boundary of the plate.

It should be noted also that, for small values of the ratio r/a, the effect of the shearing force $P/2\pi r$ upon the deflection is represented mainly by the second term on the right-hand side of Eq. (q). To this term corresponds a slope

$$\frac{dw_1}{dr} = -\frac{3}{2}\frac{1-\nu}{1+\nu}\frac{P}{2\pi rhG} \tag{t}$$

Comparing this result with the expression (i), we conclude that the factor

$$k = \frac{3}{2}\frac{1-\nu}{1+\nu} \tag{u}$$

if introduced into Eq. (i) instead of $k = \frac{3}{2}$, would give a more accurate value of the deformation due to shear in the case of a plate without a hole.

All preceding considerations are applicable only to circular plates bent to a surface of revolution. A more general theory of bending taking into account the effect of the shear forces on the deformation of the plate will be given in Arts. 26 and 39.

CHAPTER 4

SMALL DEFLECTIONS OF LATERALLY LOADED PLATES

21. The Differential Equation of the Deflection Surface. We assume that the load acting on a plate is normal to its surface and that the deflections are small in comparison with the thickness of the plate (see Art. 13). At the boundary we assume that the edges of the plate are free to move in the plane of the plate; thus the reactive forces at the edges are normal to the plate. With these assumptions we can neglect any strain in the middle plane of the plate during bending. Taking, as

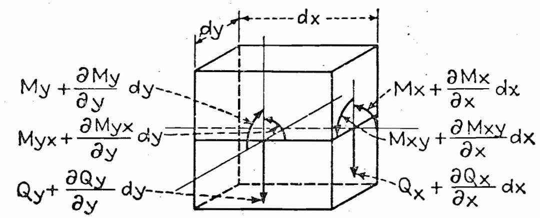

FIG. 47

before (see Art. 10), the coordinate axes x and y in the middle plane of the plate and the z axis perpendicular to that plane, let us consider an element cut out of the plate by two pairs of planes parallel to the xz and yz planes, as shown in Fig. 47. In addition to the bending moments M_x and M_y and the twisting moments M_{xy} which were considered in the pure bending of a plate (see Art. 10), there are vertical shearing forces[1] acting on the sides of the element. The magnitudes of these shearing forces per unit length parallel to the y and x axes we denote by Q_x and Q_y, respectively, so that

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz}\, dz \qquad Q_y = \int_{-h/2}^{h/2} \tau_{yz}\, dz \tag{a}$$

Since the moments and the shearing forces are functions of the coordinates x and y, we must, in discussing the conditions of equilibrium of the element, take into consideration the small changes of these quantities when the coordinates x and y change by the small quantities dx and dy.

[1] There will be no horizontal shearing forces and no forces normal to the sides of the element, since the strain of the middle plane of the plate is assumed negligible.

The middle plane of the element is represented in Fig. 48a and b, and the directions in which the moments and forces are taken as positive are indicated.

We must also consider the load distributed over the upper surface of the plate. The intensity of this load we denote by q, so that the load acting on the element[1] is $q\,dx\,dy$.

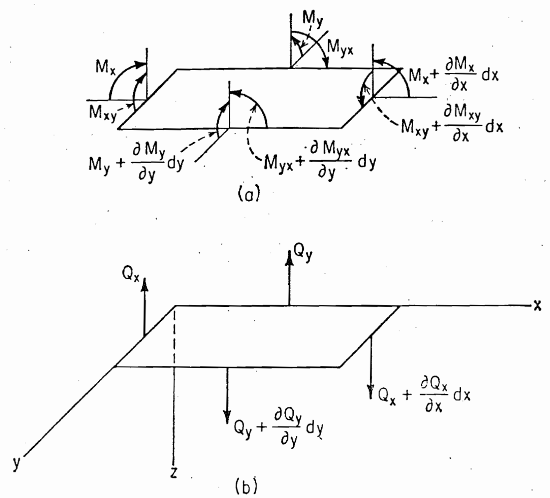

FIG. 48

Projecting all the forces acting on the element onto the z axis we obtain the following equation of equilibrium:

$$\frac{\partial Q_x}{\partial x} dx\,dy + \frac{\partial Q_y}{\partial y} dy\,dx + q\,dx\,dy = 0$$

from which

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \tag{99}$$

Taking moments of all the forces acting on the element with respect to the x axis, we obtain the equation of equilibrium

$$\frac{\partial M_{xy}}{\partial x} dx\,dy - \frac{\partial M_y}{\partial y} dy\,dx + Q_y\,dx\,dy = 0 \qquad (b)$$

[1] Since the stress component σ_z is neglected, we actually are not able to apply the load on the upper or on the lower surface of the plate. Thus, every transverse single load considered in the thin-plate theory is merely a discontinuity in the magnitude of the shearing forces, which vary according to the parabolic law through the thickness of the plate. Likewise, the weight of the plate can be included in the load q without affecting the accuracy of the result. If the effect of the surface load becomes of special interest, thick-plate theory has to be used (see Art. 19).

The moment of the load q and the moment due to change in the force Q_y are neglected in this equation, since they are small quantities of a higher order than those retained. After simplification, Eq. (b) becomes

$$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0 \qquad (c)$$

In the same manner, by taking moments with respect to the y axis, we obtain

$$\frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0 \qquad (d)$$

Since there are no forces in the x and y directions and no moments with respect to the z axis, the three equations (99), (c), and (d) completely define the equilibrium of the element. Let us eliminate the shearing forces Q_x and Q_y from these equations by determining them from Eqs. (c) and (d) and substituting into Eq. (99). In this manner we obtain

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{yx}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial x\,\partial y} = -q \qquad (e)$$

Observing that $M_{yx} = -M_{xy}$, by virtue of $\tau_{xy} = \tau_{yx}$, we finally represent the equation of equilibrium (e) in the following form:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} = -q \tag{100}$$

To represent this equation in terms of the deflections w of the plate, we make the assumption here that expressions (41) and (43), developed for the case of pure bending, can be used also in the case of laterally loaded plates. This assumption is equivalent to neglecting the effect on bending of the shearing forces Q_x and Q_y and the compressive stress σ_z produced by the load q. We have already used such an assumption in the previous chapter and have seen that the errors in deflections obtained in this way are small provided the thickness of the plate is small in comparison with the dimensions of the plate in its plane. An approximate theory of bending of thin elastic plates, taking into account the effect of shearing forces on the deformation, will be given in Art. 39, and several examples of exact solutions of bending problems of plates will be discussed in Art. 26.

Using x and y directions instead of n and t, which were used in Eqs. (41) and (43), we obtain

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \tag{101}$$

$$M_{xy} = -M_{yx} = D(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y} \tag{102}$$

Substituting these expressions in Eq. (100), we obtain[1]

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \tag{103}$$

This latter equation can also be written in the symbolic form

$$\Delta\Delta w = \frac{q}{D} \tag{104}$$

where

$$\Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \tag{105}$$

It is seen that the problem of bending of plates by a lateral load q reduces to the integration of Eq. (103). If, for a particular case, a solution of this equation is found that satisfies the conditions at the boundaries of the plate, the bending and twisting moments can be calculated from Eqs. (101) and (102). The corresponding normal and shearing stresses are found from Eq. (44) and the expression

$$(\tau_{xy})_{\max} = \frac{6M_{xy}}{h^2}$$

Equations (*c*) and (*d*) are used to determine the shearing forces Q_x and Q_y, from which

$$Q_x = \frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{106}$$

$$Q_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{107}$$

or, using the symbolic form,

$$Q_x = -D\frac{\partial}{\partial x}(\Delta w) \qquad Q_y = -D\frac{\partial}{\partial y}(\Delta w) \tag{108}$$

The shearing stresses τ_{xz} and τ_{yz} can now be determined by assuming that they are distributed across the thickness of the plate according to the parabolic law.[2] Then

$$(\tau_{xz})_{\max} = \frac{3}{2}\frac{Q_x}{h} \qquad (\tau_{yz})_{\max} = \frac{3}{2}\frac{Q_y}{h}$$

[1] This equation was obtained by Lagrange in 1811, when he was examining the memoir presented to the French Academy of Science by Sophie Germain. The history of the development of this equation is given in J. Todhunter and K. Pearson, "History of the Theory of Elasticity," vol. 1, pp. 147, 247, 348, and vol. 2, part 1, p. 263. See also the note by Saint Venant to Art. 73 on page 689 of the French translation of "Théorie de l'élasticité des corps solides," by Clebsch, Paris, 1883.

[2] It will be shown in Art. 26 that in certain cases this assumption is in agreement with the exact theory of bending of plates.

It is seen that the stresses in a plate can be calculated provided the deflection surface for a given load distribution and for given boundary conditions is determined by integration of Eq. (103).

22. Boundary Conditions. We begin the discussion of boundary conditions with the case of a rectangular plate and assume that the x and y axes are taken parallel to the sides of the plate.

Built-in Edge. If the edge of a plate is built in, the deflection along this edge is zero, and the tangent plane to the deflected middle surface along this edge coincides with the initial position of the middle plane of the plate. Assuming the built-in edge to be given by $x = a$, the boundary conditions are

$$(w)_{x=a} = 0 \qquad \left(\frac{\partial w}{\partial x}\right)_{x=a} = 0 \tag{109}$$

Simply Supported Edge. If the edge $x = a$ of the plate is simply supported, the deflection w along this edge must be zero. At the same time this edge can rotate freely with respect to the edge line; *i.e.*, there are no bending moments M_x along this edge. This kind of support is represented in Fig. 49. The analytical expressions for the boundary conditions in this case are

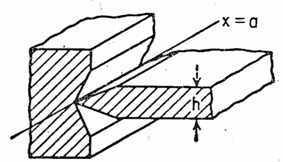

FIG. 49

$$(w)_{x=a} = 0 \qquad \left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)_{x=a} = 0 \tag{110}$$

Observing that $\partial^2 w/\partial y^2$ must vanish together with w along the rectilinear edge $x = a$, we find that the second of the conditions (110) can be rewritten as $\partial^2 w/\partial x^2 = 0$ or also $\Delta w = 0$. Equations (110) are therefore equivalent to the equations

$$(w)_{x=a} = 0 \qquad (\Delta w)_{x=a} = 0 \tag{111}$$

which do not involve Poisson's ratio ν.

Free Edge. If an edge of a plate, say the edge $x = a$ (Fig. 50), is entirely free, it is natural to assume that along this edge there are no bending and twisting moments and also no vertical shearing forces, *i.e.*, that

$$(M_x)_{x=a} = 0 \qquad (M_{xy})_{x=a} = 0 \qquad (Q_x)_{x=a} = 0$$

The boundary conditions for a free edge were expressed by Poisson[1] in this form. But later on, Kirchhoff[2] proved that three boundary conditions are too many and that two conditions are sufficient for the complete determination of the deflections w satisfying Eq. (103). He showed

[1] See the discussion of this subject in Todhunter and Pearson, *op. cit.*, vol. 1, p. 250, and in Saint Venant, *loc. cit.*

[2] See *J. Crelle*, vol. 40, p. 51, 1850.

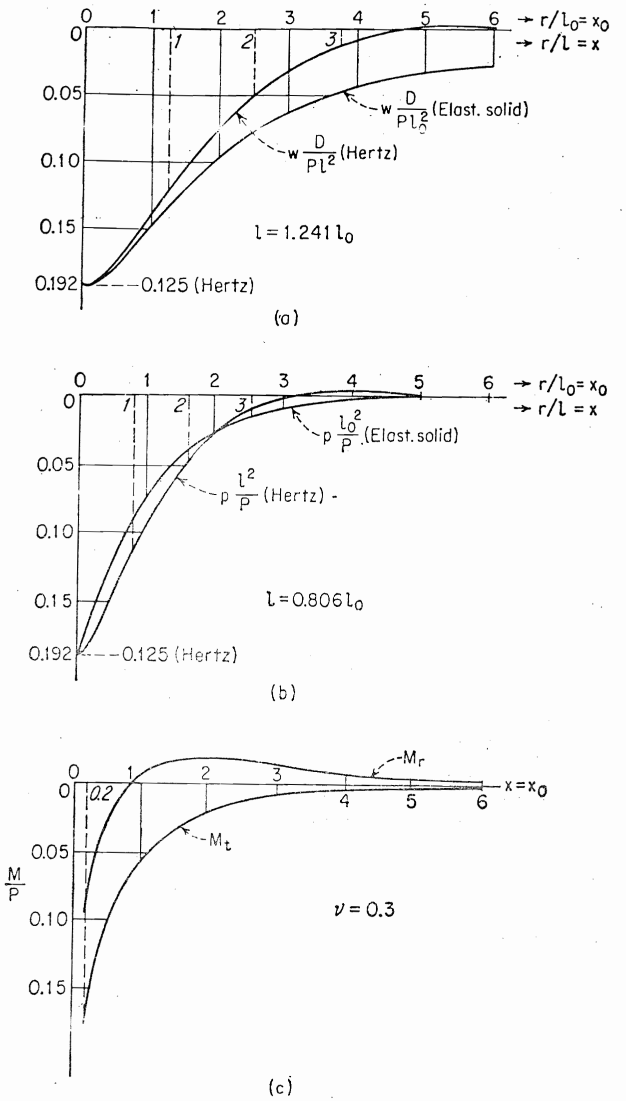

FIG. 131

Fig. 131c. It is seen that the radial moments become negative at some distance from the load, their numerically largest value being about $-0.02P$. The positive moments are infinitely large at the origin, but at a small[1] distance from the point of application of the load they can be easily calculated by taking the function kei x in the form (*i*). Upon applying formulas (52) and (53) to expression (179), we arrive at the results

[1] As compared with the characteristic length $l = \sqrt[4]{D/k}$.

$$
\begin{aligned}
M_r &= \frac{P}{4\pi}\left[(1+\nu)\left(\log\frac{2l}{r}-\gamma\right)-\frac{1}{2}(1-\nu)\right] \\
M_t &= \frac{P}{4\pi}\left[(1+\nu)\left(\log\frac{2l}{r}-\gamma\right)+\frac{1}{2}(1-\nu)\right]
\end{aligned} \tag{182}
$$

A comparison of the foregoing expressions with Eqs. (90) and (91) shows that the stress condition in a plate in the vicinity of the load in Hertz's case is identical with that of a simply supported circular plate with a radius $a = 2le^{-\gamma} = 1.123l$, except for a moment $M'_r = M'_t = -\frac{P}{8\pi}(1-\nu)$, which is superimposed on the moments of the circular plate.

Let us consider now the case in which the load P is distributed over the area of a circle with a radius c, small in comparison with l. The bending moments at the center of a circular plate carrying such a load are

$$M_r = M_t = \frac{P}{4\pi}\left[(1+\nu)\log\frac{a}{c}+1\right] \tag{m}$$

This results from Eq. (83), if we neglect there the term c^2/a^2 against unity. By substituting $a = 2le^{-\gamma}$ into Eq. (*m*) and adding the moment $-P/8\pi(1-\nu)$, we obtain at the center of the loaded circle of the infinitely large plate the moments

$$\mathrm{M}_{\max} = \frac{(1+\nu)P}{4\pi}\left(\log\frac{2l}{c}-\gamma+\frac{1}{2}\right) \tag{n}$$

or

$$M_{\max} = \frac{(1+\nu)P}{4\pi}\left(\log\frac{l}{c}+0.616\right) \tag{183}$$

Stresses resulting from Eq. (183) must be corrected by means of the thick-plate theory in the case of a highly concentrated load. Such a corrected stress formula is given on page 275.

In the case of a load uniformly distributed over the area of a small rectangle, we may proceed as described in Art. 37. The equivalent of a square area, in particular, is a circle with the radius $c = 0.57u$, u being the length of the side of the square (see page 162). Substituting this into Eq. (183) we obtain

$$M_{\max} = \frac{1+\nu}{4\pi}P\left(\log\frac{l}{u}+1.177\right) \tag{o}$$

The effect of any group of concentrated loads on the deflections of the infinitely large plate can be calculated by summing up the deflections produced by each load separately.

59. Rectangular and Continuous Plates on Elastic Foundation. An example of a plate resting on elastic subgrade and supported at the same time along a rectangular boundary is shown in Fig. 132, which represents a beam of a rectangular tubular cross section pressed into an elastic foundation by the loads P. The bottom plate of the beam, loaded by the elastic reactions of the foundation, is supported by the vertical sides of the tube and by the transverse diaphragms indicated in the figure by dashed lines. It is assumed again that the intensity of the reaction p at any point of the bottom plate is proportional to the deflection w at that point, so that $p = kw$, k being the modulus of the foundation.

In accordance with this assumption, the differential equation for the deflection, written in rectangular coordinates, becomes

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} - \frac{kw}{D} \qquad (a)$$

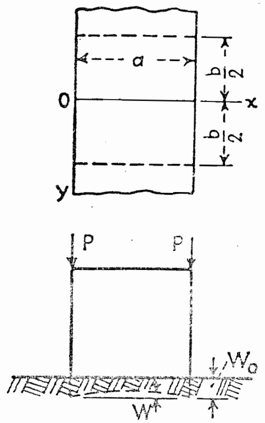

FIG. 132

where q, as before, is the intensity of the lateral load.

Let us begin with the case shown in Fig. 132. If w_0 denotes the deflection of the edges of the bottom plate, and w the deflection of this plate with respect to the plane of its boundary, the intensity of the reaction of the foundation at any point is $k(w_0 - w)$, and Eq. (a) becomes

$$\Delta\Delta w = \frac{k}{D}(w_0 - w) \qquad (b)$$

Taking the coordinate axes as shown in the figure and assuming that the edges of the plate parallel to the y axis are simply supported and the other two edges are clamped, the boundary conditions are

$$(w)_{x=0,x=a} = 0 \qquad \left(\frac{\partial^2 w}{\partial x^2}\right)_{x=0,x=a} = 0 \qquad (c)$$

$$(w)_{y=\pm b/2} = 0 \qquad \left(\frac{\partial w}{\partial y}\right)_{y=\pm b/2} = 0 \qquad (d)$$

The deflection w can be taken in the form of a series:

$$w = \frac{4kw_0}{D\pi}\sum_{m=1,3,5,\ldots}^{\infty}\frac{\sin\frac{m\pi x}{a}}{m\left(\frac{m^4\pi^4}{a^4}+\frac{k}{D}\right)} + \sum_{m=1,3,5,\ldots}^{\infty} Y_m \sin\frac{m\pi x}{a} \qquad (e)$$

The first series on the right-hand side is a particular solution of Eq. (b) representing the deflection of a simply supported strip resting on an elastic foundation. The second series is the solution of the homogeneous equation

$$\Delta\Delta w + \frac{k}{D}w = 0 \qquad (f)$$

Hence the functions Y_m have to satisfy the ordinary differential equation

$$Y_m^{IV} - 2\frac{m^2\pi^2}{a^2}Y_m'' + \left(\frac{m^4\pi^4}{a^4}+\frac{k}{D}\right)Y_m = 0 \qquad (g)$$

Using notations

$$\frac{m\pi}{a} = \mu_m \qquad \frac{k}{D} = \lambda^4 \qquad (h)$$

$$2\beta_m^2 = \sqrt{\mu_m^4+\lambda^4}+\mu_m^2 \qquad 2\gamma_m^2 = \sqrt{\mu_m^4+\lambda^4}-\mu_m^2 \qquad (i)$$

and taking the solution of Eq. (g) in the form e^{ry}, we obtain for r the following four roots:

$$\beta + i\gamma \qquad -\beta + i\gamma \qquad \beta - i\gamma \qquad -\beta - i\gamma$$

The corresponding four independent particular solutions of Eq. (g) are

$$e^{\beta_m y}\cos\gamma_m y \qquad e^{-\beta_m y}\cos\gamma_m y \qquad e^{\beta_m y}\sin\gamma_m y \qquad e^{-\beta_m y}\sin\gamma_m y \qquad (j)$$

which can be taken also in the following form:

$$\begin{matrix}\cosh\beta_m y\cos\gamma_m y & \sinh\beta_m y\cos\gamma_m y\\ \cosh\beta_m y\sin\gamma_m y & \sinh\beta_m y\sin\gamma_m y\end{matrix} \qquad (k)$$

From symmetry it can be concluded that Y_m in our case is an even function of y. Hence, by using integrals (k), we obtain

$$Y_m = A_m\cosh\beta_m y\cos\gamma_m y + B_m\sinh\beta_m y\sin\gamma_m y$$

and the deflection of the plate is

$$w = \sum_{m=1,3,5,\ldots}^{\infty}\sin\frac{m\pi x}{a}\left[\frac{4kw_0}{D\pi}\frac{1}{m\left(\frac{m^4\pi^4}{a^4}+\frac{k}{D}\right)} + A_m\cosh\beta_m y\cos\gamma_m y + B_m\sinh\beta_m y\sin\gamma_m y\right] \qquad (l)$$

This expression satisfies the boundary conditions (c). To satisfy conditions (d) we must choose the constants A_m and B_m so as to satisfy the equations

$$\frac{4kw_0}{D\pi}\frac{1}{m\left(\frac{m^4\pi^4}{a^4}+\frac{k}{D}\right)} + A_m\cosh\frac{\beta_m b}{2}\cos\frac{\gamma_m b}{2} + B_m\sinh\frac{\beta_m b}{2}\sin\frac{\gamma_m b}{2} = 0$$

$$(A_m\beta_m + B_m\gamma_m)\sinh\frac{\beta_m b}{2}\cos\frac{\gamma_m b}{2} - (A_m\gamma_m - B_m\beta_m)\cosh\frac{\beta_m b}{2}\sin\frac{\gamma_m b}{2} = 0 \qquad (m)$$

Substituting these values of A_m and B_m in expression (l), we obtain the required deflection of the plate.

The problem of the plate with all four edges simply supported can be solved by using Eq. (a). Taking the coordinate axes as shown in Fig. 59

(page 105) and using the Navier solution, the deflection of the plate is

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (n)$$

In similar manner let the series

$$q = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (o)$$

represent the distribution of the given load, and the series

$$p = kw = \sum \sum kA_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (p)$$

represent the reaction of the subgrade. Substituting the series (n) in the left-hand side and the series (o) and (p) in the right-hand side of Eq. (a), we obtain

$$A_{mn} = \frac{a_{mn}}{\pi^4 D\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 + k} \qquad (q)$$

As an example, let us consider the bending of the plate by a force P concentrated at some point (ξ,η). In such a case

$$a_{mn} = \frac{4P}{ab} \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b} \qquad (r)$$

by Eq. (b) on page 111. By substitution of expressions (q) and (r) into Eq. (n) we finally obtain

$$w = \frac{4P}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b}}{\pi^4 D\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 + k} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (s)$$

Having the deflection of the plate produced by a concentrated force, the deflection produced by any kind of lateral loading is obtained by the method of superposition. Take, as an example, the case of a uniformly distributed load of the intensity q. Substituting $q\,d\xi\,d\eta$ for P in expression (s) and integrating between the limits 0 and a and between 0 and b, we obtain

$$w = \frac{16q}{\pi^2} \sum_{m=1,3,5,\ldots}^{\infty} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{mn\left[\pi^4 D\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 + k\right]} \qquad (t)$$

When k is equal to zero, this deflection reduces to that given in Navier solution (131) for the deflection of a uniformly loaded plate.[1]

Let us consider now the case represented in Fig. 133. A large plate which rests on an elastic foundation is loaded at equidistant points along the x axis by forces P.* We shall take the coordinate axes as shown in

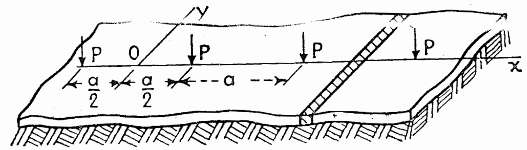

FIG. 133

the figure and use Eq. (f), since there is no distributed lateral load. Let us consider a solution of this equation in the form of the series

$$w = w_0 + \sum_{m=2,4,6,\ldots}^{\infty} Y_m \cos \frac{m\pi x}{a} \qquad (u)$$

in which the first term

$$w_0 = \frac{P\lambda}{2\sqrt{2}\,ak} e^{-\lambda y/\sqrt{2}} \left(\cos \frac{\lambda y}{\sqrt{2}} + \sin \frac{\lambda y}{\sqrt{2}}\right)$$

represents the deflection of an infinitely long strip of unit width parallel to the y axis loaded at $y = 0$ by a load P/a [see Eq. (283), page 471]. The other terms of the series satisfy the requirement of symmetry that the tangent to the deflection surface in the x direction shall have a zero slope at the loaded points and at the points midway between the loads. We take for functions Y_m those of the particular integrals (j) which vanish for infinite values of y. Hence,

$$Y_m = A_m e^{-\beta_m y} \cos \gamma_m y + B_m e^{-\beta_m y} \sin \gamma_m y$$

To satisfy the symmetry condition $(\partial w/\partial y)_{y=0} = 0$ we must take in this expression

$$B_m = \frac{\beta_m A_m}{\gamma_m}$$

[1] The case of a rectangular plate with prescribed deflections and moments on two opposite edges and various boundary conditions on two others was discussed by H. J. Fletcher and C. J. Thorne, *J. Appl. Mechanics*, vol. 19, p. 361, 1952. Many graphs are given in that paper.

* This problem has been discussed by H. M. Westergaard; see *Ingeniøren*, vol. 32, p. 513, 1923. Practical applications of the solution of this problem in concrete road design are discussed by H. M. Westergaard in the journal *Public Roads*, vol. 7, p. 25, 1926; vol. 10, p. 65, 1929; and vol. 14, p. 185, 1933.